Nonlinear System Analysis
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Lecture – 32
Interconnection between non linearity and a linear system – Sector Nonlinearities and Aizermann's conjecture

So you would have already seen how to handle non-linear systems; which are which do not have an input. Now a very clever thing which was done was that when you try to analyze these non-linear systems which do not have an input, you can actually decompose that, into 2 separate things: one a linear system and the other you pack all the nonlinearities together. And then when you want to talk about the stability of the equilibrium point of this non-linear the original non-linear system, that is equivalent to talking about the stability of this interconnection, between the linear system and that non-linearity.

So, let me try and explain what I mean by this ok. So, suppose you were to look at an equation like let us say the second derivative of y plus let us say the first derivative of y squared plus let us say 2 times y equal to 0.

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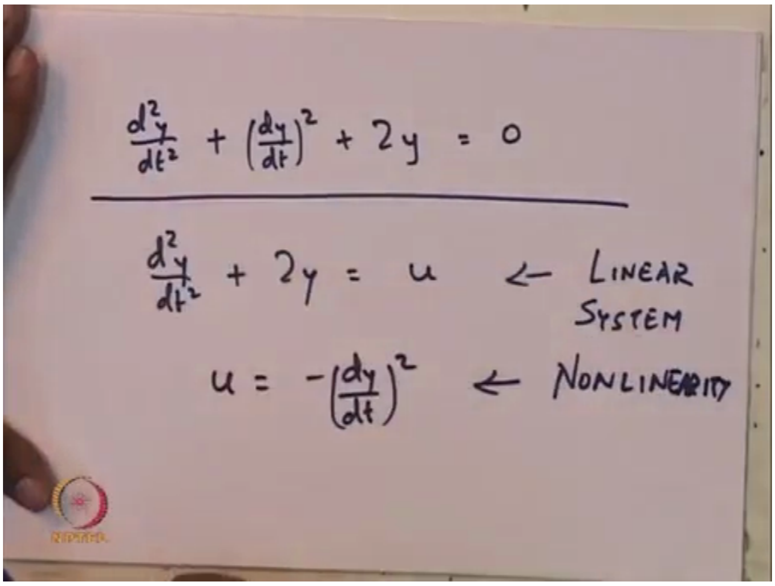

Now, one wants to analyze the this system, this system is a system without any inputs. And there are well known methods of trying to analyze the system, but what I was just talking about was this clever way of thinking about this system as an interconnection between a linear system and a non-linearity ok.

So, now let me try again give you how that is done. So, think of this equation d 2 by dt squared plus 2 y equal to u; this now is a linear system. Now this is exactly like this equation except that you could think of this dy dt squared as minus u. So, let me put that as the next equation which is u equal to minus d by dt squared and this is the non-linearity ok.

Now, the interconnection of these 2 systems then would give us exactly the same as this original non-linear system ok. So, let me let us look at what this interconnection would look like.

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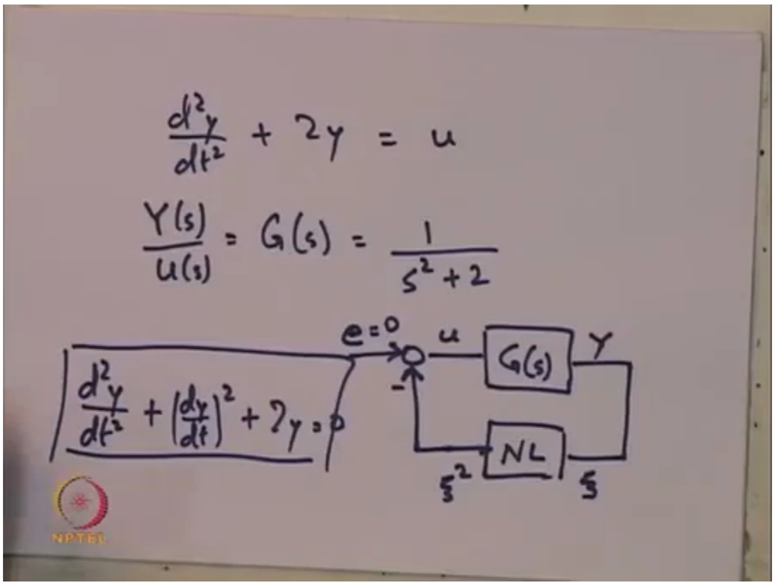

So, so, if you look at that system d 2 by dt squared plus 2 y equal to u, this is a linear system which has transfer function given by Y s by us. This is the transfer function G s which is equal to 1 upon s squared plus 2.

So, now let us think of this G s in this way and here is the input that you give which is u and the output that you get for this which is y. You take this y and put it to a non-linearity and this non-linearity has the characteristics that if the input to the non-linearity zi the output is zi squared.

