Nonlinear System Analysis
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Lecture - 08
Existence and Uniqueness Theorem of ODE Part 01

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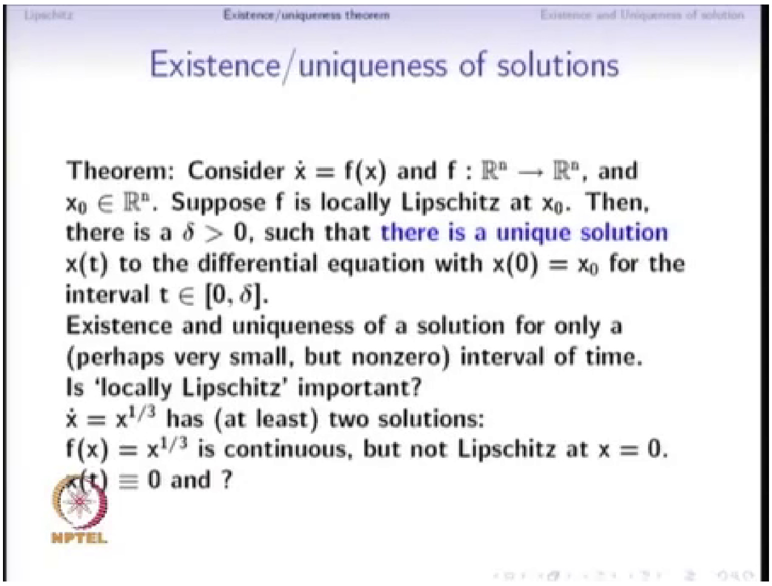

We finally come to the Existence and Uniqueness Theorem for solution to a differential equation. So, consider x dot is equal to f of x consider this differential equation, where f is a map from R n to R n and consider the point x naught in R n. Assume, f is locally Lipschitz at x naught then there is a delta greater than 0 such that there is a unique solution x of t to the differential equation, to the differential equation x dot is equal to f of x with the initial condition x 0 equal to x naught.

So, such a solution, such a unique solution exist for the time interval t belonging to 0 to delta. So, there is some number delta that is strictly positive because of which there is this interval of time 0 to delta for which we have a solution and moreover the solution is unique. That is what it says; with the initial condition x 0 is equal to the point x naught where the function f was assume to be locally Lipschitz.

So, it turns out that existence and uniqueness of a solution is being guaranteed for only an interval of time. It is possible that this interval of time is very small, but it is guaranteed to be a non-zero interval of time it is not just one point, but it is an interval 0 to delta. You can ask the question is locally Lipschitz important. After all we have spent analyzing the significance of Lipschitz, locally Lipschitz, it is relation to differentiability and continuity, we could ask is this locally Lipschitz property really crucial for the existence and uniqueness of a solution to the differential equation.

For this purpose we will see that the differential equation x dot is equal to x to the power 1 by 3 has at least two solutions. Why? Because x to the power 1 by 3, turned out to not be locally Lipschitz and hence uniqueness was not guaranteed by this theorem and we will see that there are two solutions. So, what are the solutions? This is what we will see now.

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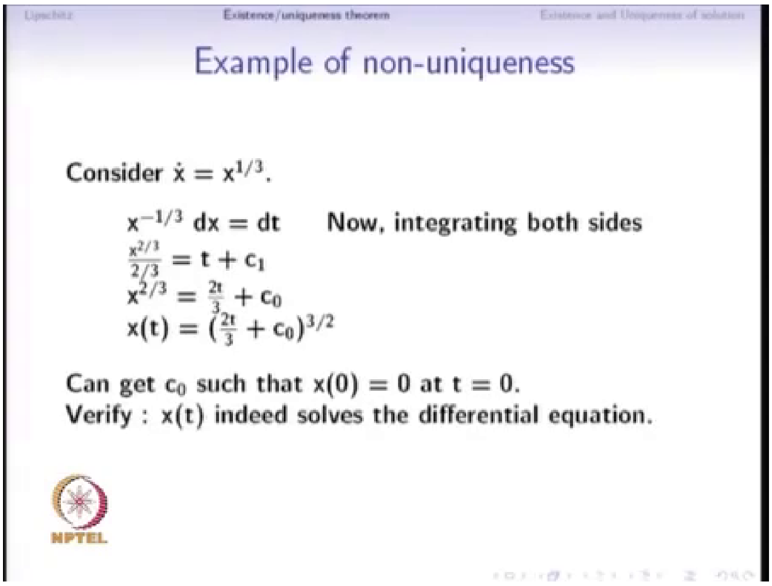

