Nonlinear System Analysis
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Lecture - 06
Lipschitz Continuity and Contraction Mapping Theorem - Part 02

Welcome to lecture number 4 of Non-linear dynamical systems, we had just begun seeing what a Lipschitz function is.

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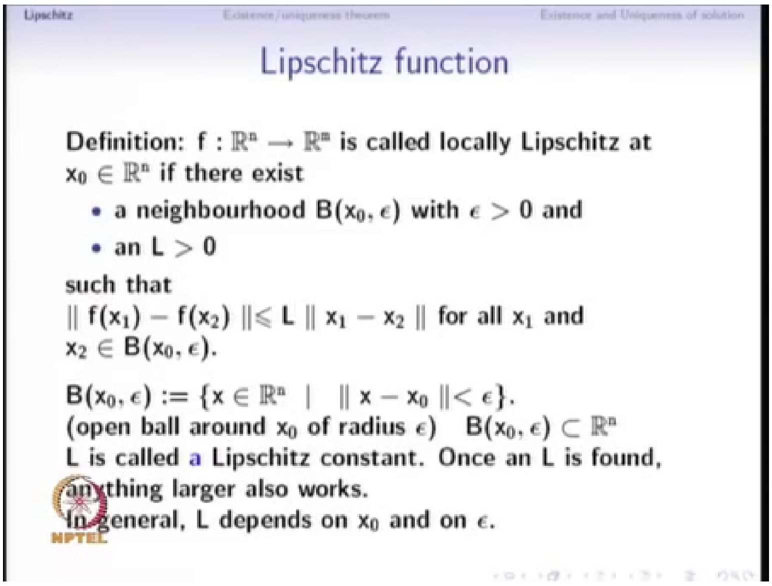

So, let us just recapitulate the definition. So, function f from R n to R m in called locally Lipschitz at a point x naught in R n; if that exists a neighbourhood B is; a neighbourhood here in this case is defined to be a ball centered around x naught with radius epsilon and the radius is greater than 0. There should exist a neighbourhood and some constant L such that this inequality is satisfied for all points x 1 and x 2 in that neighbourhood of the point x naught.

So, we are using a ball which is an open ball which means that the distance of every point in that ball is strictly less than epsilon from the center x naught; that is why it is called a open ball. So, L is called a Lipschitz constant; once an L is found, we can see that anything larger also can be put in here and that inequality and this inequality will be satisfied with a larger L also. So, in general L depends on both x naught and on epsilon.

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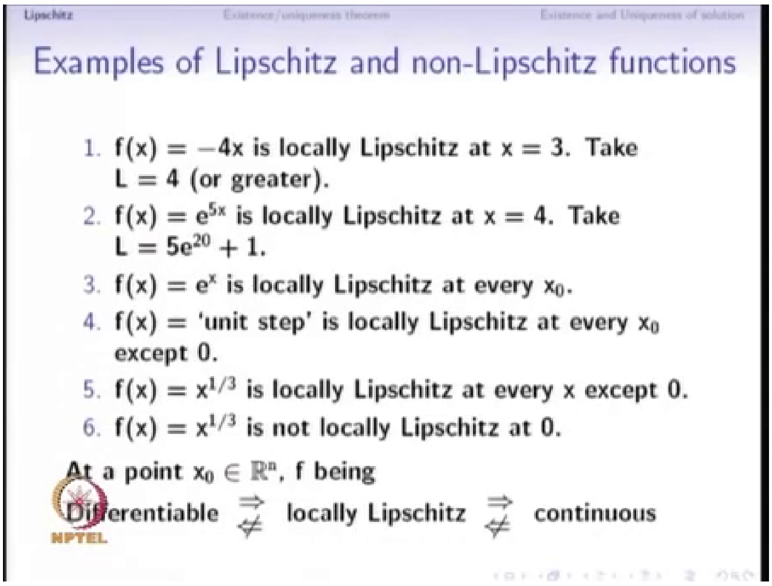

So, some examples of Lipschitz functions; f x is equal to minus 4 x is locally Lipschitz at x is equal to 3; you will later see that it is locally Lipschitz everywhere. In fact, it is globally Lipschitz. This is what we will see very soon, but we had just begun drawing a graph what is the meaning of Lipschitz for the special case.

