Nonlinear System Analysis
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Lecture – 07
Lipschitz Continuity and Contraction Mapping Theorem - Part 03

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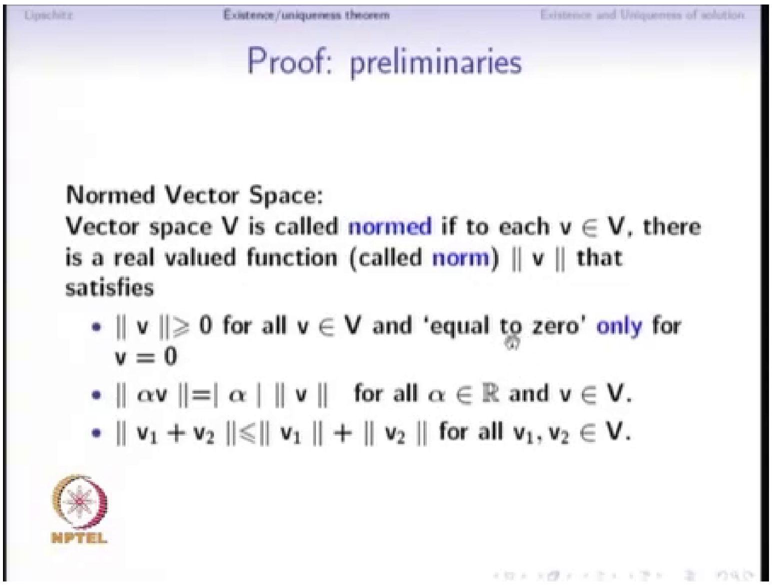

So, for this proof we need some preliminaries. So, what for that purpose we will need to use the Banach Fixed Point Theorem and these are some preliminaries for that purpose. So, we need the notion of a normed vector space. So, vector space V is called normed if to each vector v in the vector space capital V, there is a real valued function which we will call as the norm and you will denote is a norm of the by this notation such that this norm is always greater than or equal to 0..

So, norm of a vector v will be a real number and it cannot be negative that is what it says moreover. So, this is greater than or equal to 0 for all V and also it is equal to 0 only for the vector v equal to 0. This is the first condition required from the norm function. Another condition that is required is if you scale the vector v by a constant alpha, then the norm also get scaled by that same number alpha, the same number alpha if alpha is positive and absolute value of alpha in general. So, for any real number alpha and a vector v, this equality has to be satisfied and finally the triangular inequality is required to be satisfied.

So, for two vector v 1 and v 2 the norm of v 1 plus v 2 cannot be more than norm of v 1 plus norm of v 2. This triangular inequality is also required to be satisfied for all vectors v 1 and v 2. So, this is called Triangular inequality, this is called the Linearity property of the norm function and this is the definition that the notion of length of a vector is required to satisfy.

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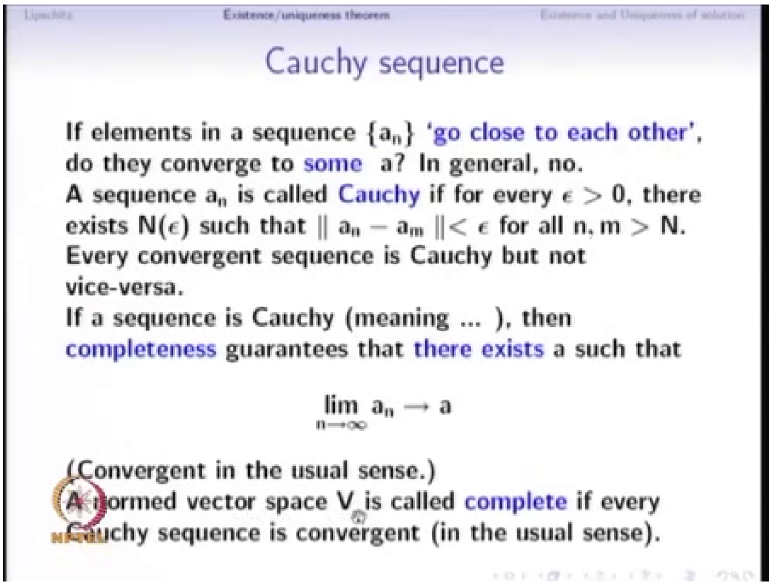

In the context of convergence, we also need what is the definition of Cauchy sequence. So, in the context of convergence we like to say that elements in a sequence are going close to each other. So, in general, if elements in a sequence a n go close to each other, do they converge to some element a? So, the answer is in general no. Just because elements a n are converging close to each other does not mean they will eventually converge to a number a..