So, consider this differential equation x dot is equal to x to the power 1 by 3. So, we can re-write this as x to the power minus 1 my 3 dx equal to dt and upon integrating both sides we see that we get this equality, where c 1 is some constant and re-writing these terms x to the power 2 by 3 is now equal to 2 t by 3 plus c 0. So, c 0 and c 1 are related by the constant 2 by 3. For convenience we have renamed c 1 times 2 by 3 as c naught.

Then upon taking suitable powers we see that x of t, x of t is equal to 2 t by 3 plus c naught raised to the power 3 by 2. So, we see that this is also a solution to the differential equation. And how do we calculate c naught? We can ask the question suppose that t equal to 0, the solution the differential equation was at 0 x satisfied at 0 is equal to 0, at t equal to 0. By substituting this we can get c 0.

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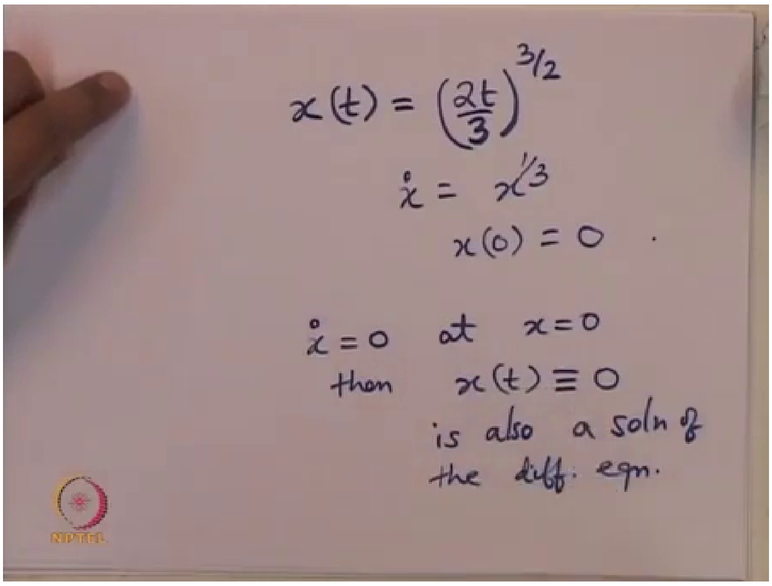

So, we see that x of t equal to 2 t by 3 to the power 3 by 2 is a solution to the differential equation. Which differential equation? x dot is equal to x to the power 1 by 3 with the initial condition x 0 is equal to 0. But, is this the only solution? No, because if x dot equal to 0 at x equal to 0, then we also know that x of t is also a solution, also a solution of the differential equation. So, we see that if x is equal to 0 at t equal to 0, then x dot is equal to 0. Why x dot equal to 0?.

Because we put x equal to 0 here and cube root of 0 is nothing but 0 and hence x dot is equal to 0, and then we conclude that x is equal to 0 for all time t. This is one solution to the differential equation. But we see that x t is equal to 2 t by 3 whole to the power 3 by 2 is also a solution to the differential equation with the same initial condition. So, for the same initial condition we see that we have two solutions to the differential equation. Let us just draw a graph of x versus time.

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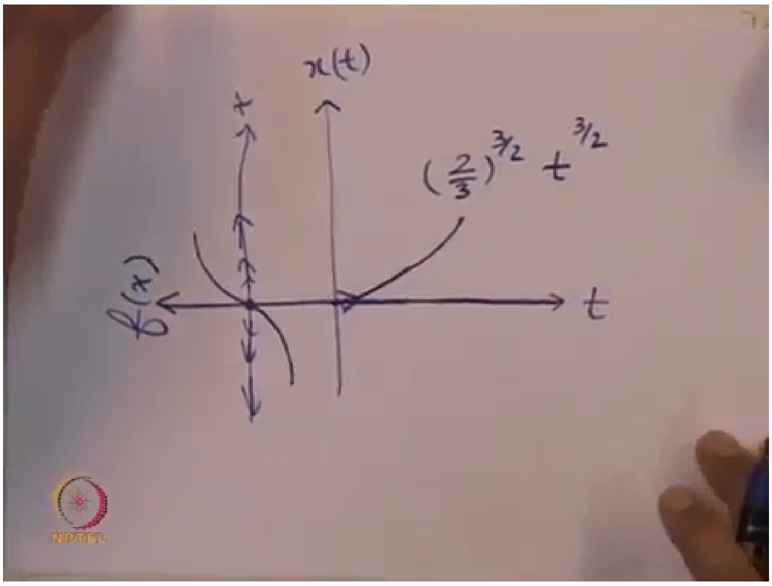