Now in this case, if we now feedback this thing with a negative feedback then if this e which is the input of this feedback loop. If this e is set to 0 then what we would be looking at is exactly the original equation that, we had which is d 2 by dt squared plus d by dt the whole thing squared plus 2 y equal to 0.

So,. So, talking about this closed loop system being asymptotically stable for example, is exactly the same as saying that the original non-linear system is asymptotically stable. Now of course, in this system we can clearly make out that the system would have the; I mean if you draw a phase plane portrait of the system then the origin is an equilibrium point.

Now, the origin is an equilibrium point if one wants to find out whether this origin is let us say asymptotically stable that is the same as in this feedback system; whether that given system this feedback system is asymptotically stable. So, if you if you set up this feedback system and set y with some value and then you let this system evolve. And if the system evolves finally, settling down to the value y equal to 0 u equal to 0. Then what that would mean is this system is asymptotically stable and what that effectively means is that the original non-linear system is effectively stable.

Now, this clever trick is I mean once this clever trick was discovered this was used again and again. So, the non-linear systems were analyzed by looking at splitting it up into a linear system and a non-linear part and looking at the feedback connection of this non-linear system and this non-linear part. Now once that was done then people came up with some sort of a conjecture.

Now, ok. So, let me let me give you what this conjecture was all about ok. So, suppose you had a system like.

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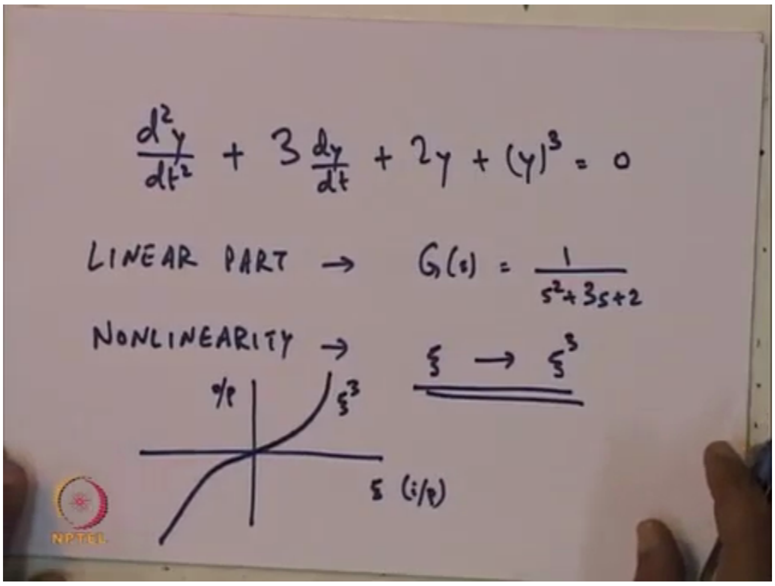

Let us say a zimilar to the earlier one, but d 2 by dt squared plus let us say 3 times dy dt plus let us say 2 times y plus y cubed equal to 0. So, this now can be split up into a linear system. So, the linear part would be given by a transfer function which is 1 upon s squared plus 3 s plus 2. And the non-linearity the non-linearity that you would interconnect with this linear system is given by something where if zi is the input the output is zi cubed ok.

Now, if one looks at this non-linearity carefully this is non-linearity. So, suppose I think of zi, which is the input to the non-linearity then the output to the non-linearity is here and this looks like something like that ok. So, that is zi cubed . Now the conjecture which was made was made by a person called Aizermann.

And Aizermann made the following conjecture that suppose you have a non-linearity and let us assume that this non-linearity is memoryless. So, what do I mean by memoryless? What I mean by memoryless is that this non-linearity, it only the output of the non-linearity only depends upon the instantaneous input.

Now, the earlier non-linearity that we looked at where the input was zi the output was zi cubed. And this output was independent of what the past values of inputs were yeah. So, it is a sort of the output only depended upon the instantaneous input and. So, if one could think of this non-linearity as a memory less non-linearity ok.

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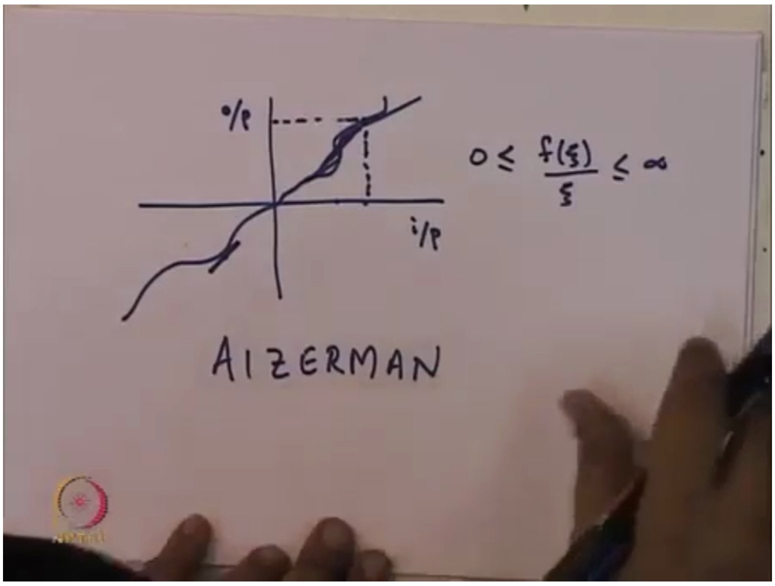