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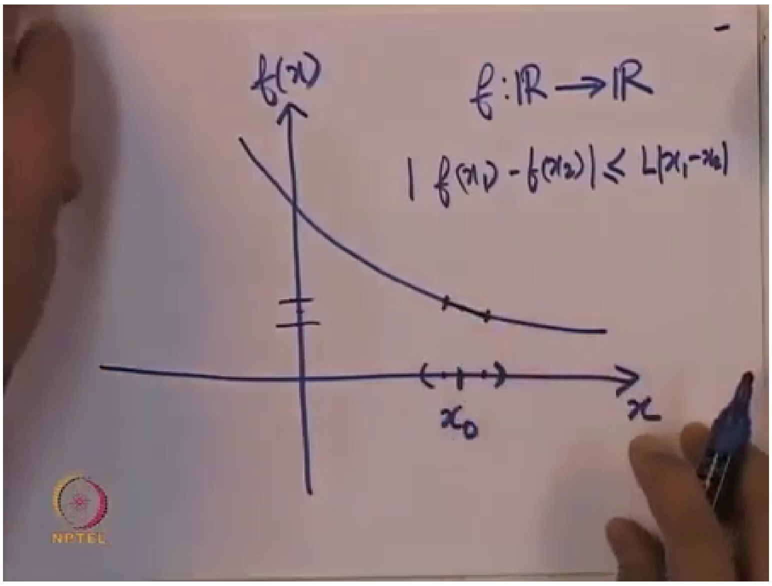

So, here is the function. So, suppose we are interested in checking if the function is Lipschitz at this point locally, to say it is locally Lipschitz at this point means we should be able to find a ball; in this case it is an open interval such that we take any two points inside this ball and we look at the corresponding values and those values we take here.

So on one side of the inequality f x 1 minus f x 2 less than or equal to L times x 1 minus x 2; this inequality just means that if we connect those two points by a line, then the slope of this line should have absolute value at most L. So, can we find a number L such that L puts a upper bound on the absolute value of the slope? Even though this is decreasing the slope is not positive here, but we look at the absolute value of the slope and that should be bound from above by a number L.

So, we will quickly see that a discontinuous function will not satisfy Lipschitz property at the point of discontinuity.

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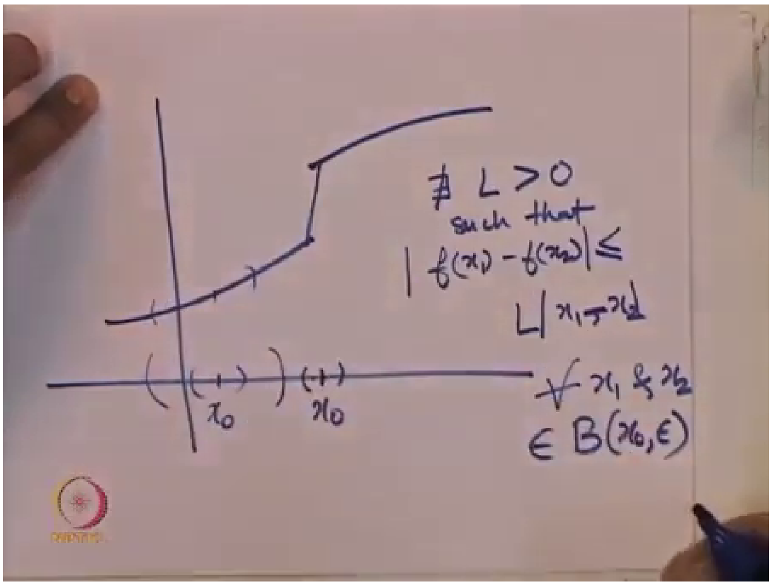

At other points, it could satisfy the property of Lipschitz, but at the disc at the point of discontinuity; suppose this is the function and at x naught, we have this discontinuity. So, if at x naught how much our small neighbourhood we take; we are forced to take points both from the left and the right and when we connect these two points from the left and the right we see that this line could have a slope which becomes more and more vertical. The line connecting these two points becomes more and more vertical, it will become vertical if and only if you take the point x naught and just before it.

But you cannot find; there does not exist a number L greater than 0 such that; such that f x 1 minus f x 2 is less than or equal to L times x 1 comma x 2 in which this inequality satisfied for all x 1 and x 2; for all x 1 and x 2 in the ball. So, for how much our small epsilon we take because we are forced to take points both from the left and the right. It turns out that, this line connecting the point from the left and connecting the point from the right; this particular line has a slope that is not bounded in absolute value [FL].

So, because of that we see that there does not exist an L such that this inequality satisfied for all x 1 and x 2 in the ball. For a particular x 1 and x 2, you might be able to find the l, but that L will work only for that x 1 and x 2 in that ball, but we want this inequality to to be satisfied for all points x 1, x 2 inside the ball of radius epsilon strictly greater than 0. So, this is how we see that if it is discontinuous; it cannot be locally Lipschitz at the point of discontinuity. At another point x naught, suppose this is another point x naught at this point it can very well be locally Lipschitz; as long as in a in this case as long as we take a ball that does not contain this discontinuity, we can see that the function is locally Lipschitz.

