# Nonlinear System Analysis

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## Lecture - 03
## Math Preliminaries Part 02

We will continue with our previous discussion where we had stopped with the definition of a definition of interior point of a set right. And, the way it was defined was a point x of point x in a set A is said to be an interior point right, if you can find a neighbour route such that all its points belong to the set A right.

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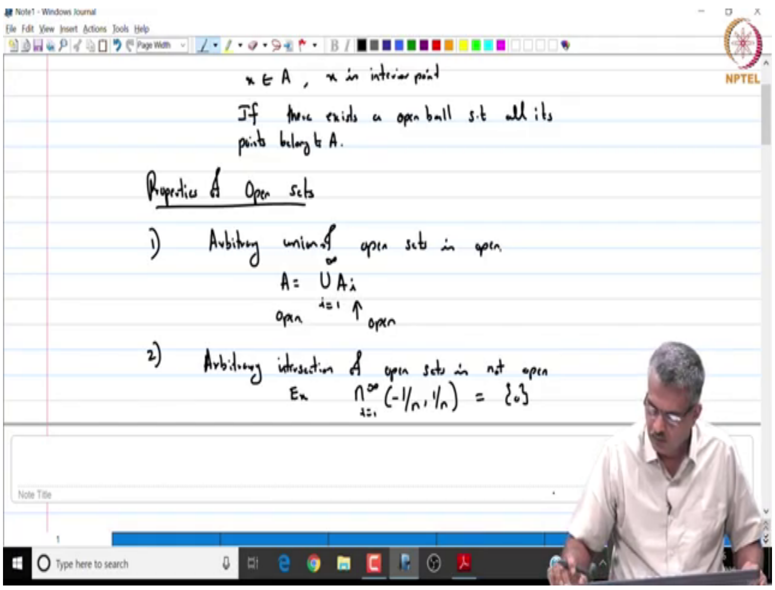

So, could be like this a an open ball right, we do not define what is a neighbourhood but, we defined a open ball such that all its points belong to A. And, then once you had a notion of interior point then the collection of a set is said to be open, if it contains all its interior points.

So, now we know the definition of open set; so, let us see some properties of open sets right. So, this was interior point. So, the first property is arbitrary union of open sets is open right so; that means, if you have a collection of open sets where A i is open right and then you take arbitrary union of open sets. So, if I call this as some A then so, each A i is open then arbitrary union of open sets is also open ok. We will not prove this, we will it is a property right one can prove that. But, what is important is that you can show that arbitrary intersection; arbitrary intersection of open sets is not open, its not necessarily open.

Example you take a intersection over the intervals of the form open intervals of the form 1 by n minus 1 by n 1 by n where, n is a integer right. So, this is nothing, but the singleton right and we know that this is we will later on show that this is not an open set. Next we will move to the notion of closed sets right. So, just like we defined the notion of open set using an interior point right, we will define a notion called an adherent point.

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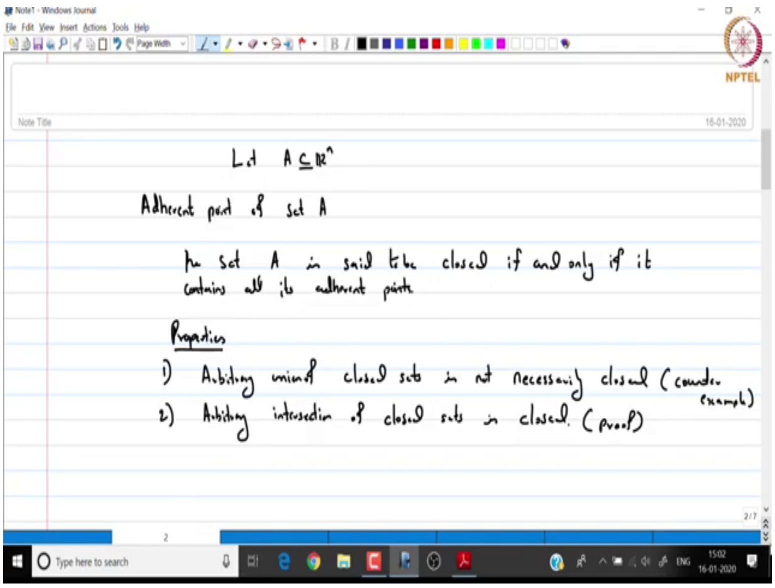


So, let A be subset of R n and we want to define the notion of adherent point adherent point of set A right.