So this if it does converge, then we will call the sequence convergence. In order to arrive at that we will define this property called Cauchy. So, a sequence a n is called Cauchy if for every epsilon greater than 0 there exists some capital N possibly depending on epsilon, such that if we want that a n and a m are smaller than epsilon, then that will be satisfied for all n and m that are greater than this capital N..

So, the importance of the statement is it says that elements a m and a n are going close to each other; if you want them to be close to each other closer than a number epsilon, then all we have to do is take m and n greater than some capital N. So, when somebody specifies the epsilon greater than 0, we are able to find number n such that this inequality satisfies for all n and m greater than capital N..

So, in general if epsilon is made smaller, then capital N might have to be made larger. So this quantifies, this makes precise the notion that elements in the sequence a n are going closer and closer to each other. So, every convergent sequence is Cauchy, but turns out that the converge is not true. So, if a sequence is Cauchy it just means that elements are going close to each other, but it does not imply that there is a number a to which it converges. So, if we assume an important property called completeness, then we will also be able to guarantee convergence.

So, what is the definition of completeness? See if a sequence is Cauchy, then completeness would guarantee that there exists a number a such that a n converges to a and this later convergence is what we will say convergent in the usual sense. So, this is what we will use for the definition of complete of a norm vector space. A norm vector space v is called complete if every Cauchy sequence is convergent, convergent in the sense of this limit..

So, we so we see that if every Cauchy sequence in other words are sequence in which all elements are going closer and closer to each other, if we are going to say that it converges to a number in that particular set in that particular vector space V, then this vector space V is going to be called complete.

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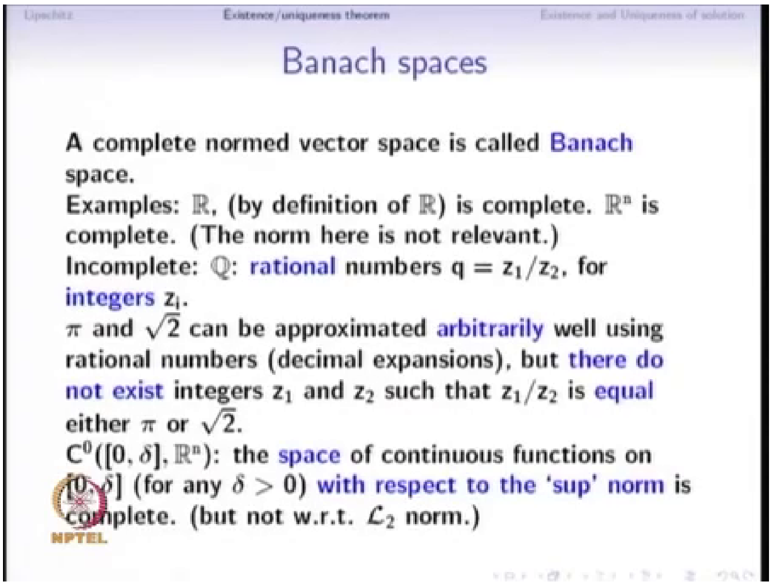

Finally, a complete normed vector space is called a Banach space. So, what is a Banach space? It is a norm vector space, which is also complete. In other words every Cauchy convergence sequence is also convergent.

Some examples of complete spaces, a set of real numbers R. This is by definition of R. This is complete. R n is also complete where n is some finite number. So, if we take n tuples of real numbers, then this is also complete vector space. So, which norm here it turns out is not relevant, yeah we could take for examples the standard Euclidian norm. We could take the Euclidian norm which is very standard.

