Nonlinear System Analysis

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Lecture - 04
Math Preliminaries Part 03

Welcome back. So, we will continue with our, the exercises on Math Preliminaries that we are doing. So, we will do some more examples to get a feel of the different notions of that we have introduced right. So, let us start with an example right.

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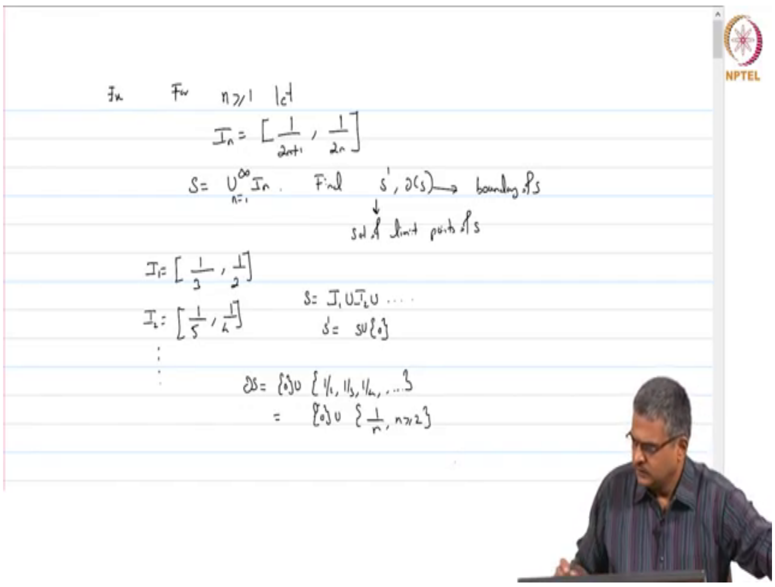

So, for n greater than 1, let us define an interval I n as 1 upon 2 n plus 1 and any set of integers greater than 1. So, and define the set S right as union of n equal to 1 to infinity I n.

So, we need to find S prime and the boundary of S. So, this is set of a limit points, limit points of S and this is the boundary of S right. So, let us enumerate or let us write down these sets for a few few values of n. So, far I n equal to 1, you have 1 by 3 and 1 by 2 and then you have 1 by 5 and 1 by sorry 4 and so on right. So, let us see what will be the; so, as n increases right, so this number on the left the left limit is decreasing right.

And so, what do we get? So, S. So, suddenly as n increases you are approaching 0 here right and so even the right hand side is decreasing. So, it suddenly 0 is a limit point because about 0 you take a any neighbourhood around 0 or for every neighbourhood around 0 right it contains an element belonging to the set S. So, S is I 1 union I 2 so on right and in each of this as I as n goes becomes larger and larger right, so this approaches the value 0 right both the left and right end points right.

So, S prime certainly consists of S because it is a union right, so and all this are closed sets. So, each yeah say and S has all the all these intervals right. So, and each of these intervals is a closed set. So, S prime will have S and also it will have the singleton 0 because 0 is the also the limit point of the set S even though it is it does not belong to S right. So, what about the boundary of S? The boundary of S right will contain all the points right.

So, if you see here the highest number that we get is 1 by 2 right and then if you see the pattern 1 by 2 then 1 by 3 1 by 4 1 by 5 and so on. So, the boundary of S consists of course, it contains the 0 also and then union 1 by 2 1 by 3 1 by 4 and so on right. So, we can write this as 0 union to the set 1 by n, where n is greater than 2 right. So, this is the boundary of the set S. Now we saw that we are seeing how to calculate the boundary of a set. So, let us take another example.

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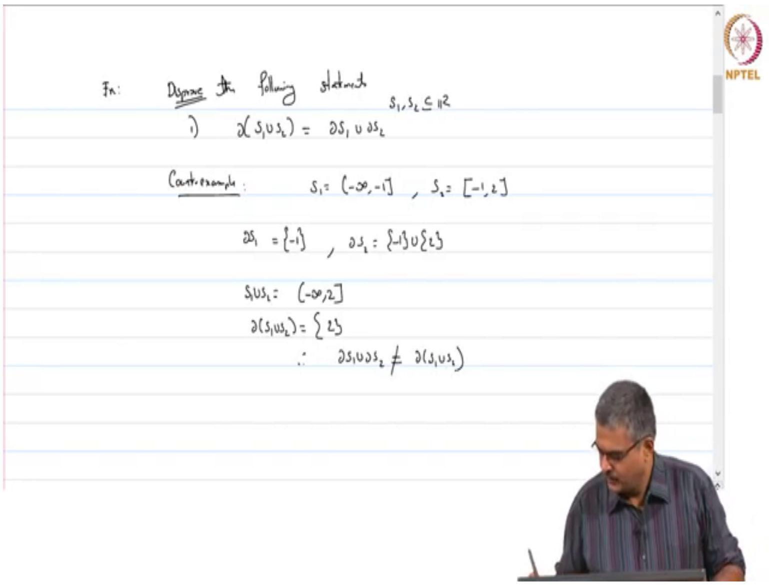