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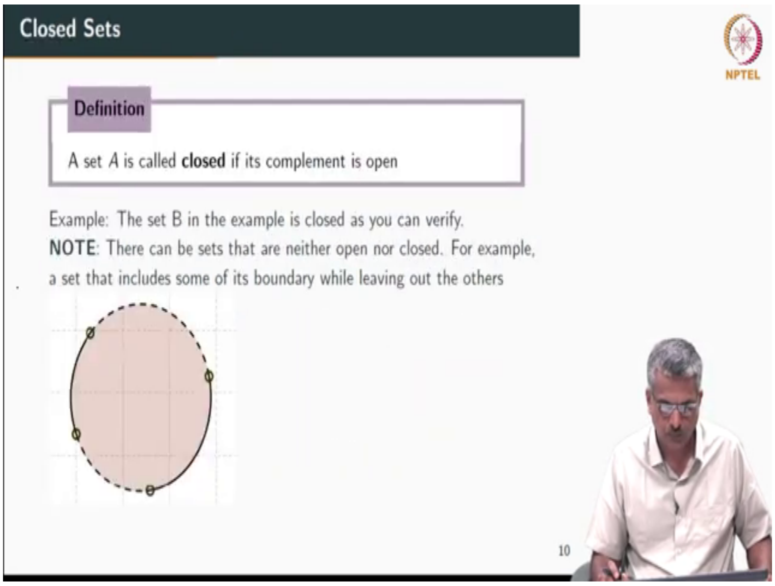


So, let us see that. So, one way of defining closed set we saw that the complement of open set is closed set, but we want to do it from basic definition right yeah.

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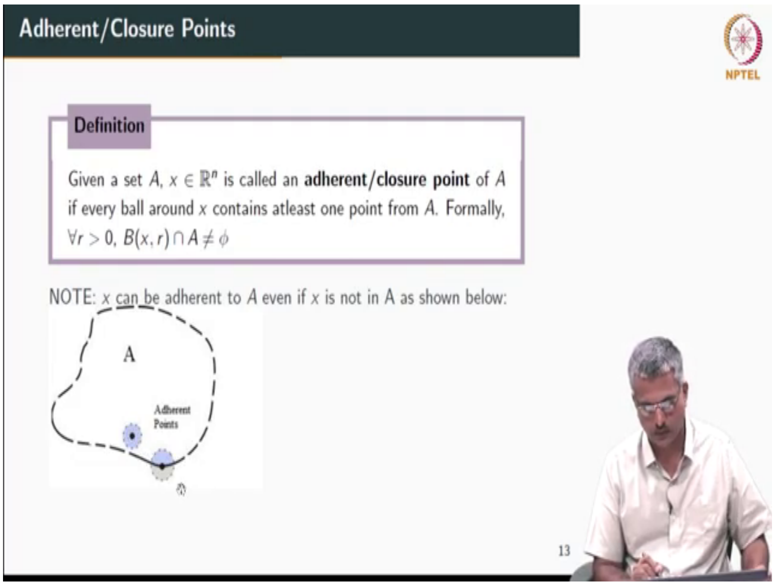


So, given a set A x belong to R n, note that x need not belong to A x is any point in R n; it is called an adherent point or a closure point of A. If every ball around x contains at least one point of A right so; that means, given a point x you should be able to come up with an R right. So, a ball centred at x and radius R such that its intersection with A is non-empty right. So, here the key points out that the one is the x need not belong to the set A, x is a point in R n and you have to you have to show that for every ball, every ball right around x contains at least one point from A unlike the notion of interior point where we had to just come up with one ball right.

So, here you have to show every ball around x contains at least one point of A right then x is said to adhere to the set A. So, in this figure right you can see that there is a set A and by I denote by dotted lines to signify that the points on the boundary do not belong to the set A right. So, now obviously, if you take a point in inside the set, if you take the point inside the set then every neighbourhood of this point right contains an element of the set A. However, if you take a point here right then no matter how small the ball you choose we will never be able to sorry.