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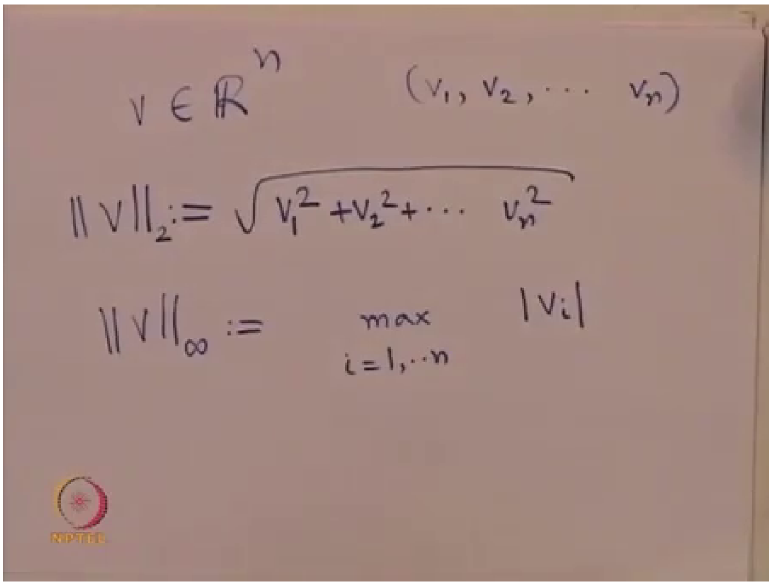

Suppose a vector V is in R n, then we can define this is one of the definitions V 1 V square plus V 2 square plus V n square. This guarantees are it is positive and then we take square root, it is also linear in V. So, this is one of the norms of R n. What is R n? It is a n tuple every element in R n is a n tuple in which n is finite. So, this is one definition of norm and this is called the two norm. It is also called the euclidian norm. We could also have taken V so called the infinity norm as the maximum for i equal to 1 up to n of the absolute value of the i th component V V 1 up to V n, there are n components in the vector V..

For each of those vectors we take the absolute value and for each of the components we take the absolute value and we look at the maximum for i varying from 1 to n of this absolute value that is called the infinity norm.

So, we will see some more notions of norm. So, with respect to any of the norms it turns out that R n is complete. What are examples of incomplete spaces? So the set q of rational numbers, this is incomplete. What is rational q is a standard notation for the set of rational numbers. Any number q small q, in this capital Q can be written as the ratio of two integers z 1 by z 2. So, if a number q can be written as ratio of two integers, then q is said to be a rational number and capital Q is a set of all such rational numbers. So, it turns out that q is not complete. For example pi and square root of 2, these are some numbers which can be approximated arbitrarily well, using rational numbers to be able to approximate arbitrarily well.

For example, we know that square root of 2; can have a decimal expansion. If you take more number of digits in a decimal expansion, then we get a more accurate representation of the number square root of 2 and similarly for pi. But just because we can approximate square root of 2 and pi by a very good approximation using decimal expansions, that does not imply that there are integers z 1 and z 2 such that z 1 by z 2 is equal to either pi or square root of 2..

In fact, they do not exist z 1 and z 2 such that z 1 by z 2 is equal to pi. Similarly they do not exist integers z 1 and z 2, such that z 1 by z 2 is equal to square root of 2. In other words, pi and square root of 2 are so-called irrational numbers. However we know that these irrational numbers can be approximated arbitrarily well, this this proves that q is not complete.

Another important set, another important space that is complete is the so-called space of continuous functions on this interval 0 to delta, where delta is strictly greater than 0 with respect to the sup norm. With respect to the sup norm, it turns out that this space of continuous function is complete. So, this space of continuous functions on this interval functions from this interval to R n is also denoted by C by this notation..

So, the 0 here means that it is just continuous; differentiability is not being guaranteed. So, I should emphasize that it is important that with respect to the sup norm only C 0, this is one is complete. There could be some other norms with respect to which it is complete, but here it is no longer the case that the norm is not relevant which norm we are taking that will very crucially decide whether this space of continuous function is complete or not, that statement depends on the norm with respect to which we are asking the question.

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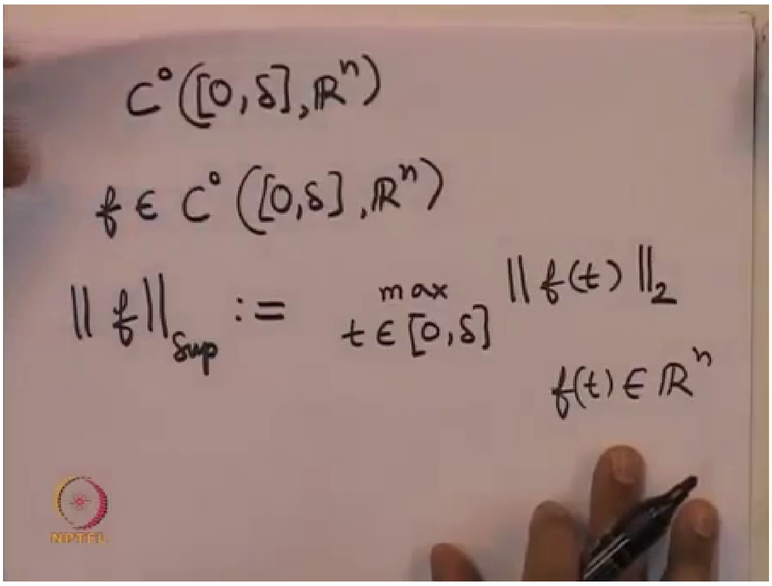