So, let us see some examples. So, f x equal to minus 4 x is locally Lipschitz at x equal to 3. So, take L equal to 4 take the Lipschitz constant equal to 4 or anything larger. Consider a function f x equal to e to the power 5 x; this is locally Lipschitz at point say x equal to 4. For this particular point, we can take the Lipschitz constant L equal to 5 times e to the power 20 plus 1.

So, notice that we are taking the slope of the function evaluated at x equal to 4 and we take something that is slightly greater. How much greater we take; decides on how big the open ball around the point x equal to 4 is, but since we are interested in just an open ball of radius greater than 0. We can take the Lipschitz constant slightly more than the slope at the point x equal to 4. Consider the function f x is equal to e the power x; this is locally Lipschitz at every point x naught whichever point x naught we take; we were able to find the Lipschitz constant L that will work for all points inside a suitable ball.

The unit step is locally Lipschitz that every point x naught except the point x equal to 0. So, what do we mean by locally; what do we mean by the unit step?

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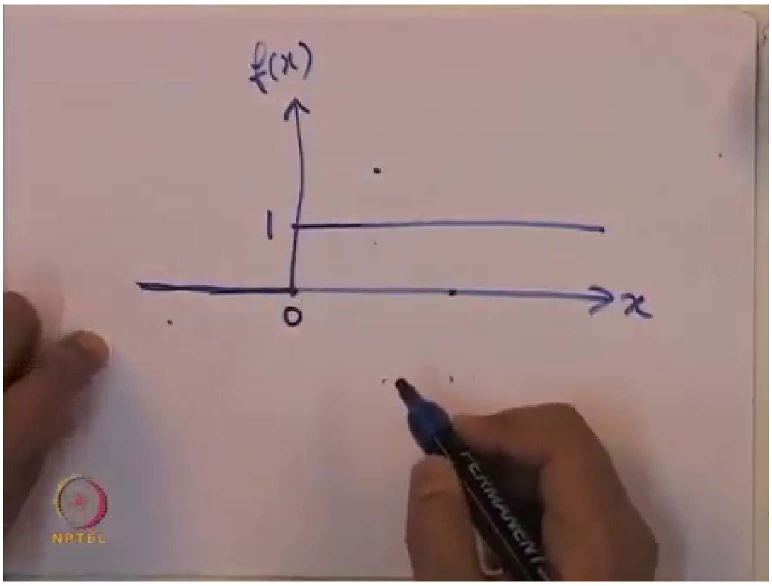

This is like our step function, this is equal to 0 up to here; then suddenly it jumps to the point 1. So, this particular function up to x equal to 0, it is equal to 0 for x greater than 0; it has jumped to 1 and this is locally Lipschitz at every point except x equal to 0. At x equal to 0, we saw that there is this discontinuity and hence there will not be a constant L that will work for that inequality for all points in a ball how much ever small the ball maybe.

So, in other words the unit step is locally Lipschitz at every point x naught except the point x naught.

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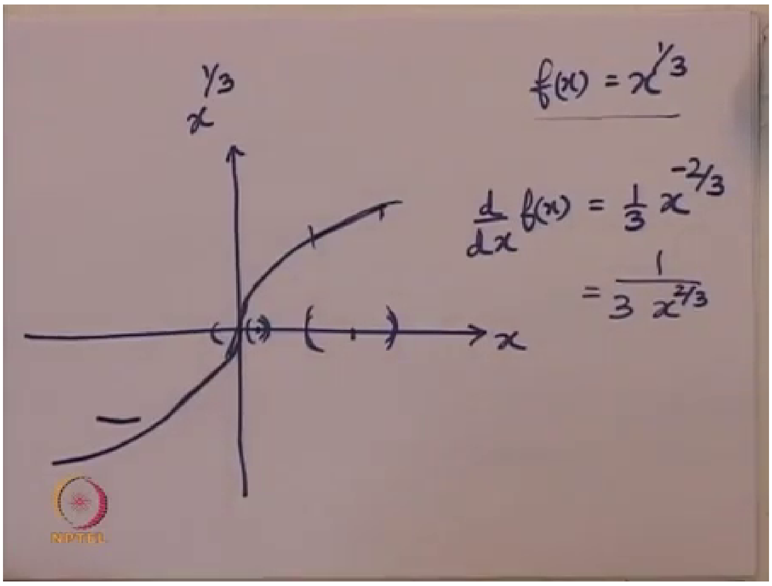

This particular function f x is equal to x to the power 1 by 3 is locally Lipschitz at every x again except; except x equal to 0; this is a very important example we will see this again and again. So, let us just draw the graph of f x is equal to x to the power 1 by 3. This particular graph we see that; this is how the graph looks. So, we see that at x equal to 0; this particular curve becomes vertical almost vertical. Why? Because if we try to evaluate D by D x of f of x; we get 1 by 3 times x to the power minus 2 by 3; yeah which is nothing, but 1 over 3 times x to the power 2 by 3.