So, typically if you look at a non-linearity which is memory less, it might be you know something like that ok.

So, this here is the input to the non-linearity and this is the output to the non-linearity. So, when you have this as the input then the output is this value here and zince it just depends on the ok. So, this curve of course, is a slightly twisted curve. So, it looks like you know with this input there might be multiple values, but let us assume that the curve was straight enough. So, that such a such a zituation never arose ok.

So, the output purely depends upon the instantaneous value of input. Now the conjecture was made by this person Aizermann and his conjecture was the following. You see if you look at this non-linearity at each point there is some tangent to the curve . Now one way that you could talk about this nonlinearities you could limit the non-linearity. So, this particular non-linearity for example, one could say that 0 is less than equal to f of zi by zi which is less than equal to infinity.

So, this is a non-linearity whose slope instantaneous slopes lie between 0 and infinity ok. So, of course, I am sorry, but initial curve probably had had slopes which are greater than infinity, but this one is some trajectory or this is a non-linearity which has its slopes lying between 0 and infinity ok. Now if one looks at such nonlinearities.

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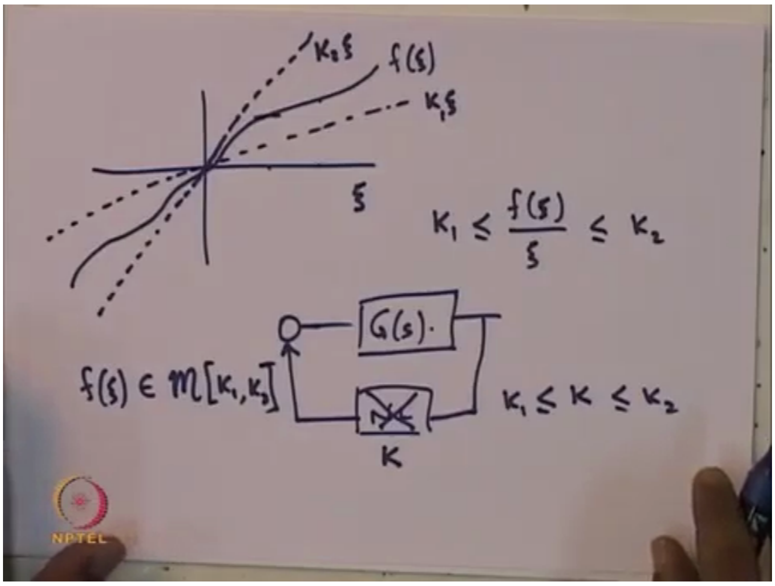

So, so, let me let me give some more examples of zimilar non-linearity. So, you could have a non-linearity like that.

And then clearly all the slopes of this non-linearity lie between this line. So, if I call the input zi. So, this line is K times zi or K 1 times zi and there is another slope here. So, this is K 2 times zi. And so, this particular non-linearity one could say satisfies the characteristics that f zi by zi is less than is greater than this particular slope K 1 and its less than this other slope K 2.

Now, suppose we interconnect this non-linearity. So, this non-linearity is interconnected with a linear plant G s ok; in this feedback structure. And then suppose this linear plant was such that if instead of the non-linearity if instead of the non-linearity you used a linear gain K. And this K was between K 1 and K 2 and the resulting linear system was stable. Then Aizermann's conjecture was that if you put the non-linearity instead of that linear gain then the resulting system would also be stable would be asymptotically stable.

So, the idea being that if you are at this point and this was the input then there is a some slope here correct. Now this slope when you put that linearity the resulting system is stable or asymptotically stable ok. Now if the resulting system is stable then the guess was that for the non-linearity because locally it is like this linearity here therefore, it will behave nicely.

So, now, as a result of course, the system evolves and you go to some others zi and at that zi there is some other slope. Now this slope when you put in as the linear thing then again you get a asymptotically stable system asymptotically stable linear system.

Now, working in this way the guess was that if you had a non-linearity which lie which lay in some sector. So, this particular non-linearity f of zi is set to lie in the sector. So, it is a non-linearity which lies in the sector K 1, K 2. So, if you have a non-linearity which lies in the sector K 1 K 2, then if the linear plant is such that when you put feedback gain any gain between K 1 and K 2. And the resulting system is asymptotically stable, then if instead of that linear feedback you put the non-linearity in there then the resulting system is going to be asymptotically stable.