And we will disprove the following statements, disprove the following states statements. A first statement is boundary of so, so S 1 and S 2 are subsets of R and we want to disprove this following identity S 1 union S 2 all right is right. So, you want to disprove that; that means this is not true. So, we will have to disprove we will come we will have to give a counter example. So, a counter example is as follows. So, you can cook up a following example take S 1 as minus infinity minus 1 and S 2 as minus 1 2 right.

So, boundary of S 1 will be will be just minus 1 because minus infinity does not belong to S 1 whereas, boundary of S 2 consists of both minus 1 and right. And what is S 1 union S 2? So, it consists of all points from minus infinity to 2 right and the boundary of S 1 union S 2 is 2 right. So, you see that therefore, boundary of S union boundary of S 2 is not equal to boundary of S 1 union S 2.

So, to disprove you just need to come up with a counter example. Let us take another example and what do we want to disprove right.

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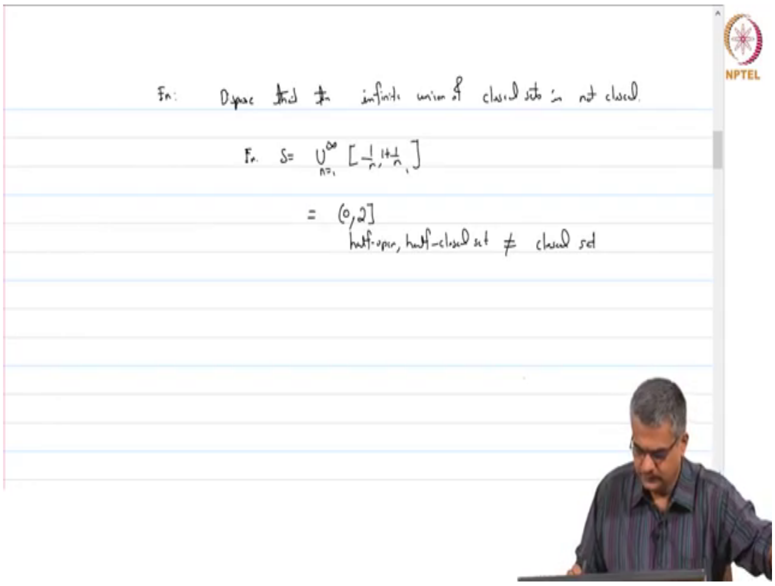

So, disprove that the infinite union of closed sets is not closed right, whereas, the intersection of closed sets is closed, arbitrary intersection of open sets is not open, whereas, arbitrary union of open sets is open right. So, in the open sets also you had something to disprove right with an counter example and here you will come up with a counter with a example to show that the infinite union of closed sets is not closed.

So, just we need to come up with an example. So, the set that; so the this is the example and the example is you take union of sets of the form 1 by n 1 by n plus 1 right. So, it is a closed intervals or closed sets right. So, you have arbitrary union of closed sets. What happens? Say let me call this as S. So, as n approaches very large value this number becomes closer and closer to 0, but it never becomes 0, it approaches 0. So, it this will be open 0 on the left hand side. Whereas, on the right side you get the maximum for n equal to sorry this should not be n plus 1; it should be 1 plus 1 by n right.

So, for n equal to 1 you get the maximum which is 2, whereas, for any other value of n you get a number which is less than 2, right. So, it will take closed 2. So, what we got is the union of closed sets is a half open half closed set right. This is not equal to the; this is not a closed set, it is called a neither close nor open set right. So, we have shown that the arbitrary union of closed sets is not closed. That takes us to the next notion. So, let S be a set in R right, let me take an example.

