Nonlinear System Analysis
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Lecture - 05
Lipschitz Continuity and Contraction Mapping TheoremPart 01

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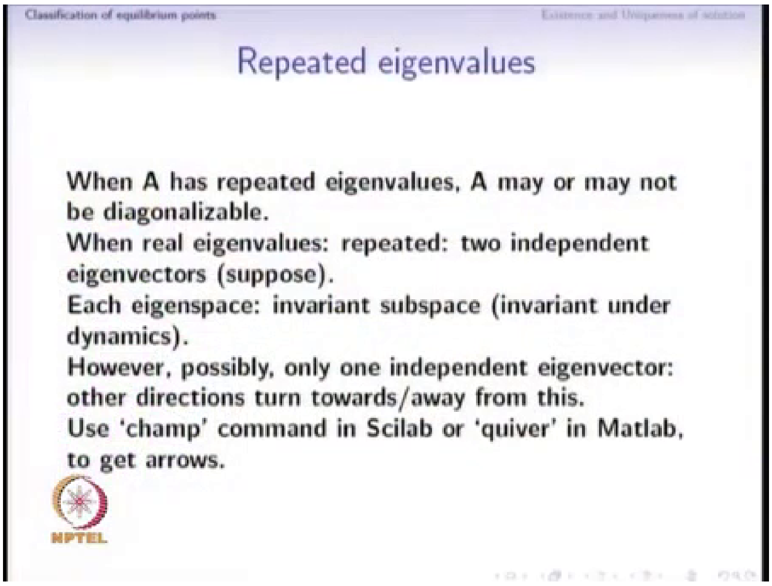

So, the next important question we will start studying now is when does there exist a solution to the differential equation.

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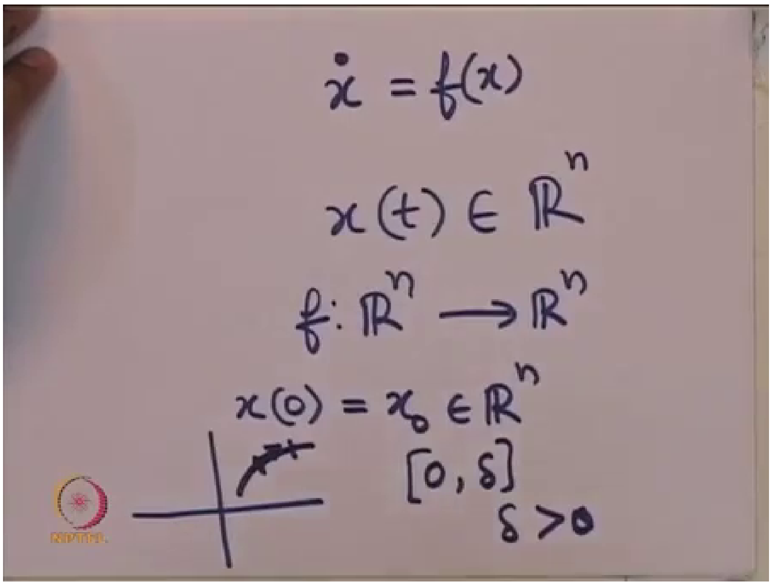

If we are given with a differential equationx dot of t is equal to f of x in which now x has n components, at any time instance t x has n components and hence f is a map from R n to R n. And for this situation suppose we are given with the initial condition x naught, x at time t is equal to 0 is some vector called x naught which is an element of R n.

We are interested in the question; suppose, if this is our space R n this is our point x naught then the direction is given here by fevaluated at the point x naught, x naught we are interested in answering the question when does there existed trajectory that starts from the point x naught at t equal to 0. And there is a unique trajectory for some time duration for a time duration 0 to delta in which delta is some positive number possibly very small, but for this duration of time we have a unique solution to the differential equation x naught is equal to f of x. So, this is the question we will answer in the next few lectures starting from now.

