Nonlinear System Analysis
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Lecture – 30
Supplementary lecture: Comparison Lemma and Lyapunov Stability

So, in the proof of one of those we had used something called the Comparison Lemma. So, I will do small proof of that and also in addition I will give you a little more physical interpretability of the solutions, especially of the condition 5. So, that we had for stability of which was like A transpose P plus P A is less than 0.

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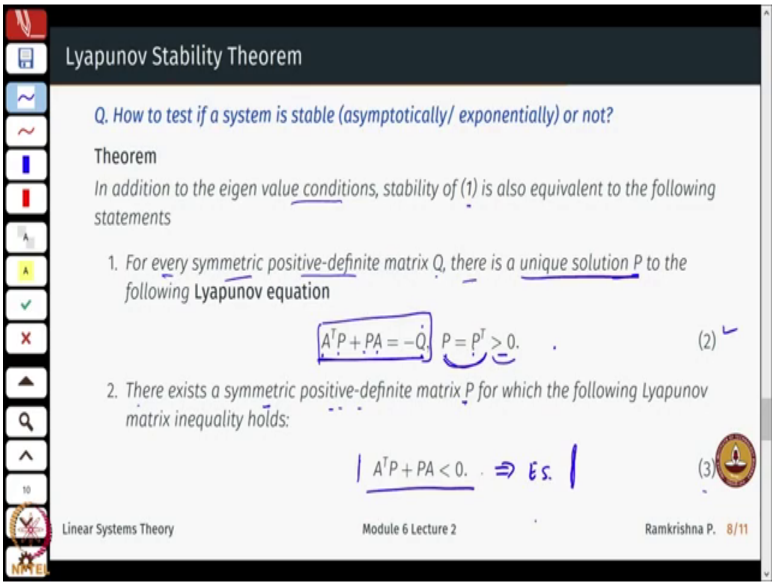

So, essentially conditions which look like this. So, we will try see if there is any physical interpretation to this condition number 3.

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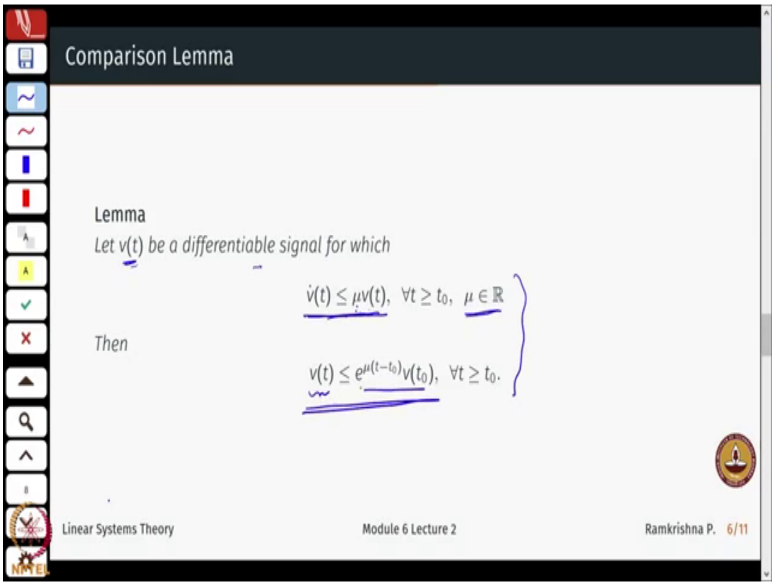

So, we will start with proving the comparison lemma. So, the statement I will derived again. So, given v t which is a differentiable signal which satisfies a condition like this, that v dot t is less than or equal to mu v of t. Then essentially, we of t solution of it looks look something like this ok. So, I will just do small proof of that and it is sometimes use full to know how this proof techniques would derive.