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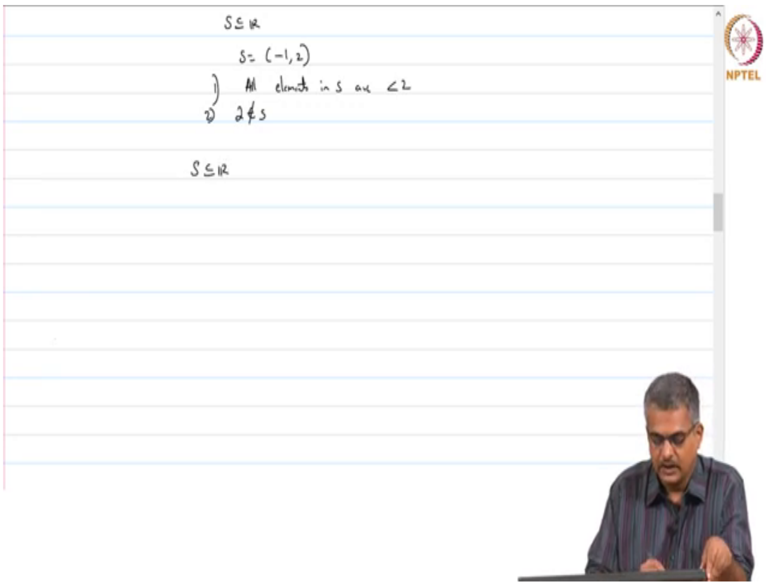

So, let S be say minus 1 and 2 you have taken open interval right. Now you can see that the number 2 right, all elements in the set S are less than the number 2 ok; however, the number 2 does not belong to S. So, the first observation is all elements in S are less than 2 right. Why less strictly less than because 2 does not belong to the set S and 2 does not belong to the set S 2 does not belong to S. Same thing can be said about the number minus 1 right.

So, there is no number right which is greater than sorry the; it is lesser than minus 1 right. So, minus 1 is the least number in this set and again minus 1 does not belong to the set S. So, let us capture this in a notion called supremum of a set. So, this notion holds for only subsets of the real line right not subset of R n. So, S is subset of R right. Let us now move on to a another notion that we will need a need frequently in our analysis and that is the notion of a supremum of a set.

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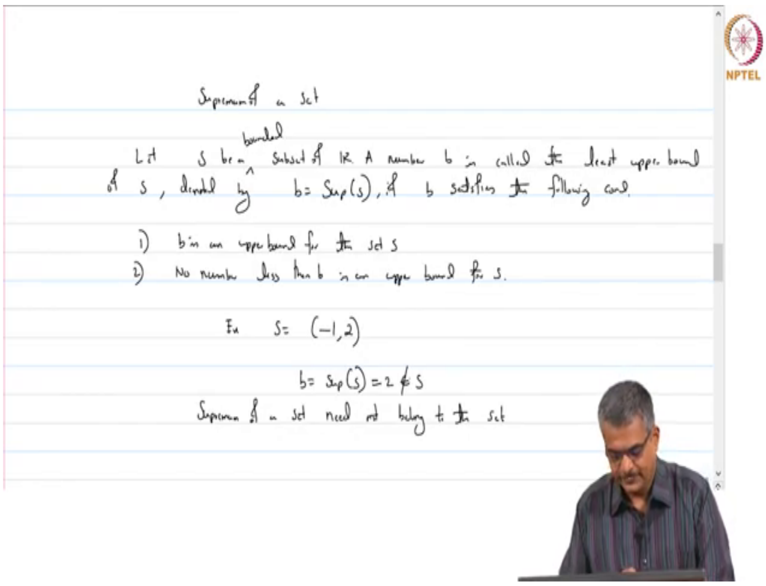

So, let S be a subset of R right is let be a bounded subset of R. A number b is called least upper bound of S denoted by b is supremum of the set S sup of S we call, if b satisfies the following conditions if b satisfies the following conditions. What is this? The first is b is an upper bound for the set S; it has to be an upper bound right. Second is no number less than b is an upper bound for S. So, if you can find a number b which satisfies this 2 properties that is b is the least upper bound for S and no number less than b is an upper bound for S then b is said to be the supremum of the set S right.

So, to give an example take S to be say minus 1 2 it is a bounded set in the real line right. So, the upper bound is 2 because there is no number which is lesser than b and forms an upper bound for the set S right. So, here in this case b is supremum of the set S right. So, it satisfies both the this. As I said 2 is there it satisfies the first condition, it is an upper bound for the set S because all numbers are less than 2 and the second condition also holds there is no number other than 2 which satisfies the upper bound condition ok.

Note that what differentiates this between the notion of maximum value of a set and the supremum is that supremum of a set need not belong to the set.