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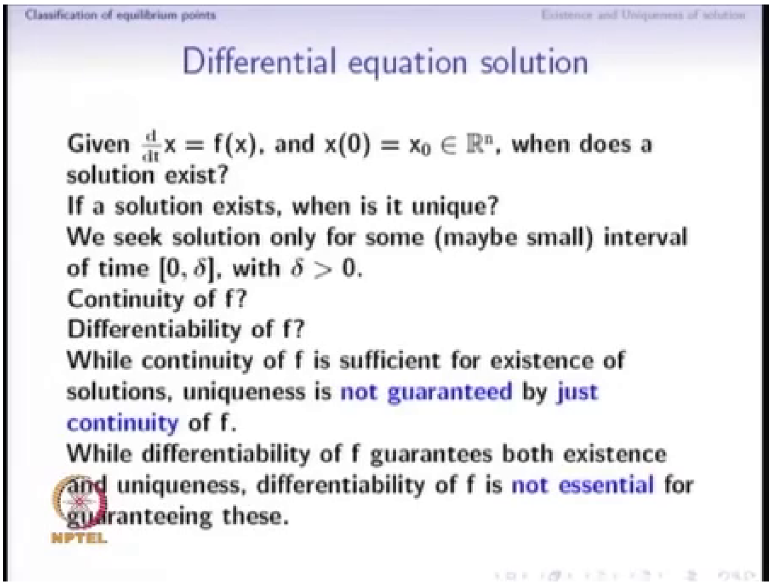

So, let us look at this differential equation. So, given d by dt of x is equal to f of x and the initial condition x 0 is equal x naught, an element in R n when does the solution exist? Then we will ask, if a solution exist when is it unique? Under what conditions on f at the point x naught do we have a solution and when is it unique? So, please note that we are interested in a solution possibly for a very small interval of time. It might be difficult to guarantee existence and uniqueness of solutions for a large duration of time, but we are interested only for an interval 0 to delta, in which delta is greater than 0 possibly quite small.

So, we ask is continuity of f the important property here or is it differentiability of the function f at the point x naught that is required here. So, it is important to note here that while continuity of the function f is sufficient for existence of solutions, uniqueness of the solution is not guaranteed by just exist ah by just continuity of the function f.

On the other hand, while differentiability of the function f guarantees both existence and uniqueness of the function f, both existence and uniqueness of solution to the differential equation x naught is equal to f of x, this differentiability of f is not essential for guaranteeing existence and uniqueness of the solution.So, keeping note of this we we we can ask what is the important property required for existence and uniqueness of solution to a differential equation. It appears to be a property that isslightly more strong condition than continuity, but might not be as strong as differentiability of the function f at the specified initial condition x naught.

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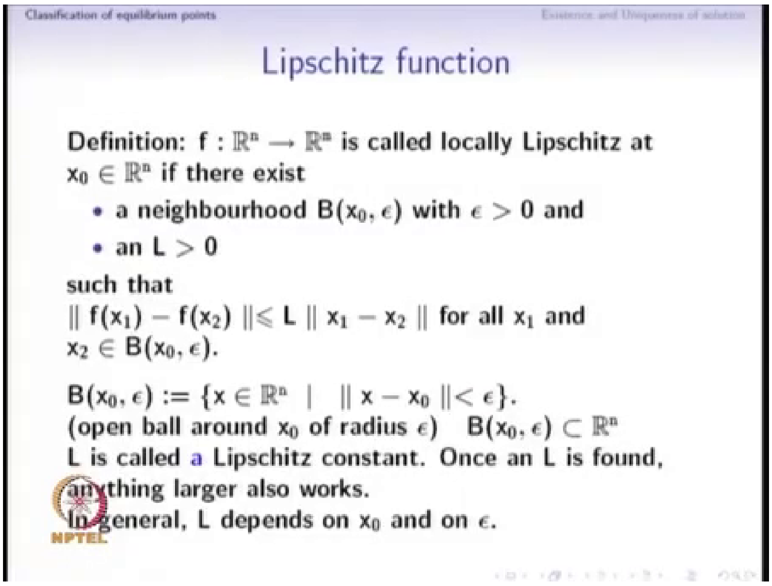

So, it turns out at this property is a important property calledLipschitz condition on the function f. So, what is definition? This definition isvalid for a function f from R n to R m even though in our case f is always from R n to R n. We will define this definition ofLipschitzfor a case when f is a map from R n to R m. So, it is said to be locallyLipschitzat a point x naught if there exist a neighbourhood B of x naught of radius epsilon you will see a precise definition of a neighbourhood very soon. A neighbourhood B of x naught comma epsilon with epsilon greater than 0 and a constant L greater than 0, such that an inequality is satisfied.