The space of continuous function from the interval 0 to delta to R n, these are functions which are not necessarily differentiable, but they are just continuous and hence 0 has been put here. They take their the domain is from 0 to delta, it takes its values time varies from 0 to delta and for each time instant in this interval it gives a vector in R n. So, this is the space of continuous functions from this interval to R n.

And we are saying sup norm, what is a sup norm? If f is an element of this set, then the sup norm they are going to define as the maximum when t varies from 0 to delta of f of t. So, at each time instant f of t is an element of R n. Yeah, this is the meaning of f is a element of continuous functions from this interval to R n. So at each time instant t, this is some vector and for that vector we defined already the Euclidian norm, the two norm and this two norm is defined for each time t in this interval and we will look at the maximum of this particular function as t varies from 0 to delta, this maximum is set to be the sup norm of the function f..

So, sup norm is no longer relevant to a particular time instant even though f of t 2 norm was relevant to which time instant the two norm was been taken. So, the sup norm is a norm on the space of continuous functions from this interval to R n. So with respect to this sup norm, it terms out that this is a complete space, but with respect to which norm is it not complete. For example we can also define the so-called L 2 norm.

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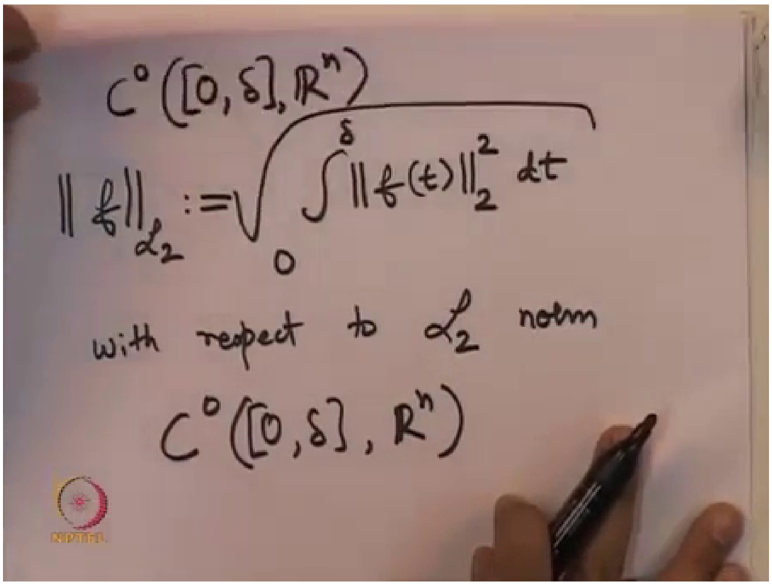

So, what is the L 2 norm? For the same space of functions we can define f L 2 norm as integral from 0 to delta of. So, we can take the two norm of the vector f of t at a time instant t and the two norm, we take the square and we integrate from 0 to delta. Upon integrating from

0 to delta, we get some number that is no longer dependent on the time t and its square root is set to be the L 2 norm of the function f, where f is an element of this.

So, this L 2 norm also is no longer dependent on t, but f of t before we took the two norm here is indeed a function of time t and after we integrate from 0 to delta, this is no longer dependent on time t. So, this is the L 2 norm. So, with respect to with respect to L 2 norm, this same space of functions is not complete. Just because elements are going very close to each other does not mean that it will also converge to a function that is continuous and close to each other for that we are using the notion of L 2 norm. So, with respect to the sup norm however two functions if they are close to each other, they will eventually converge to a continuous function again inside the space.