So, in this case you can see that here which is 2 does not belong to S. So, a number can be the can be a supremum of the set even if it does not belong to the set S right. So, similarly 1 can define the notion of infimum of a set, infimum of a set right.

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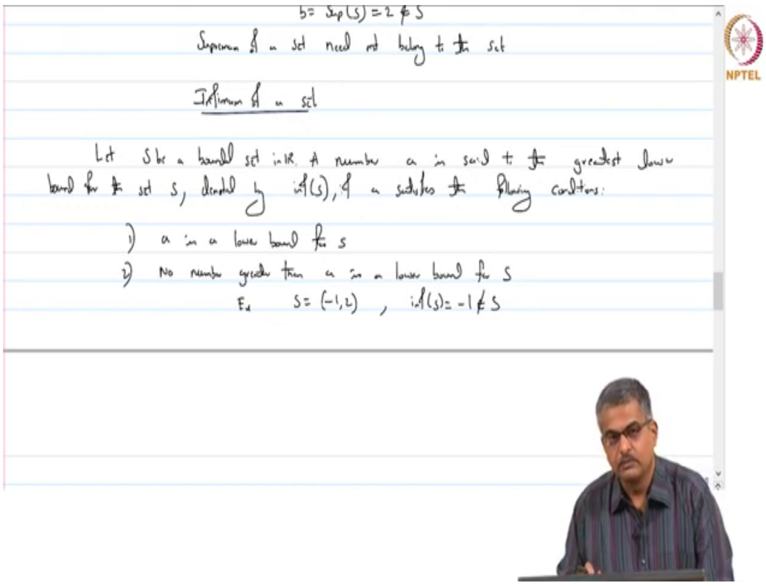

So, again let S be a bounded set in R. A number say a is said to be the greatest lower bound. So, unlike in the supremum you had the least upper bound. So, we have here the greatest lower bound for the set S denoted by infimum of S right, if a satisfies the following conditions.

And what is that? a is a lower bound for S; second is no number unlike here there was no number less, here no number greater than a is a lower bound for S. So, the same example, we took the example as minus 1 and 2 open interval. So, this is so, here the infimum of S is minus 1 and this number does not again belong to the set S right because that is the lowest bound for the set S right there is no number greater than minus 1 which is a lower bound for the set S.

So, again here as a set the number minus 1 does not belong to set S. So, the infimum of a set need not necessarily belong to the set S right. So, let us take some more examples to illustrate the notion of supremum and infimum right.

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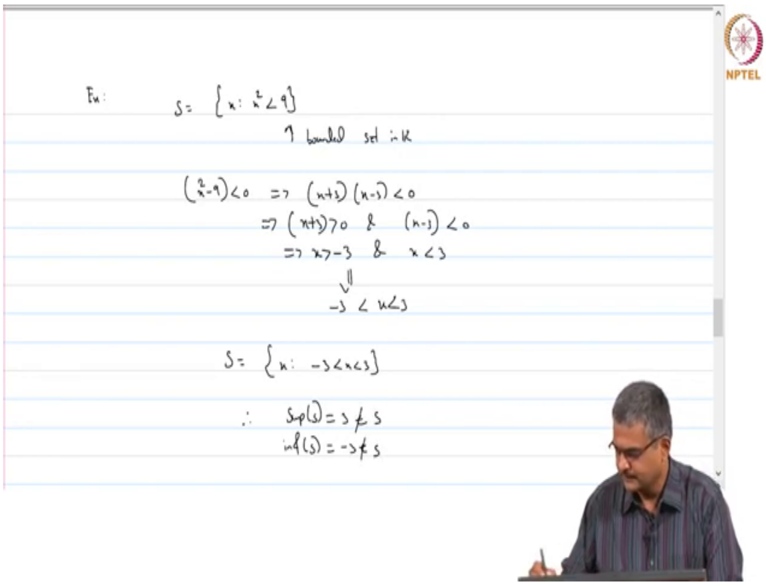

So, I have a set S which is set of points such that x square is less than; again this is a bounded set bounded set in R right. So, we can write x square minus 9 less than 0 which implies, x plus 3 x minus 3 is less than 0.

So, it implies this has to be positive and this term has to be negative; x plus 3 has to be positive and x minus 3 has to be negative which implies x is greater than minus 3 and x is less than 3 right. Together this implies that x is greater than minus 3, but less than three. So, the set S consists of all x which satisfies this unique quantity right. Therefore, the set S is upper bounded the least upper bound for the set S is 3 and the infimum of S is minus 3 and both do not belong to the set S.