So, here is this x that was 0 until t equal to 0 and from here it is growing as 2 by 3 to the power 3 by 2 times t to the power 3 by 2. This is a graph of x versus time t. And in addition, the differential equation also has equivalently equal to 0 as a solution to the differential equation. In other words, at this point t equal to 0 there is this solution that comes out of x equal to 0 and it also continues at x equal to 0 has another solution.

At this point the vector field we see that it is pointed, sorry this is something we have to plot like this. So, the graph of x to the power 1 by 3 is like this. we see that of course, there is this instability, but at x equal to 0 itself the arrow had length 0 because the graph of f crossed the x axis at x equal to 0, and hence if it is at the point x equal to 0 then it continues to be at the point x equal to 0. This is what out vector field diagram told us.

But here we see that in addition to continuing to be at 0, there is also this possibility that it comes out, it emanates out of the equilibrium point without requiring a perturbation. While this figure says that this equilibrium point is a unstable equilibrium point, we see that upon perturbation there are points that are trajectories that are going away from the equilibrium point.

But here is an example because of this non-locally Lipschitz property at the point x equal to 0 without a perturbation also we see that there is a solution that emanates out of the equilibrium point. Thus making us ask what is what exactly is a definition of an equilibrium point. When, there are solutions that can emanate out of the equilibrium point even without requiring a perturbation.

For these purposes, we will from now on eventually assume that a function f when studying a differential equation satisfies locally Lipschitz at every at every point x naught, but this is an example where without locally Lipschitz property it turns out that there can be non-uniqueness of solutions to the differential equation.

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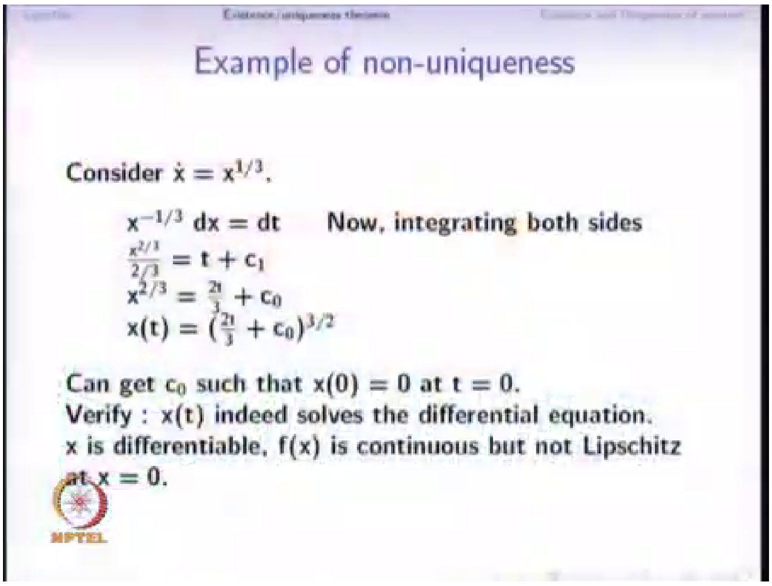

So we see that the solution x is differentiable, we are we were able to get explicitly the solution x as a function of time. We can also differentiate this as a function of time and see that it solves this differential equation. f of x is continuous but it is not locally Lipschitz at x equal to 0, that is why the previous theorem would not guarantee existence of solutions, existence and uniqueness of solutions. And here is an example where uniqueness fails.