What inequality? ff at x 1 minus f at x 2 norm, this distance is less than or equal to L times x 1 minus x 2, and this inequality is required to be true for all x 1 and x 2 in the neighbourhood, in that neighbourhood of the point x naught. So, this neighbourhood is been called as a ball B, centered at x naught and of radius epsilon. So, this is the precise definition of the ball. So, B x naught comma epsilon is defined to be the set of all points x such that distance of this point x from epsilon is strictly less than epsilon. It is not more than, it is not more than epsilon away from in the point x naught. Even equal to epsilon away we are not including into the ball B x naught comma epsilon and hence this is called an open ball around x naught of radius epsilon.

Around which point the ball is centered? That is centered around the point x naught and. What is the radius? That is epsilon. And we are saying it is an open ball because this distance is strictly less than epsilon. So, this ball is contained in R n because we are taking all points in R n that satisfy this condition. So, for this, so for some ball around the point x naught with a radius strictly greater than 0 we should be able to guarantee that this inequality is satisfied for all x 1, x 2 inside this ball.

So, this number L a positive number L is said to be aLipschitz constant, it is not unique because if you have found a constant L such that this inequality satisfied for all x 1, x 2 in the ball B x naught comma epsilon then you can take a number larger than L; for that larger L also this inequality would be satisfied and hence we see that thisLipschitz constant is not going to be unique.

But in general this Lipschitz constant L will depend on x naught and on epsilon. It will depend on the point x naught itself and also on the radius epsilon, radius epsilon of the ball of the open ball around x naught.So, using this definition ofLipschitz function it is possible to specify under what conditions solution to a differential equation exists and when it is unique.

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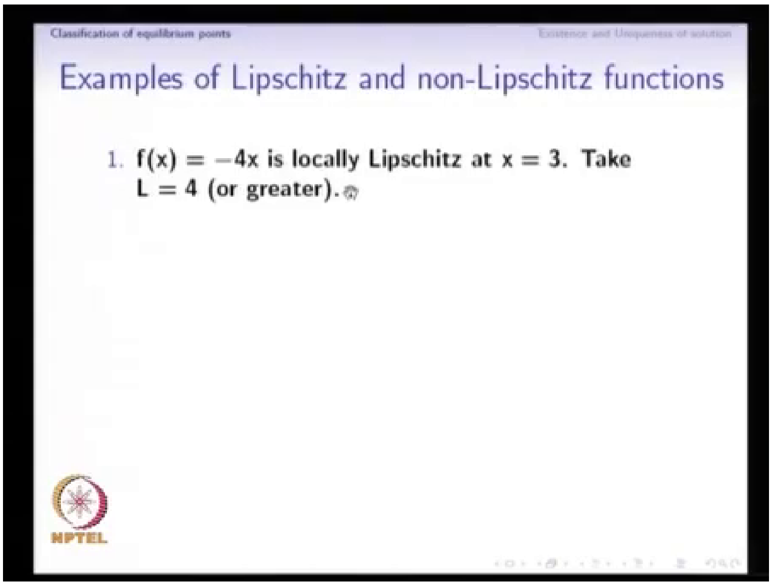

So, we will see some examples of aLipschitz function and of some non-Lipschitz functions.So, the line f x is equal to minus four x is locallyLipschitz at the point x equal to 3. If it isLipschitz then we are, we should be able to give a number L such that that inequality satisfies and here we can take L is equal to 4. So, notice that we can take the slope of the function f, absolute value of the function f or we can take something larger.

To understand theLipschitz function we will take graph of a function f for the situation that f is a map from R to R.