If you take a point on the boundary right, then you take any ball around this point, it will contain at least one point of the set A right. So, in that way the point on this sitting on the boundary is also an adherent point. Note that point does not belong to the set A because as I said set A does not contain the boundary right. So, it contains all points inside this dotted line. Now, so, once we know the notion of an adherent point right then we can define the notion of closed set right. So, a set A right a set A is said to be closed if and only if it contains all its adherent points right.

So, just like we defined the open set using interior points, we can define the notion of a closed set using adherent points. Again, properties of closed set right 1, is arbitrary union of closed sets is not necessarily closed, it need not be closed whereas, arbitrary intersection of closed sets is closed right. So, for this you need to come up with a counter example whereas, here you will have to give a formal proof right. So, we will not dwell into that.

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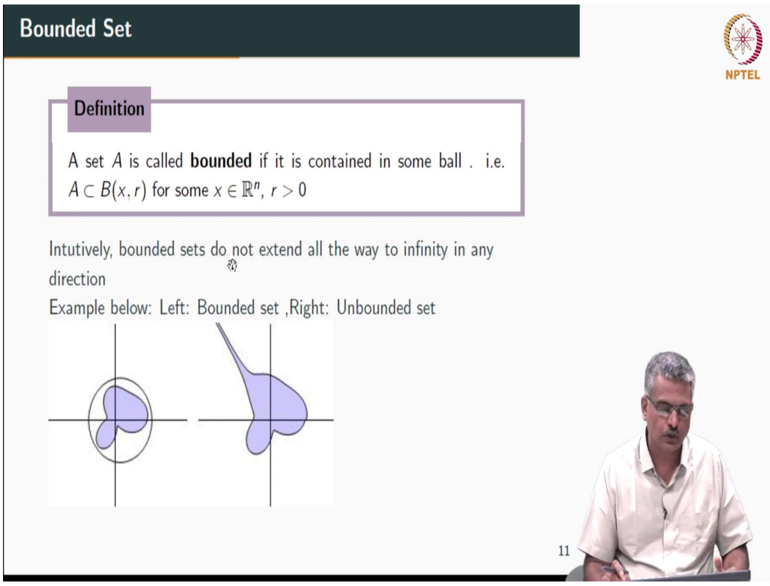


So, now once we have defined the notion of open set and closed set we require what is known as bounded set. So, a set A is called bounded if it is contained in a some ball with some radius and with some centre x right. So, here in this case you I have two examples right. So, this is a case of a this is a set A and you can bound it with an open ball, with some centre and some radius R whereas, this set right it you can see that it is unbounded, because this part it continues infinitely right it opens up infinitely.

So, you will never will be able to find a centre x and a radius R such that this set is contained in a ball right. So, once you have the notion of a closed set and a and a bounded the notion of boundedness, one can defined what is known as a compact set. But, before we go that we need another notion called the notion of limit point of a set.

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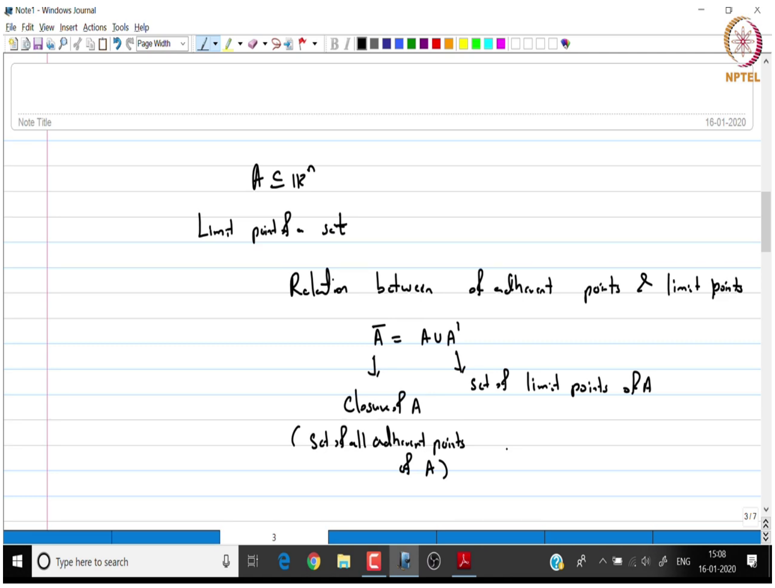


So, again A is a subset of R n. So, we want to define the notion of limit point of a set, it is a another way of characterizing a closed set. So, we will see that, we will see the definition a point x in R n is said to be a limit point of a set A if for if every neighbourhood of x right contains a point of A distinct from x right.