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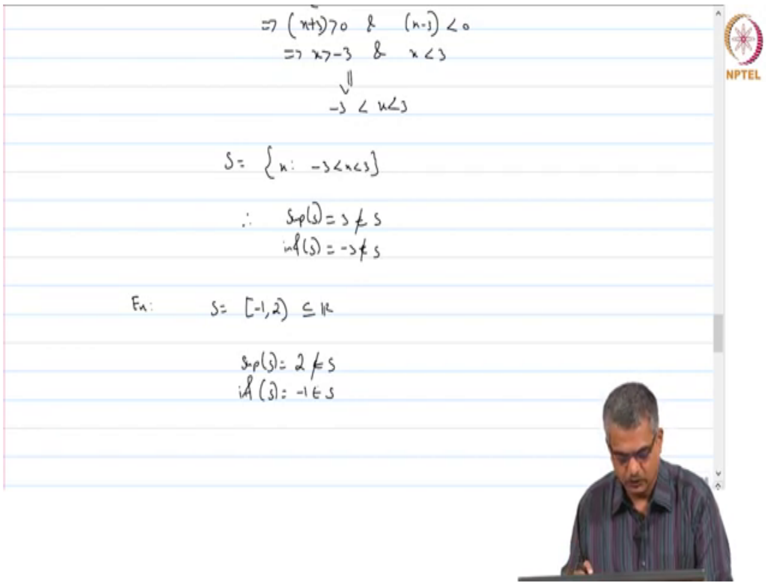

Another example: suppose S was minus 1 2 with a open on the right side. So, this is again a bounded set. So, the supremum of S is 2 which does not belong to the set S, where as the infimum of the set S is minus 1 which belongs to the set S right. In fact, the we say the infimum is achieved right. So, that is the minimum element. So, we talk about max and min when the elements belong to the set right, but if they do not belong we use the word supremum and infimum accordingly and we will take one more example.

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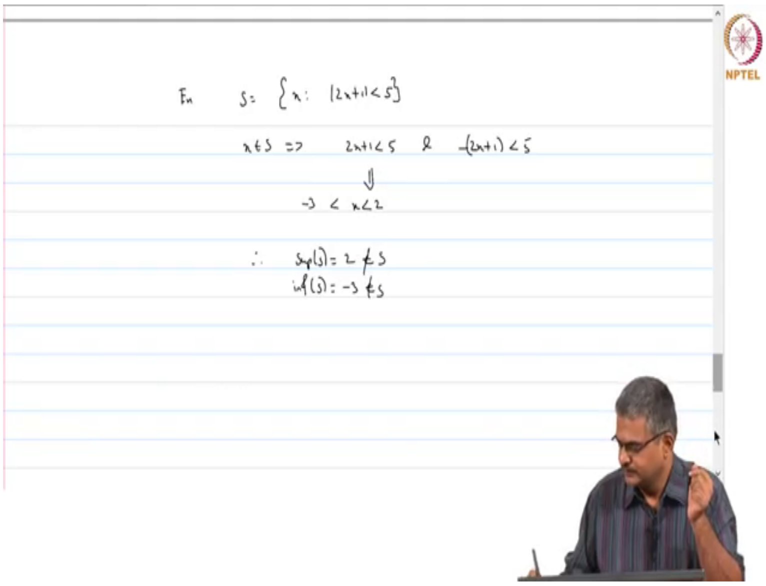

So, S is set of points such that the absolute value of 2 x plus 1 is less than 5 right. So, x belongs to S implies 2 x plus 1 is less than 5 and also minus of 2 x plus 1 is less than 5. So, if you do further simplification this implies x is less than 2, but greater than minus 3 right. So, again the supremum of S is 2 which does not belong to the set S and the infimum of S is minus 3 which again does not belong to the set S right. So, that brings us to the notion of supremum and infimum. We move on to one more notion that is the notion of a continuity of a function.