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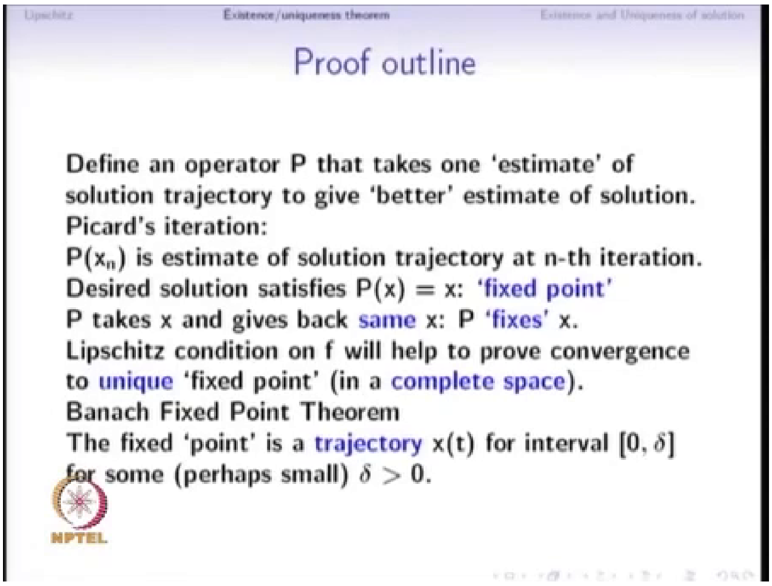

So, because of the importance of this result of existence and uniqueness of solution to differential equation under locally Lipschitz property, we will because of its importance we will see the proof. So, the outline of the proof will be as follows. We will define an operator P that takes one estimate of the solution trajectory and gives a better estimate of the solution. So, this operator P takes a trajectory, a trajectory which is a solution to the differential equation on the interval 0 to delta and it gives a better estimate of the solution.

So, this is going to be called Picard's iteration, because of Picard's work in this area. So, P x n is an estimate of the solution trajectory at the nth iteration. We will define P such that the desired solution will satisfy P x equals to x in other words x will be a so called fixed point. Why do we call it fixed? Because this desired solution the solution to the differential equation takes x and gives back the same x. In other words, while it takes different x n and gives back x n plus 1.

Possibly x n plus 1 is different from x n. While this is possible in general we will construct P such that the desired solution x will satisfy P x equal to x, in another words P fixes x. So, the Lipschitz condition on f will help to prove convergence of this iteration convergence, convergence of this iteration to a unique fixed point, but this fixed point would be unique provided we are looking for a so called complete space. This is a these are something that we will define precisely. So, for this purpose we will use a so called a Banach fixed-point theorem.

So, please note that the point in this context is a trajectory x of t for the interval 0 to delta. Possibly the interval delta, the interval 0 to delta is a very small interval in other words delta is only slightly more than 0, but it is positive which means that the interval is not just a point, but it is the interval of time of length delta.

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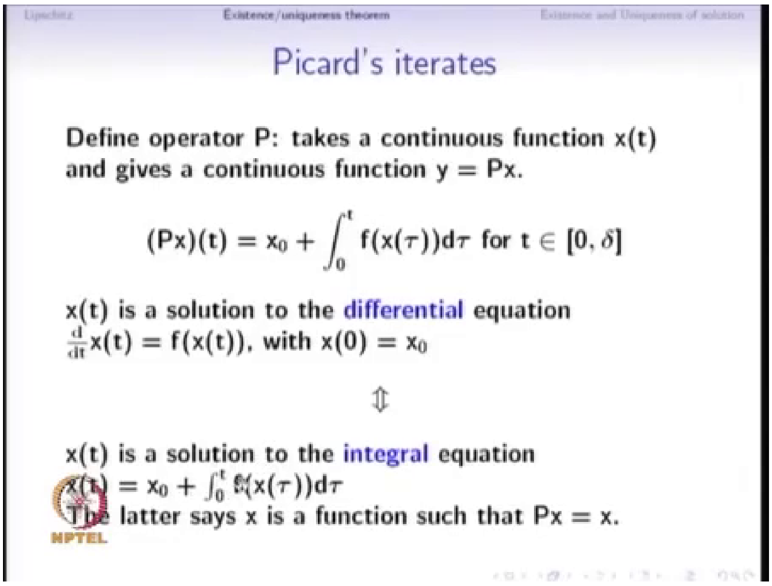

So, how will we define this Picard's iterates. So, define the operator P that takes a continuous function x of t and gives another continuous function y is equal to P of x. How will we define it? P x is again a function of time. P x at any time t is defined as x naught plus the integral from 0 to t of f of x tau d tau, where t belongs to the interval 0 to delta. So, t comes in here and this integral is being added to 0. So, we see that for this t equal to 0 this particular trajectory is also equal to x naught. So, these are different different functions of time that start from x naught.