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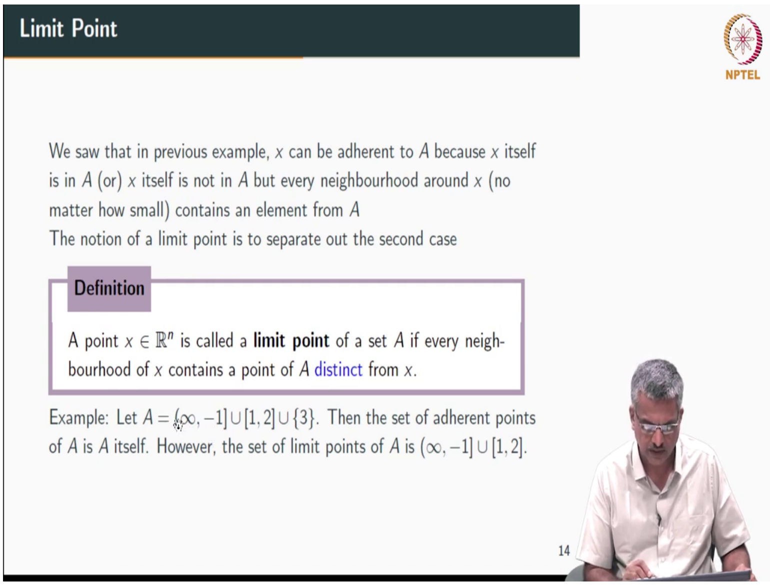


So, unlike an adherent point where every neighbourhood of x contain a point belonging to A and that point could be x itself in the in the definition of adherent. Whereas, in the definition of limit set the difference is that the every neighbourhood of x should contain a point distinct from x other than x you should be able to find a point from the set A right, then that set that is point qualifies to be a limit point right. So, here I have the an example to know capture all the whatever notions we have studied.

So, let us so, here the in this case the set is union of 3 sets here minus sorry infinity to it should have been minus infinity, it is a typo here minus infinity to minus 1 union 1 to 2 union the singleton. So, we denote the singleton by this curly bracket right. So, if you want to find the adherent points of this, the adherent points are; obviously, here all points from minus infinity to minus 1 again there is a typo here, it should be minus infinity all points between 1 and 2.

And, also the point 3 itself right; however, for the limits points its the whole of this interval, the whole of this interval ok. But, its it does not include the boundary point because if you take the boundary, if you take the singleton 3 right then you take any neighbourhood around that point right then you will not be able to find a point of that set other than the point itself. So, the definition fills and the boundary the isolated point 3 does not qualify to be a limit set right.

So, here is an example which illustrates the difference between the notion of adherent points and limit points. Again the one can characterize the close set as a set A is said to be closed, if it contains all its limit points. So, what is the relation between the set of adherent points and limit points? Sometimes it is also called accumulation points. So, the relation is that A closure this is the closed set is nothing, but A union the set of limit points. So, this is the closure of A closure of A and this is the set of limit points of A right or this is nothing, but set of all adherent points of A.

So, basically in A prime that is set of limit points an element x is said to be belong to the set of limit points, if x adheres to A minus the point x itself right. So, we have the notion of now the adherent points limit points interior points right and, also we have also seen the complement of the set right.