So, we see that as x tends to 0; this quantity becomes unbounded and hence and hence we see that the slope is not bounded about the point 0. So, how much our small neighbourhood we take about the point 0? When we connect two points, we see that it could have a slope that is unbounded; we are not able to find a number L such that the Lipschitz; the inequality in the Lipschitz condition definition that inequality will be satisfied for all points inside that ball, such a number L we will not be able to find. That is why we will say this particular function is not locally Lipschitz at the point x equal to 0.

At another point say here we are able to find a ball such that there will be a number L; in other words whatever ball we take as long as we do not include the point 0. When we connect this, we can take the farthest and the nearest, we can see where the slope is maximum and we can choose the number L accordingly; that number L will work for everything. We have to see where the absolute value of the slope is maximum and we take a Lipschitz constant that is equal to that or more and that will work for all points in that ball.

Hence, we see that as long as we do not take the point x equal to 0; how much ever close we take if it is away from x equal to 0, we can find a ball such that [ther/there] there is a number L satisfying the Lipschitz inequality. And hence except the point x equal to 0, there is a number L that will satisfy the Lipschitz inequality; hence it is locally Lipschitz at every point except the point x equal to 0.

Some more examples; so the same function f x is equal to the power 1 by 3 is not locally Lipschitz that x is equal to 0. So, we could just we could conclude that at a particular point x naught in R n; if f is differentiable, then it is indeed locally Lipschitz at that point x naught. If it is locally Lipschitz at a point x naught, it implies that it is continuous locally Lipschitz at a point x naught means it is continuous at that point x naught.

Conversely, if the function f is continuous at a point x naught; it does not imply that it is locally Lipschitz at that point x naught.

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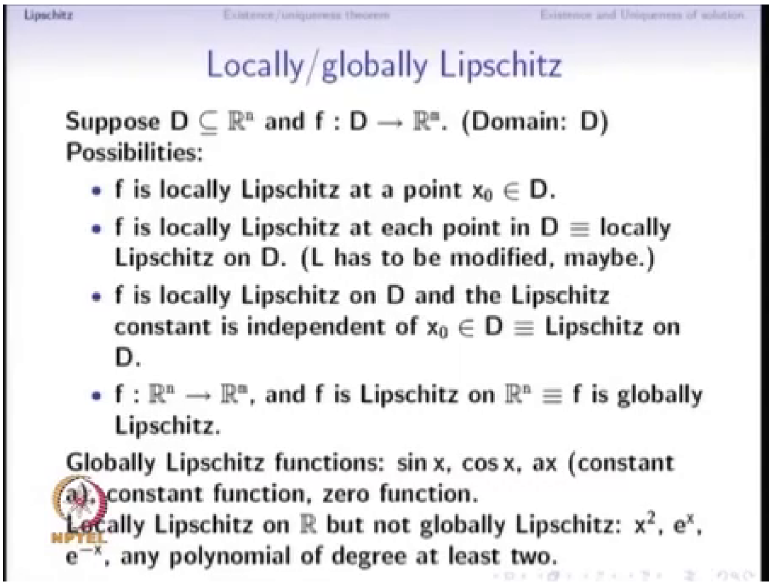

And if the function f is locally Lipschitz at the point x naught; it does not mean that is differentiable at the point x naught. So, we can see some examples about this; just to see why differentiability is not assured by locally Lipschitz property, we will see that locally Lipschitz only require that the slope is bounded.