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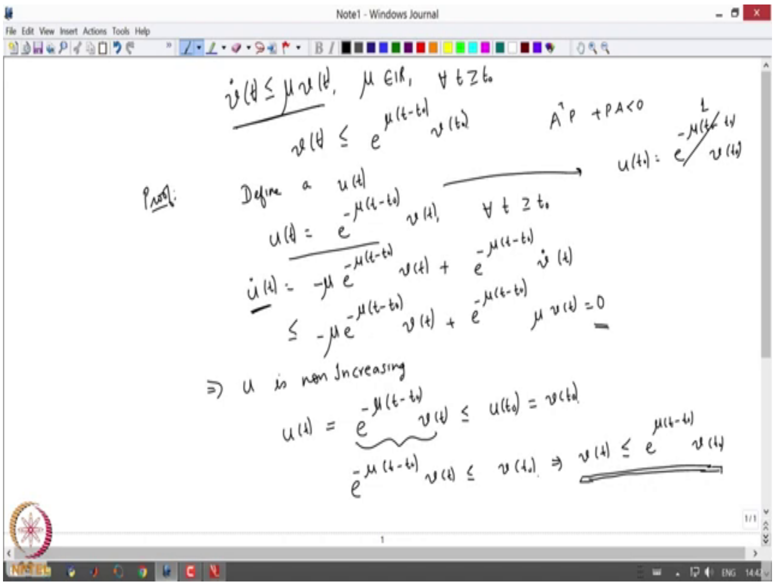

So, what are we given is v is a differentiable signal in such a way that v of t is equal sorry, is less is less than or equal to mu times v of t this mu could be anything in R and for some or sorry, for all times t greater than some defined initial time t 0. So if this holds then v of t that is less than or equal to e power mu t minus t naught, v at t naught ok. This the inequality was essentially useful in proving the condition there is a fifth condition on the Lyapunov stability. Namely, A transpose P plus P A is less than 0 ok. So, we will do little proof of this.

So what does the proof say ok? Let us define a new signal u of t which looks like this u of t is e power minus mu t minus t naught v of t again for all times t greater than or equal to t naught ok. So I will just take the derivatives on both sides. This u dot t is minus mu e power minus mu t minus t naught, v of t plus same thing as e power minus mu t e minus t naught v dot of t ok. What do I know of v dot of t. So, v dot of t is satisfies is this is actually give to me.

So, this will be less than or equal to minus mu e power minus mu t minus t naught v of t plus e power minus mu t minus t naught. What is v dot? v dot is mu v of t. So, this is equal to 0 ok. So, what do we know that? So u dot is equal to 0, this means that; u is non increasing all right. And therefore, u of t is which is this is how we defined it right. e power minus mu t minus t naught v of t starting from here ok. So, this is less than or equal to the value of u at t naught ok. Now what is the value of u at t naught? Just substitute for t naught here u of t naught is e power minus mu, t naught minus t naught times v at t naught.

So, u at t naught so, this is this goes to 1, now this is v at t naught ok. Now take a look at this. So, what do I have e power minus mu t minus t naught v of t is less than or equal to v t naught ok. This is essentially what I wanted to prove right. So, I just can just rearrange and right this as a following right. And this what do you wanted to prove and this is this concludes the proof ok.

So, it is a very nice intuitive proof here and of course, we saw ho this was instrumental for us in deriving stability conditions for linear systems ok. Now so, if I again go back to this thing here the equation number 3, this essentially solved it.