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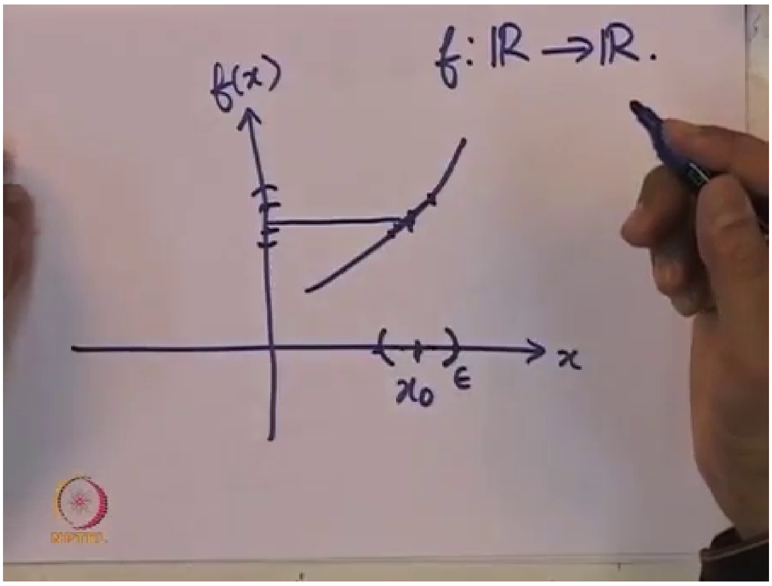

Suppose, this is our point x naught and this is the graph of the function. So, what does this say? f is said toLipschitz at the point x naught if there exists a ball of radius epsilon which means at this point is x naught plus epsilon, this point is x naught minus epsilon and both these points are not included in the ball because it is an open ball. In other words this interval is an open interval. So, for this particular ball we require some inequality to be satisfied. So, we take all the points, take any points x 1 and x 2 in this ball and we look at the corresponding distance between them and when we connect, so ah it is required to draw a larger figure to be able to see what theLipschitz condition is specifying on the function f.

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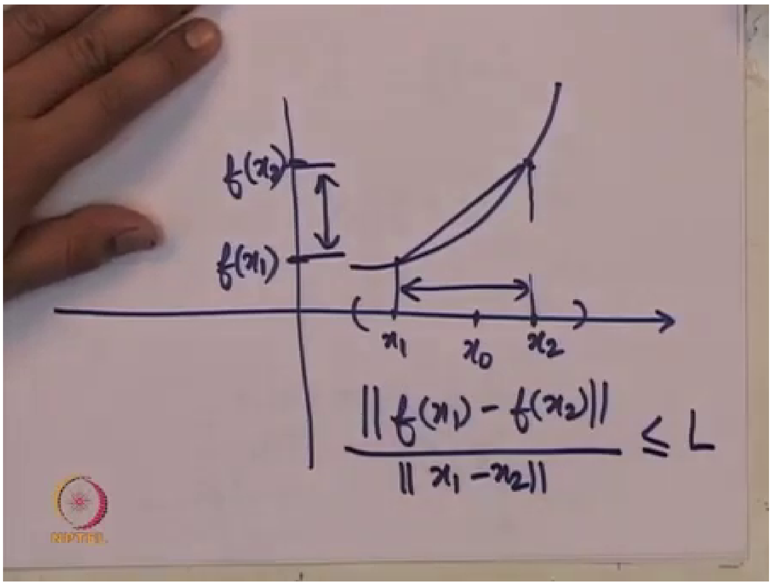

This is the point x naught, this is the ball in in this case it is an interval of width two epsilon and open interval of width two epsilon in which the center is x naught. Suppose we take x 1 here and x 2 here, they do not have to be on opposite sides of the point x naught.

So, what is being specified is, this is the value at x 1, this is the value of f at x 2, and the distance between f x 2 minus f x 1, the distance between f x 1 and f x 2 that distance is nothing but this gap this gap divided by this gap this ratio in absolute value should not exceed capital L, f x 1 minus f x 2in absolute value; in this case it is just absolute value more generally it is a normshould not exceed L. There should exist a number L such that this inequality satisfied for all x 1, x 2 in the ball around x naught of radius epsilon and open ball. In this case it is just an open interval.

So, this particular ratio is nothing, but absolute value of the slope of this line that connects this point and this point. Which point? The point with x 1,f x 1 here and x 2 and f x 2 here, then we connect these two a by a line then the slope of this is precisely this, but without the absolute values.Once you take the absolute values, then it is the absolute value of the slope of this line and theLipschitz condition on f at the point x naught says that there should exist a ball around the point x naught of radius epsilon and a number L such that the line has slope of absolute value at most L. There should now exist one number L such that this slope is bounded from above by L, the absolute value of the slope

So, this property of Lipschitz condition is a key property. We will see examples ofLipschitz and non-Lipschitz conditionfunctions and it will play a key role for existence and uniqueness of solutions to a differential equation. This is what we will see in detail from the next lecture.

Thank you.