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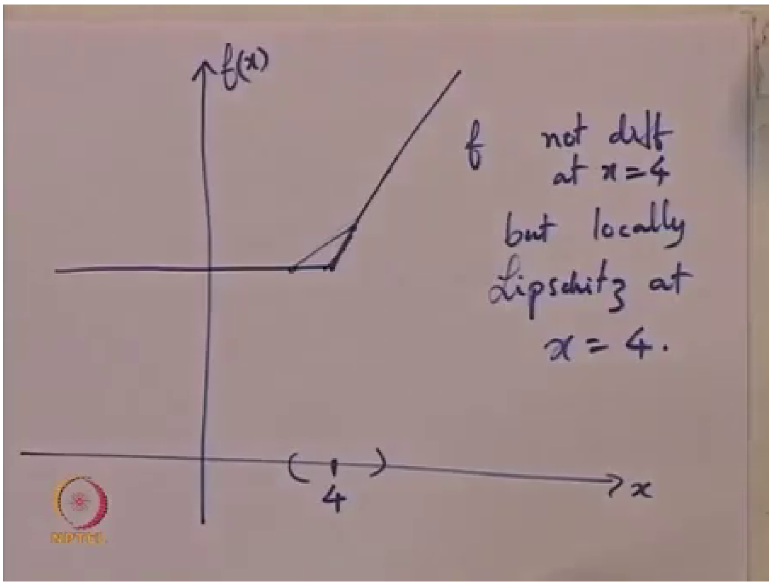

Slope between any two points; if we take the unit ramp, so it is continuous now. So, we see that the derivative does not exist at this point because the left hand line limit of the derivative and the right hand limit of the derivative are not equal to each other; hence at the point 0, the function f is not differentiable.

But we see that since the slope is bounded; we take any two points; we connect them by this line the slope is bounded. So, I will just draw a bigger figure; take this particular example. So, at the point let us say 4; we have drawn a figure such that the graph of f versus x is continuous, but it is not differentiable at this point because the left hand limit of the derivative and the right hand limit of the derivative are not equal to each other.

However, you take any two points and we connected; we see that this line is guaranteed to have a slope that is less in absolute value than the slope of this line yeah. So, we could take any two points here maybe on the same side and the line slope in absolute value is an upper bound for the absolute value of the slopes of any two points; close to the number 4.

In other words, we can find a ball here such that we can find the number L that will satisfy the Lipschitz inequality for all the points in that ball. In other words, here is an example f that is not differentiable at x equal to 4, but locally Lipschitz at x equal to 4. In other words, if somebody tells us that this particular function f is locally Lipschitz at the point x equal to 4 and the one asks us does it imply that it was differentiable at the point x equal to 4? The answer is no. So, this is what the statement here says.

So, if the function f is differentiable; it does imply that it is locally Lipschitz at the point x naught. If it is locally Lipschitz at the point x naught it is also continuous at the point x naught, but if it is continuous at the point x naught; it does not imply that it is locally Lipschitz at x naught. We saw x to the power 1 by 3 as an important example and if it is locally Lipschitz at a point x naught; it does not imply that it is differentiable at the point x naught, for that we saw an example now.

So, we will just spend a few; we will spend one slide on the difference between locally Lipschitz and globally Lipschitz. So, for this purpose we need to see to what extent the number L depends on x naught and epsilon. So, consider a domain D; a subset of R n and f a map from D to R n. So, f need not be defined on the whole of R n; it is defined on a domain d. So, domain is an open and connected subset of R n in this case. So, to say that D is a domain in R n; it means D is open and connected subset of D, open and connected subset of R n.

So, for a function f that is a map from D to R n; there are various possibilities, f could be locally Lipschitz at a point x naught in D. We saw the definition for locally Lipschitz at a point; we could also have a situation where at every point in D, f is locally Lipschitz at that point this we will say is we will use a word f is locally Lipschitz on d. So, what is the significance here? L might has to be modified depending on the point x naught. If x naught is another point; the slope absolute value of the slope might be larger because of which L might have to be made larger. So, at each point in D; f is locally Lipschitz, but the Lipschitz constant L is having to be modified depending on the point x naught perhaps L has to be modified.

We will say f is locally Lipschitz on D; if it is the case that f is locally Lipschitz on D and the Lipschitz constant is independent of the point x naught in D, then we will say f is Lipschitz on D. The word locally is no longer relevant, since we can find one Lipschitz constant L that works for the entire domain D. Finally, when the domain D is a whole of R n; if f is Lipschitz on R n that is there is a constant L that works for every point x naught in R n; then we will say f is globally Lipschitz.

So, examples of globally Lipschitz functions are sin x, cos x, a constant a times x. The constant function itself and the zero function, it is possible to decide for each of this functions a Lipschitz constant L that works for the entire domain of these functions; entire domain in this case is whole of R.

Examples of function which are locally Lipschitz on r but not globally Lipschitz R x square into the power x into the power minus x. In fact, any polynomial of degree two or more. So, these all examples R locally Lipschitz on r, but not globally Lipschitz why because this x square if slope could become very large depending on the point and hence there is no one number L that works on the whole of R. Similarly e to the power x and e to the power minus x, I can have slopes which are very large in absolute value.