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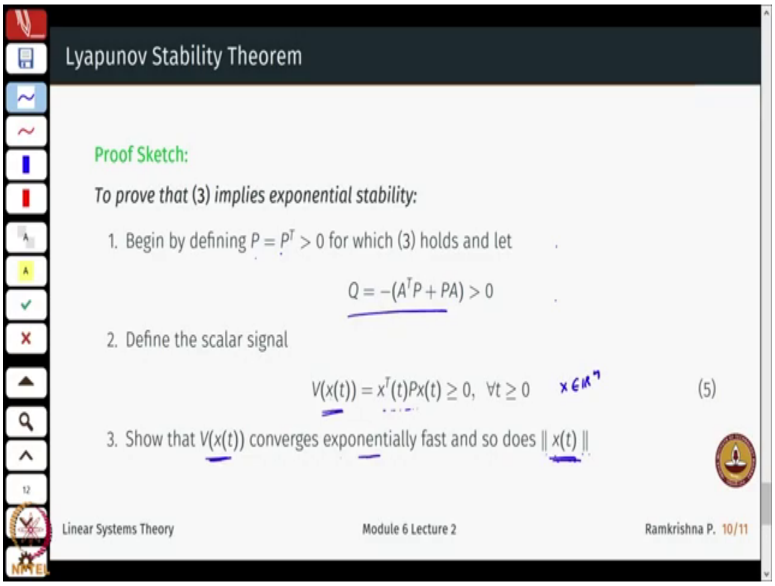

How did the proof go the proof required defining a scalar signal v of x t as x transpose p times x t it is a quadratic function it is always greater than or equal to 0 and we showed that the time derivative of v dot or the time derivative v dot was less than or equal to 0 or which also had some bounds. And therefore, since v dot was less than 0, v was converging exponentially fast and so does the solution ok. So, what does this mean in terms of does it have any anyphysical interpretation right.

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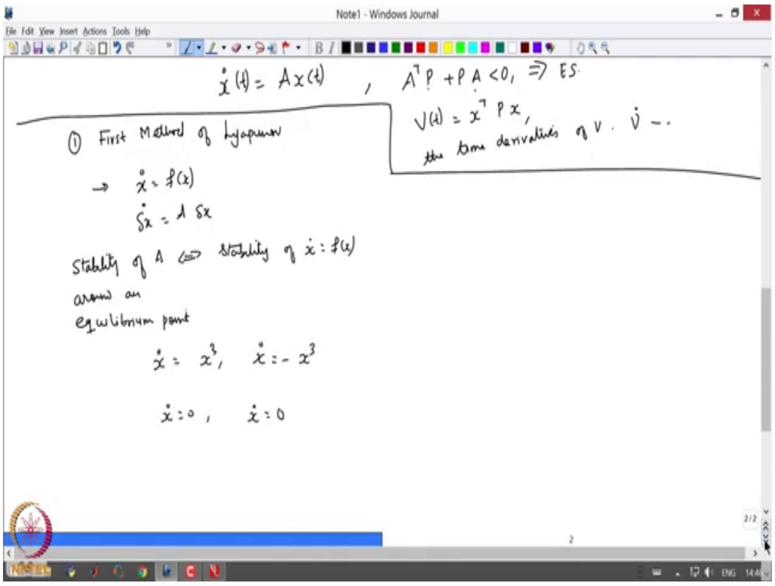

So, here we sought with the autonomous LTI system, x dot is a x of t. And then we were we had this condition A transpose P plus P A is less than 0. Not only that we defined a function V dot sorry, V as sorry, V of t is x transpose Px and then we were interested in the time derivatives of e of V of V right. So, we computed V dot and then derived a bunch of properties based on based on the system dynamics based on this condition. And what we showed essentially that this condition satisfaction of this condition was actually equal to or actually implied exponential stability ok.

So, if we you diagnose a bit and then we look at some literature from how do they proof stability in the non-linear case. The first one is what they call as the first method of Lyapunov. And this essentially involved linearization of the non-linear system of what we solved right. So, stability so, we start with non-linear system x dot is f of x we have some linearization delta x dot is a delta x and stability of A had some direct implications on this stability of x dot is fx, again this is all around an equilibrium point.

Of course, we here rule out condition when A has a 0 eigenvalue. A (Refer Time: 09:41) example to that was x dot equals to x cube and x dot is minus x cube which both had the same linearization, namely x dot is 0 and x dot is 0 and we could not say anything about the stability of the non-linear system ok. So, that is that we delt with quite extensively this is also something call this second method of Lyapunov.