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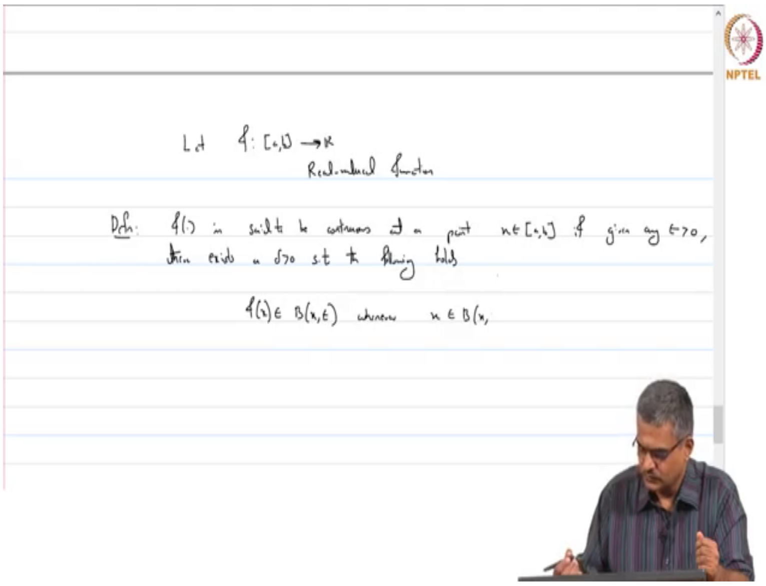

So, let f be a real valued function right, so, real valued function. So, you might have been familiar with the notion of continuity using the notion of limits the by using the notion of left limit right limit. So, when both are when both the limits exists and they are equal we say that that f is said to be continuous at that point, but let us not use the basic like the calculus approach. But we will do it more from the set from the point set topology notions that we have discussed so far in this course all right.

So, definition: f is said to be continuous at a point if given any epsilon greater than 0. So, an epsilon is like a challenge thrown to you and if you can come up with given any x belong to a b continuity to x, if given any epsilon greater than 0 there exists there exists a delta greater than 0, such that the following holds. What is that? f of x right belongs to a ball with centre x and radius epsilon whenever x belongs to a ball with centre x and sorry. We next move on to the notion of continuous function right.

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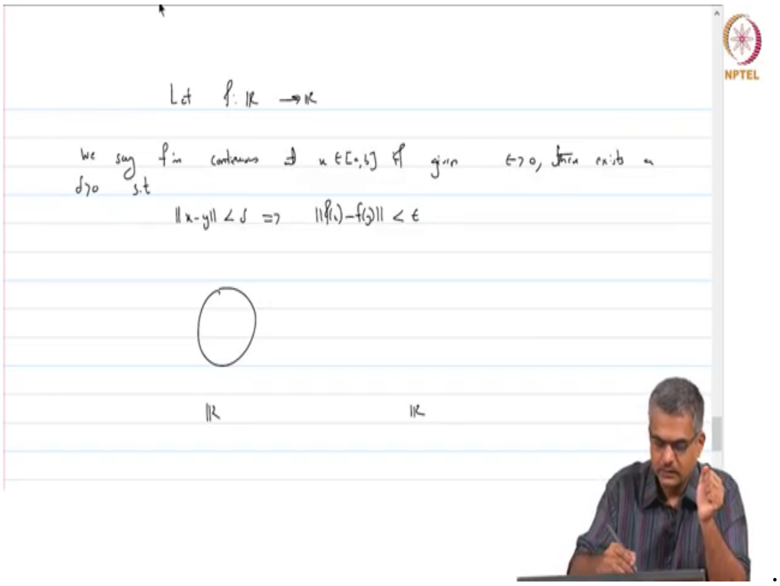

So, let f be a function real valued function. We say f is continuous at x belong to a b, if given any epsilon if given epsilon greater than 0, there exists a number delta greater than 0 such that whenever norm of x minus y is less than delta implies norm of f of x minus f of y is less than epsilon.

So, what is definition says is that let us picturize this right. So, you have a the real space right, its it is mapping from an interval in the so, I can even talk about instead of an interval I can take function from R to R right and let us now move on to the notion of a continuity right of a function.

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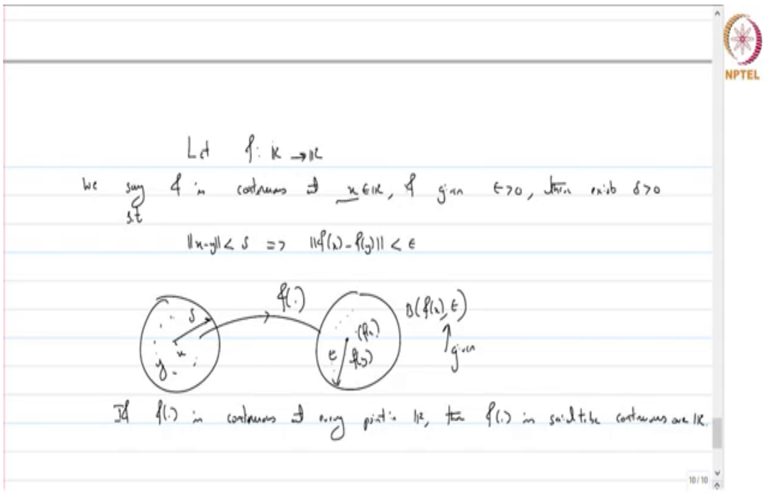