So, what is the significance of this operator P with our solution to the with our differential equation? We see that x t is a solution to the differential equation d by dt of x is equal to f of x with this initial condition x 0 is equal to x naught, if and only if x t is a solution to the integral equation, to this integral equation, x of t equal to x naught plus integral from 0 to t f of x tau d tau. So, by differentiating this right hand side, this is something we can quickly check. Why is solution to this integral equation also solution to the differential equation?

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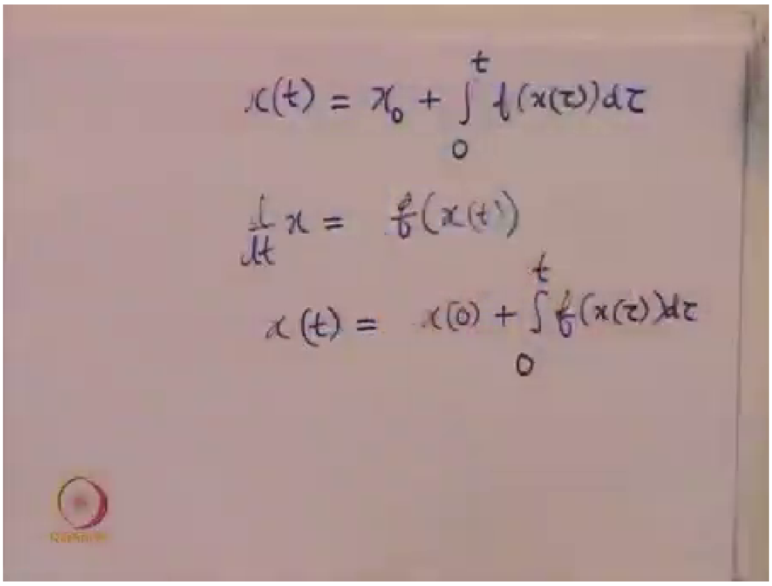

We see that x appears on both sides. Here is x and x is also inside this integral. So, d by dt of x is equal to dt of this is a is equal to 0 and since t appears only here this is nothing, but f evaluated at the end point. So, this is what our differential equation was. In other words a solution to the integral equation is exactly a solution to this differential equation also. When we integrate this on both sides we see that x of t also satisfies this initial condition plus integral from 0 to t of the right hand side.

In other words, solution to the differential equation when integrating both sides this differential equation we obtain exactly the integral equation. Thus, solution to the integral equation is a solution to the differential equation and solution to the differential equation is also a solution to the integral equation. However, we see that here also x appears on both sides and here also x appears on both sides, and it is not clear that going from a differential equation to the integral equation is genuinely a improvement.

We will see that obtaining an integral equation allows us to use Picard's iteration. So, the latter what is important is that when x satisfies this integral equation that x is also a fixed point of this operator P. Why? Because the right hand side is nothing but P of x and what x we have put in here is exactly what is also here, that is why this particular integral equation says that P of x equals to x.

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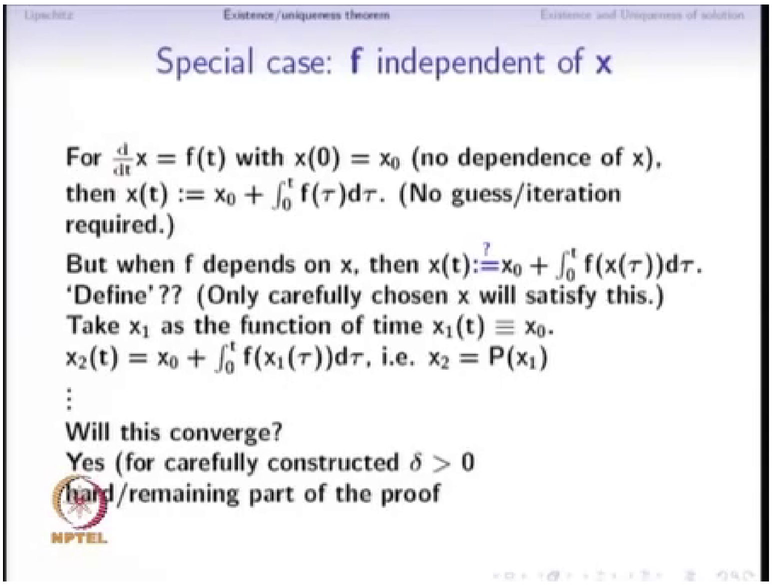