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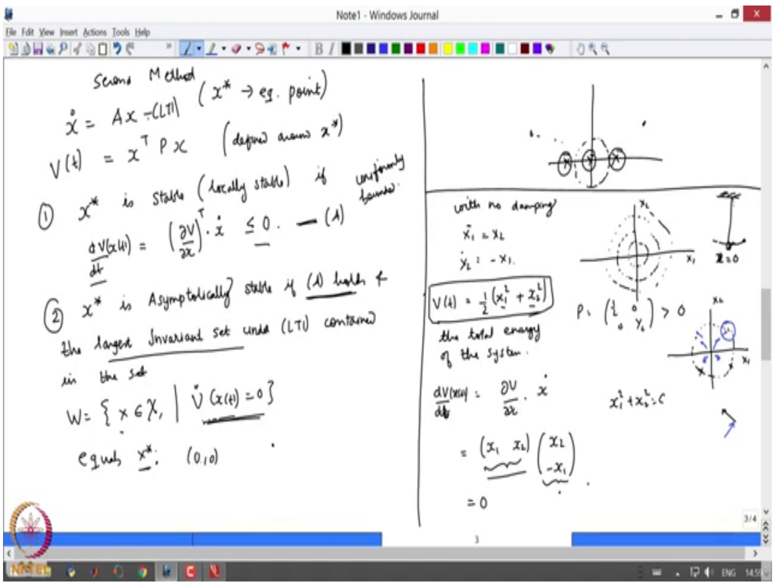

We will not analyze this thoroughly in the context of non-linear systems, but we will try to interpret that in our in within context of linearize system. So I have (Refer Time: 10:23) with this dynamics x dot is A x right. So, this could be linearization of a non-linear system around and equilibrium point and so on. So, let V of t be so, this is a positive definite function x transpose P x. So, this is usually in the non-linear setting will say we will define it.

So, this function is defined in the small labourhood around the equilibrium point. But in the linear setting well we can assume it to be if it is the system is linear by nature we can assume it to be valid all for all x ok. So if x star is an equilibrium say x star is an equilibrium point. So, the theorem says that x star is stable or if I am looking of a non-linear system I am just looking at locally stable. Locally stable because, the non-linear systems could have say isolated equilibrium points where for example, this could be stable, this equilibrium stable, this could be a stable and so on.

So, I am just looking at it is behavior around an equilibrium point within a small neighborhood and therefore, I call it locally stable there is not globally stable because all trajectories if the trajectories starts here or here they may not necessarily converge to the origin or even at this point at this equilibrium is if I start and so on ok. So, this system is stable if I am looking at this quantitative dv by dt, x of t which is dv by dx times x dot ok.

You just put little transpose here if I just say this I am just taking the gradient of this vector. If this quantity is less than or equal to 0, then the system is locally stable. Second, so x star is asymptotically stable if ok, I will call this condition A, if A holds and the largest invariant set under ok, this linear systems I will ok, let me call this LTI. Under this dynamics of LTI systems contained in the set.

So, let me define the set W as all x belonging to some total there to the state space such that V dot of x t equal to 0 equals x star right. Which means that the only solution that starts, only solution that starts in this set remains in w for all sets right and that actually coincides with the equilibrium point. So we slowly see what this mean in the context of some physical systems too ok.

So, let us take in the case of simple pendulum right. So, and say just look at the linearized version of this. So, it is dynamics around this equilibrium point x equal to 0 with no damping would look something like this right x 1 dot is x 2 and you normalized all parameters to be 1.

x 2 dot is minus x 1 ok. Now how does the phase space of this look like? The phase space of this looks like this. So, you just concentric circles around the origin. So, here in the origin is the equilibrium point or here or here or here ok.

Additionally, if I now define this function right. So, V of t so, let me call this half x 1 square plus x 2 square. So, this is exactly this function with P being the 1 by 2, 0, 0, 1 by 2. And this you can see is symmetric and greater than 0 right. So this if you has also the interpretation of the total energy of the system. The kinetic energy plus the potential energy like, half m v square and half k x square kind kinds of. So this is the total energy of the system.