So, let f be a mapping from R to R right. We say f is continuous at x belonging to R, if given epsilon greater than 0, it is like a challenge thrown to you. There exists delta another number greater than 0, such that whenever norm of x minus y is less than delta implies norm of f of x minus f of y is less than epsilon. Of course, here you can replace the norm by absolute value because they are from real line to real line, but this definition holds for in general from mapping from R n to R n space.

So, so we say that f is continuous at a point x, if given epsilon greater than 0; somebody gives you an epsilon and you come up with a number delta such that for all y which are in the delta ball around x, the function f of x lies in a ball of radius epsilon centred at f of x.

So, let us picturize this. So, you have a x here you have a delta ball right and then you map all this through this function f, you end up with a ball with centre as f x and radius as epsilon. So, f is said to be continuous at x right, if you if this is given to you epsilon is given to you right x is a test point and epsilon is given to you. If you can come up with a delta a number a positive number such that for all y in this ball open ball right note that this is an open ball because these are strict in quality for all y in an open ball with centre as x and radius delta the function gets mapped to.

So, this is a f of x sitting here and then all other points mapped to f of y. And how do they map? Well, they all lie in a ball with centre as f x and radius as epsilon. So, if you can show for any epsilon that is thrown out to you, if you can find out a delta such that this in quality holds then f is said to be continuous at x. If it holds for every point in the real line then f is said to be continuous over the real line right. If f is continuous at every point in R, then f is said to be continuous over the real line right. You can replace the real line by subsets of R by intervals closed intervals and so on.

So, once we have the notion of continuity, we can define the notion of a path simply connected set.

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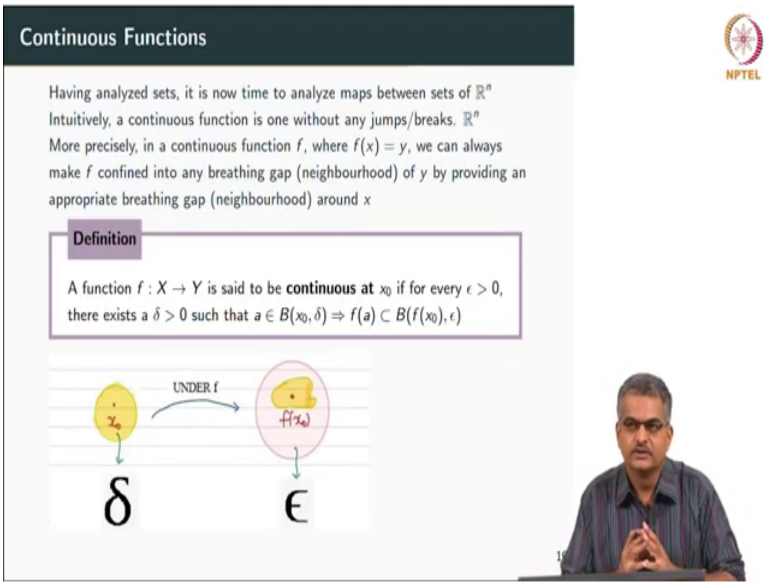

So, we already saw the notion of connected set. Connected set roughly captures a set is in 1 piece right. We said that we defined connectedness by defining the notion of a set being separated. So, 2 sets a and b are said to be separated if their a if a union b is the whole space and if a intersection b closure is a null set and a closure intersection b is also null set. So, we say that a and b are said to be separated. If they are not separated then they are said to be connected. Once we know the definition of connected we need a notion of simply connected.