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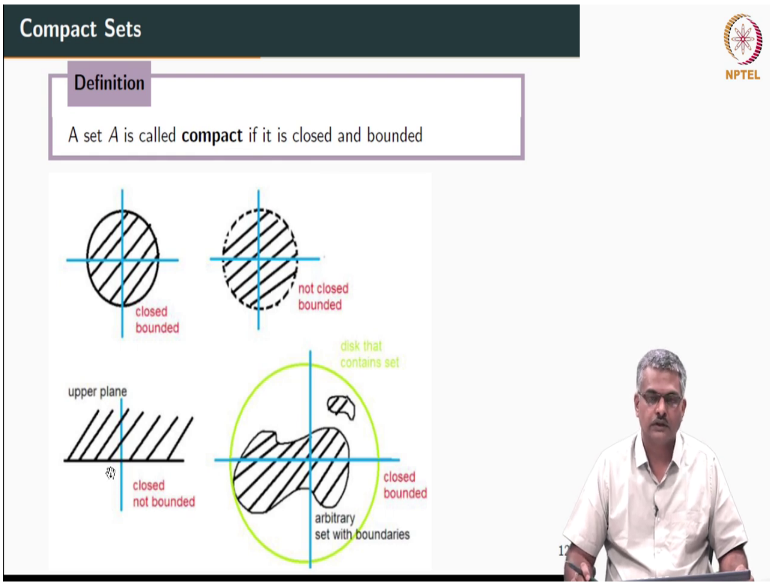


So, we will move further and define the notion of compact set, a set A is said to be compact if it is closed and bounded. Note that this is not the this works this definition works in Euclidean space and not in some other topological space right. So, there is a more formal definition of compact set, but since we are going to work in Euclidean space this definition will work right.

So, set A is said to be compact if it is closed and bounded. So, you can see here I have some examples right. So, in R 2 you have the closed balls right, it is closed right; that means, it in the set includes all its boundary points. And, it is bounded because you can always find a open ball with some centre and some radius such that this set is contained in that. And, here I have a example which is closed, but the set is unbounded the upper half right.

So, I can never find an open ball which contains this set here. So, the notion of a compact set is very important in we will use that when we do the stability analysis right. So in fact, we will use that to characterize the notion of region of attraction of a equilibrium point.

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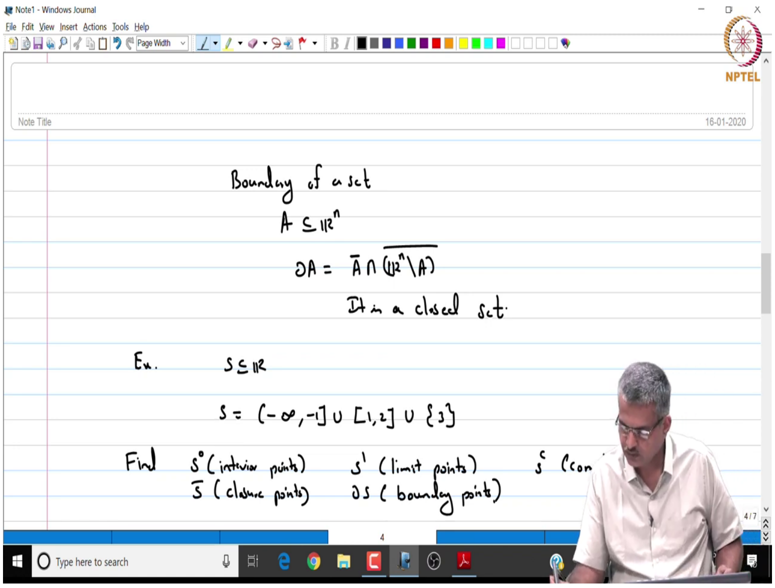


Next, we will move on to the notion of a boundary of a set even though we have been talking about boundary of a set, we have not formally defined it right. So, again A is a set subset of R n and we want to define boundary of a set. So, the boundary of a set is nothing, but its denoted by this symbol is nothing, but take the closure right intersect with its closure of its complement.

So, the complement is A right and then take the closure. So, this is how the boundary of a set is defined right and its quite straight forward to see that it is a closed set because it is a intersection of closed sets, it is a closed set. So, by boundary we mean we pictorially we understand the boundary, but it is important that we need to go beyond pictures and apply the definition, that pictures are only the starting point to understand a the definition, but one should move beyond the visualization.

So, at this stage I think now we are in a position to take up an example and identify the interior points, adherent points, limit points, boundary points and the complement right. So, I have the example here. So, let S is a be a subset of R ok. So, S is given by this interval so, we have to we need to find S interior. So, let me denote the interior points by the symbol then S it is the closure points, then S prime the limit points and the boundary points.