Now when I say v dot dv by dx sorry, dv by dt of x of t. So, I am computing this quantity right, dv over dx times x dot. So, what does this means? So, what is dv over dx? If I take the gradient of this. So, I will get x 1, x 2 and here I have x 2 and I have minus x 1, this essentially is 0 ok.

So, what does this mean? If I look just this quantity which is a definition of energy that the rate of change of energy is constant right, that the energy does not change with time. That is also from the physical interpretation right. If I just have a loss less pendulum and I subjected to some initial condition it will just keep on oscillating infinitely ok.

Now, I can also look at it from the phase space right. Just take any point in the phase space. So, just take one initial condition say over here and this is our trajectory will go right in the circle. So, at any point here if I compute the energy here it will be the same as computing the energy at this point or this point or this point. Why because even if I look at the solutions the solutions would be of the form x 1 square plus x 2 square is constant ok.

So, that also is consistent with the phase space right. So, this will give me tell me that this function x 1 square plus x 2 square is constant along the system trajectories. So, this is x 1, this x 2 right. And therefore, the v dot is 0. Now a third interpretation right of so, this essentially can be looked upon as dot product of 2 vectors. So, how does a gradient vector look if I radius to plot?

This gradient vector will simply look this right, something like this, something like this, something like this. So, essentially if I look at the so, if I just zoom in over here. So, the gradient vector will be something like this and the original phase face of the system will be just perpendicular to at in such a way the angle between these two is 90 degrees right.

And then they are like perpendicular to each other and therefore, the dot of the product will be 0 right. So, that is another interpretation of it is a gradient of the energy function times the phase plain or the vector field of the system orbit looks like ok. Now the second thing here is to do with asymptotic stability right [vocalized -noise].

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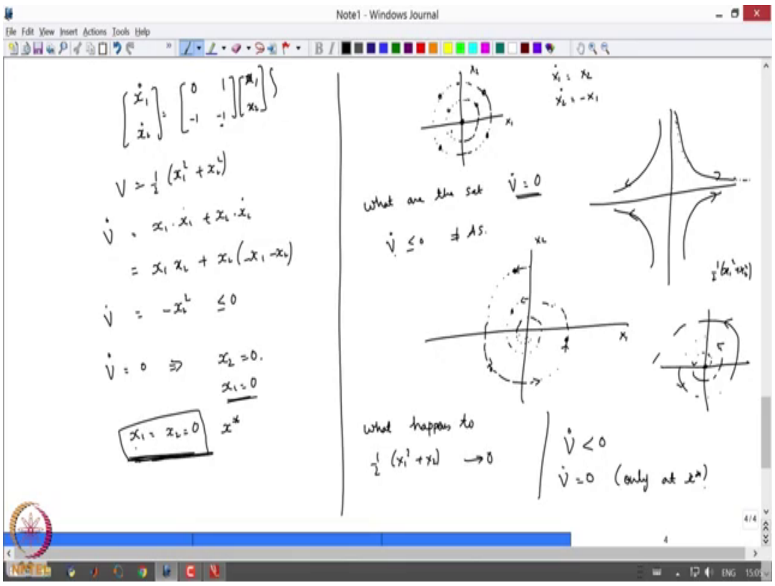

So, what does the theorem say that additionally so, let us say I take a system which is non x 1 dot, x 2 dot and I just add some friction term or some damping term 0, 1, minus 1, minus 1, 0 and see sorry, x 1 and x 2 ok. Now if I again look at V as half x 1 square plus x 2 square I compute V dot that is x 1 times x 1 dot plus x 2 times x 2 dot. So, this will be so, x 1, x 1 dot is x 2 plus x 2. What is x 2 dot? x 2 dot is minus x 1, minus x 2.