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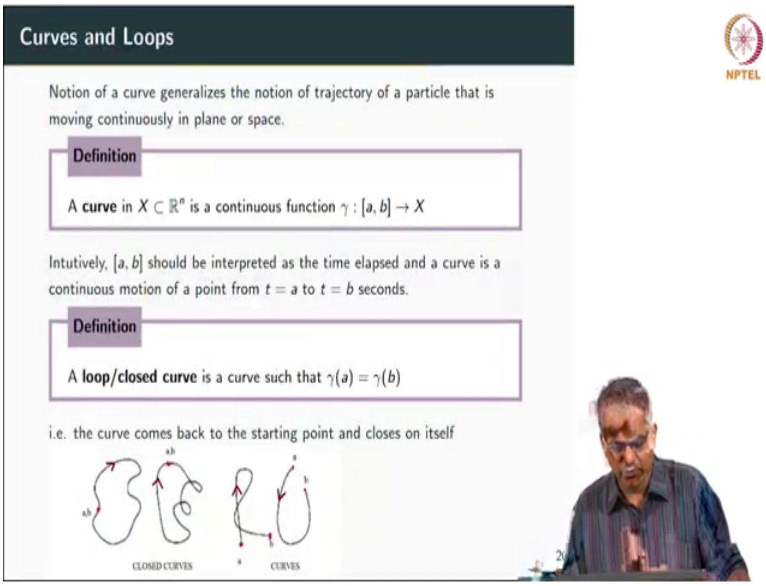

So, this notion of a simply connected is useful when we talk about non existence of periodic solutions right. So, for that we need a; so, we already defined the notion of a continuity. So, let us define 2 more notions that is the a curve, a curve in a subset x of R n is said to be is nothing but a continuous function from an interval a b to the set x right. So, once you have the notion of curve, we say a curve is closed if the end and if the start and end points coincide right.

So, a curve is a curve or a loop is a closed curve or loop is such that the start point of the curve and the end point of the curve are the same. So, here we have examples of a closed curves and these are just curves these are these are not closed.

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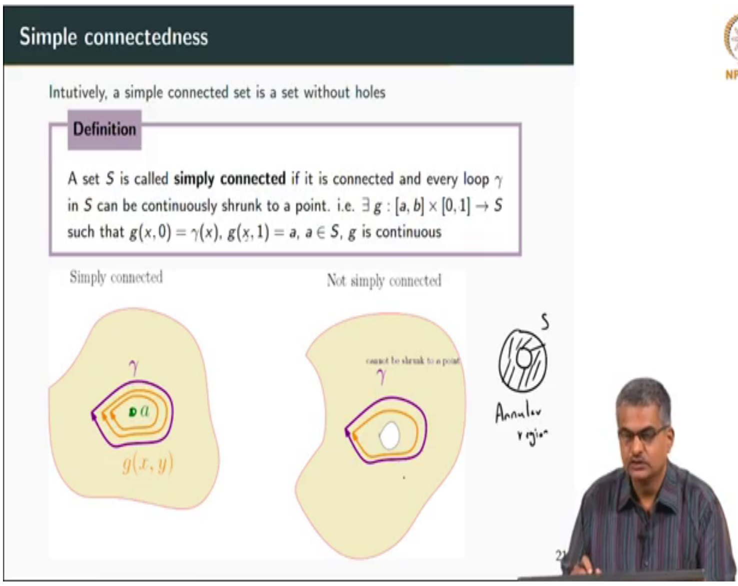

So, once we have this notion of a loops or closed curves, we can define the notion of simply connected set right. So, a set S is called simply connected if it has if it is connected that is the first requirement and every loop gamma in S can be continuously shrunk to a point in the set right. So, you know more mathematically, if there exists of continuous function g which maps from the interval a b Cartesian product 0 1 to the set S, such that g at 0 is the curve gamma and g at x at x comma 1 that is at when it takes the upper limit of this set 1 g of x 1 is the point the singleton point singleton set that is the point a, where a belongs to set right

So, note that there are three important things in this definition. One is the set has to be connected right then you have to come up with a function which is continuous, a function g which is continuous. So, what it does? It takes loops and shrinks it to a point belonging to the set S right. So, this is also 1 of the key things to remember that the you should be able to shrunk it to a point, but that point should belong to the set S right. So, it is easy to visualize.

Now, with this definition what is meant by a not simply connected set right. So, a set with a whole right like a the annular region right. So, something a set like this where it contains all this shaded points, but not the unshaded point. So, this is an annular region. So, the annular region is not a simply connected set because no matter what curve you what close curve you take right, you will never be able to find out a continuous function which shrinks this curve to a point belonging to a set right. So, this is a as I said this is going to be this notion of simply connected set is going to be used when we talk about periodic solutions right. So, we will end this lecture on the math preliminaries right.

Thank you.