So, we will first see the special case when f is independent of x. Maybe f is allowed to depend on time. Only here we will assume that f is a function of time explicitly and there is no dependence on x. So, consider this differential equation d by dt of x is equal to f of t with this initial condition x naught. So, there is no dependence of f on x. Then, we are able to integrate this, x of t we can then define as x naught plus this integral from 0 to t of f of tau d tau.

So, there is no guess or iteration required to define this. Such a definition x will always satisfy this differential equation, but when f depends on x then we are not able to define this. Why? Because x appears both on the right hand side and left hand side of this integral equation. So, the word define can no longer be used here. So, only a carefully chosen x will satisfy this equation and that is indeed the solution to our integral equation.

So, what we will do now is we will take x 1 as some function of time it turns out that within a small neighborhood of the actual solution which solution we take will not matter. So, we will take x 1 of t equivalently equal to x naught. So, we will take the function x 1 which is always equal to x naught value. x naught is a point in R n which corresponds to our initial condition. There is one particular function which is always equal to x naught for all time t that we will take as our initial x 1. Then we will define x 2 as x naught plus this integral with f evaluated at x 1 instead of x 2.

So, now, since we have x 1 here which we know and x 2 which we do not know we are allowed to use x 2 defined by this right hand side. In other words, x 2 is equal to P times x 1. So, we can similarly define x 3 as P times x 2 and in general x n plus 1 as P times x n. So, the question arises will this converge, will this converge to a solution, in other words, will it converge to a fix point of the operator P.

So, the answer is it will converge for a carefully constructed delta that will be strictly greater than 0 and for ensuring that is delta is strictly greater than 0 and that it exists we will be using the Lipschitz locally Lipschitz property of f at the point x naught. So, this is the hard and remaining part of the proof which we will proceed and do now.

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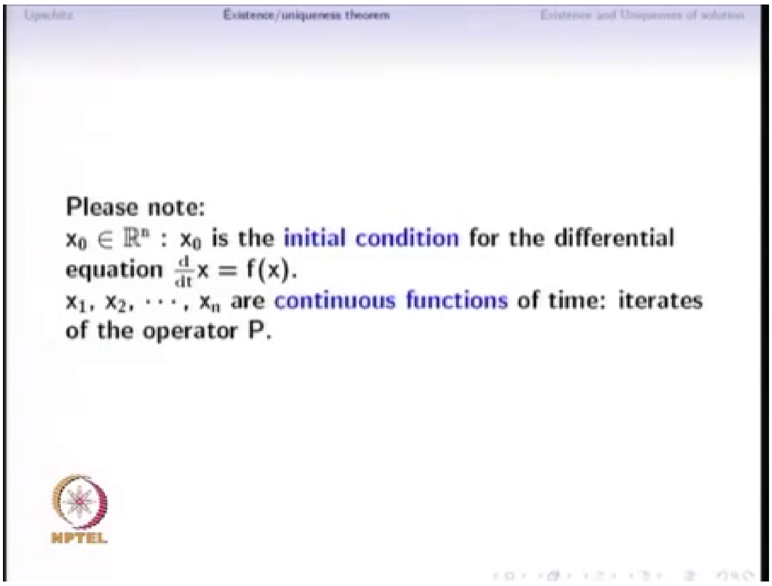

So, please note an x naught is a point in R n. This is a initial condition for the differential equation d by dt of x is equal to f of x. On the other hand, x 1, x 2, up to x n are continuous functions of time and these are iterates of the operator P these are no longer points in R n, but they are functions which take their values in R n for different time instance.