So, this is minus x 2 square and this less than or equal to 0 ok. So, the statement says that well the system is stable as long as dv by dt is either less than or equal to 0. So in this case it was just equal to 0. So, this also means that so, this statement a here that the condition A means here that your solutions are uniformly bounded.

Now, that the first condition which expressibility. Second condition which expressibility is that first that the system should be stable and it should also be in such way that as t goes to infinity the solutions go to the to the equilibrium point which is the origin in this case.

Now why is this step here important ok. So, what I am looking at? I am looking at the largest in variant set such that V dot equal to 0 ok. What does it mean by V dot being equal to 0. V dot being equal to 0 implies I am looking at x 2 being equal to 0 ok. And in terms also if x 2 is 0 also have a x 1 is also equal to 0 ok.

So, what is and why I am I obsessed with a largest in variant set right. So, here there is a statement says that the largest in variant set such that V dot such that V dot equal to 0 should only be the origin which is 0 comma 0 in this case.

So, in this case, the only possibility if for V dot for being equal to 0 is x 1 is x 2 equal to 0 ok. So, why is this important to check this additional condition? Why and why I am I actually calling it the larger invariant subset. Now if I go to the previous example where I had no damping x 1 dot is x 2 and x 2 dot is minus x 1. What do I have is well my phase plane is something like this ok.

Now, what are the points or what are the set or what is the set such that V dot equal to 0. Now I see that at this point V dot is 0, at this point V dot is 0, at this point v dot v dot is 0, this point also V dot is 0, this point also V dot is 0. If I start with this initial condition so, at each point in this curve the V dot equal to 0 in the x 1 x 2 plane.

And therefore, this system even though V dot is less than or equal to 0 is not asymptotically stable. Because additionally, I need to verify a condition like this right that the largest invariant sets such that V dot equal to 0 is only the origin which is actually happening in the second case right. The when V dot equal to 0; that means, x 2 equals to 0 if I applied it in here, I get x 1 is also equal to 0 and that the only possibility here is x 1 equal to x 2 equal to 0 which is exactly my equilibrium point ok.

Now, in the that was about why we are obsessed with the largest invariant set. Now if I look at the phase space in the second setting where I have some damping in the system the phase space is usually are the ones which go to the origin in the x 1, x 2 right. This so, possibly in this direction. So, they will spiral to the origin again based on what their eigen values are if it is a stable focus and so on.

So if I what happens to this function half x 1 square plus x 2 square which essentially is the total energy of the system. At this point it has a certain values and at this point the value diminishes, it diminishes further here, further diminishes here until it actually reaches to 0. So from calculates it also means that V dot is less than 0 and V dot is equal to 0 only at x star ok.

Now, the third condition right of an unstable system, unstable system ok. You, if you are looking at saddle point then this is how my phase space will look like or if I have say in some cases if the it might be a spiral away from the origin and with this direction of the arrows ok. Now what happens to the energy well at this if I start next to the origin then you see that the energy, the value of half x 1 square plus x 2 square it increases in a and becomes un bounded right here also here right.

So, stability can also in a way be interpret at as a certain energy function decreasing along the trajectories of the system and eventually going to 0. So, when my pendulum is swinging it switches between it is kinetic and potential energies and at some point of time if there is dissipation the energy will all dissipate and just go 0. You can also talk of rate in terms of an electrical circuit of an are LC circuit the energy will oscillate between the electric and magnetic part. And from the r will keep on dissipating energy until it reaches the state of 0 energy.

In stability will amount to the energy growing un bounded at least mathematically. Physically well you have some physical limitation that the energy cannot grow to infinity ok. So, that is the little interpretation of the Lyapunov stability in terms of the physics of the system. That is what the motive of this entire supplementary material was to actually give energy give a little energy between stability and the loss of physics which essentially comes from the loss of a conservation of energy and I hope this explanation was useful to and it will be more useful when you even do non-linear setting of Lyapunov functions.

Thanks.