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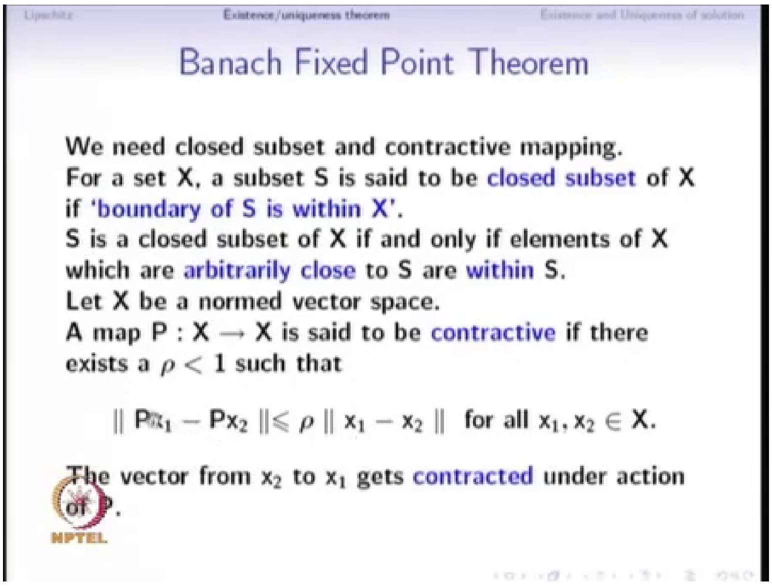

So, finally using these notions we can prove we can state the Banach fixed point theorem. So, for that we just need the notion of close subset and the contractive mapping. So for a set X, a subset S, a subset S of X is said to be a close subset of X; if the boundary of S is within X. This is how we understand a close subset more precisely. S is a close subset of X if and only if elements of X which are arbitrarily close to S are also within S. If you take an element of X which is very close to elements of S which is close to one or more elements of S, then it is not necessary that this elements of X should also be within S, but if it is indeed within S, then f is said to be a closed subset of the set X.

And what is contractive? So, if X is a normed vector space A map P from X to X is said to be contractive, if there exists a number rho that is strictly less than 1, such that this inequality is satisfied for all x 1 and x 2 in x. What is this inequality? P x 1 minus P x 2 is at most rho times x minus x 2. So, what is the importance of this inequality? x 1 minus x 2 can be interpreted as a vector from x 2 to x 1 and P x 2 to P x 1 is the vector upon action under P..

So, this vector is getting contracted under the action of P. Why contraction? Because this rho is some number strictly less than 1. So, we will say this map is contractive, if there exist such a number rho, such that this inequality satisfied for all x 1 and x 2 in x, the number the number rho should not have to be modified depending on which x 1 and x 2 we take in capital X. For such a map P, we will say it is contractive if such a number rho exists.

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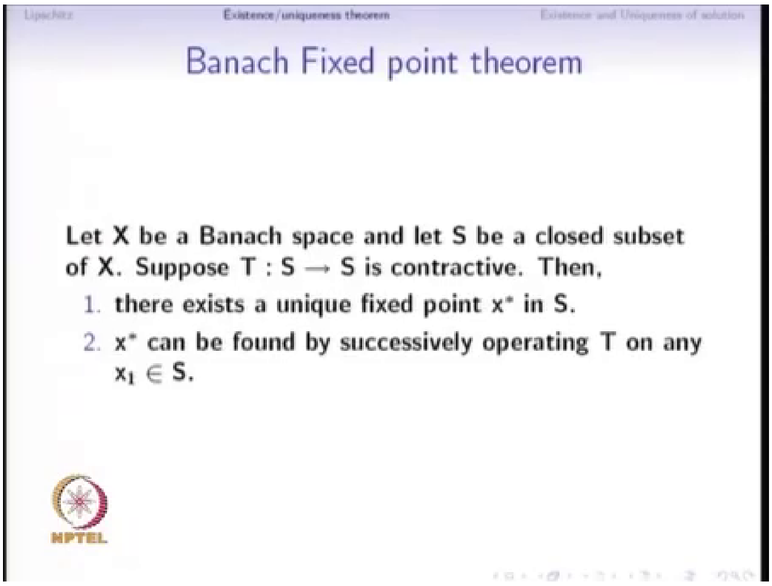

We are ready to state the Banach Fixed point theorem using the notions we have just now defined. Let X be a Banach space and let S be a close subset of X and then let T a map of S to S, let T be contractive. Then first of all there exists a unique fixed point x star in S, fixed point for the this operator T. Moreover that fixed point x star can be found quite easily what is easily it can be found by successively operating T on any x 1 in S. We take any x 1, then you take T times X 1, then we take T times T of x 1, then we do these iterations, we will converge to that unique fixed point.

We will use this Banach Fixed point theorem for the proof of the existence and Uniqueness theorem of the solution to the differential equation in the next lecture.

Thank you.