So, this was Aizerman’s conjecture. So, perhaps I should just sort of formally write it down ok.

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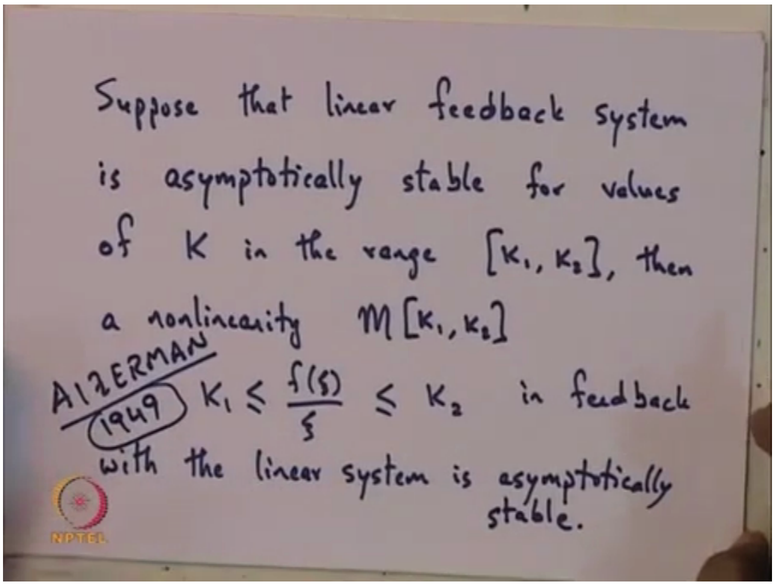

So, suppose that the linear feedback system is asymptotically stable for values of K, in the range K 1 to K 2. Then a non-linearity that belongs to this sector, which is equivalent to saying that the non-linearity f zi by zi this is less than it; I mean K 1 is less than equal to this and which is less than equal to K 2 ok.

Then a non-linearity of this kind in feedback with the linear system is asymptotically stable asymptotically stable. Now this is Aizermann's conjecture and this was given roughly in 1949. Now initially it was not clear, whether this way of looking at nonlinearities as approximation of linear systems is going to work. And in fact, in the literature for some time this used to be called the method of linearization; because the idea was that you have this non-linearity. And at instantaneous at various instance for a given input there is an output and there is also the slope of the of the output according to the characteristics of the non-linearity.

And then Aizermann's conjecture essentially said that if for all those slopes that you get in the non-linear in the non-linear function; I mean if for all of those if the non-linearity is in that particular sector. The and for those for those gains it turns out that the given closed loop system is stable then the closed loop system is stable, if you put in the non-linearity. It turns out that aizermanns conjecture is false. And this was proved by a several people over the years several people prove this thing.

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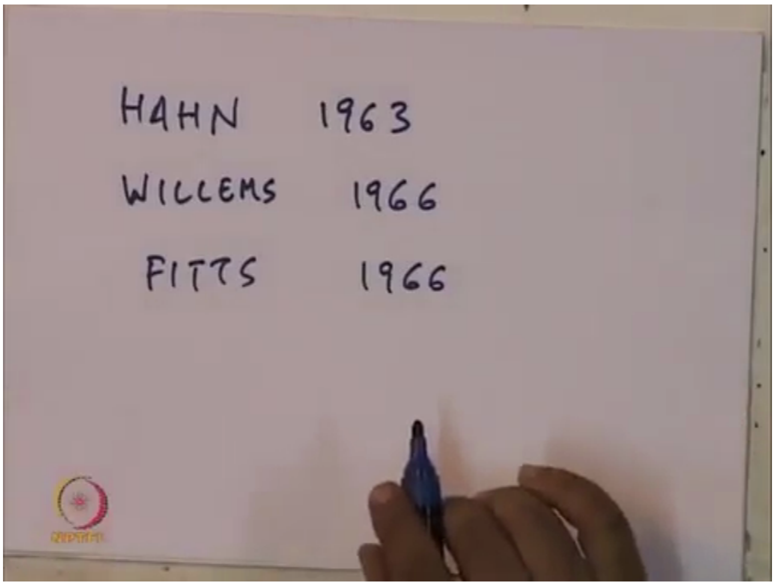

So, some of the some of the more famous names include HAHN who did this in 1963, then WILLEMS. So, WILLEMS I showed a counter example that this does not hold WILLEMS did this in 1966. Then there was FITTS who did this also in 1966.

So, they constructed all sorts of examples which showed that this conjecture of Aizerman's is not correct ok. So, let me now give an example of a system where this actually does not hold.