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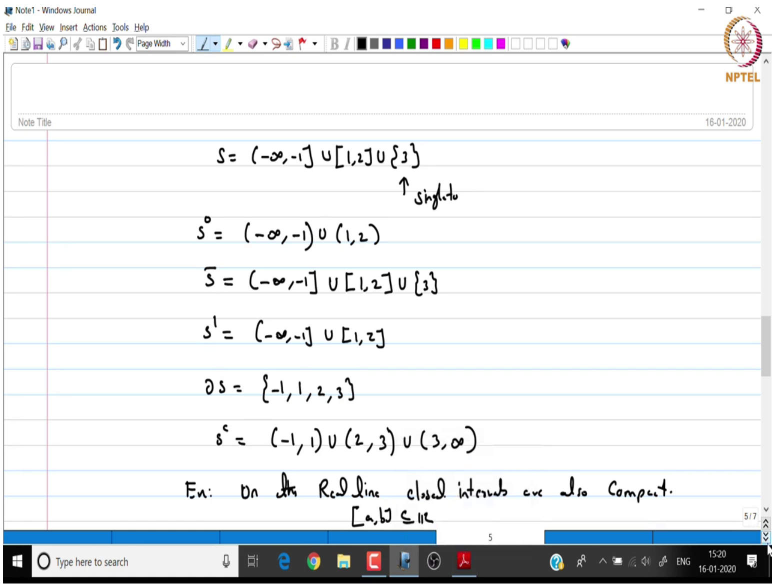


And, we also need the complement, complement of S right. So, let me write down the set once again. So, it consists of union of 3 sets so, note that this is a singleton set. So, S interior right should contain all points from minus infinity to open set minus 1 right because, if you take any point about minus 1; if say minus 1 qualifies to be an interior point. Then any ball around that right will contain all its I mean the minus 1 does not qualify to be an interior point right.

Now, what about this? So, 1 and 2 you exclude them right and we also know that the singleton is a closed set. So, we exclude the and it is a close singleton is a closed set so; obviously, it does not qualify to be a interior point right. So, the interior of set S is from minus infinity to minus 1 union 1 to 2. So, here as I said the minus 1 does not qualify to be an interior set because, if you take any neighbourhood right; however, small not all its points belong to the set S right.

Because, no matter how small ball you choose around minus 1 you will always get some points to the right of minus 1 which are not in the set S right. So, the minus so this points do not qualify to be an interior point. So, you remove them and the way you remove is you put an you make it as an open set right. So, we did not open intervals by the round brackets. What about the S closure? So, the set of closure points of S; so, certainly it will include all points from minus infinity to minus infinity to minus 1 union.

It will include the points 1 and 2 because about 1 every neighbourhood will at least contain should contain at least a one point from the set right and obviously, the number 1 is there in that set. So, it will contain all points between 1 and 2 including 1 and 2 and it also includes the singleton because you take any neighbourhood around the singleton 3; it obviously, contains the number 3 whereas, S the set of limit points, the only change is that it will not include the singleton. What about the boundary points? So, the boundary points are minus 1 right, minus infinity is not a boundary point.

It does not belong to the set right so, it is not a boundary point here. So, it will be minus 1 then 1 and 2 and 3 and finally, S complement is all points from R you remove the set S right. So, you have from minus infinity to minus 1 so, minus 1 is included; so, I need to exclude minus 1. So, it is from minus 1, 1 is included so, I need to exclude 1 union 1 to 2 is in set so, I need to exclude 2. So, 2 to 3 because 3 is there so, I need to exclude 3 union 3 to infinity sorry this should be yeah this is ok. So, this is the complement of the set S right.

So, here is a example which captures the notion of interior points of a set, the closure of a set, the limit points of a set, the boundary of a set and the complement of a set right. So, in R n we have seen that closed and bounded sets are compact right. In fact, on the real line the closed intervals are compact. So, on the real line closed intervals are also compact so, that is in interval of the form are compact. Now, the natural question is are there are two questions that arise are there sets which are neither open nor closed? That is the first question, then the second question is are there sets which are both open and closed? Well, the answer is we will answer the first second question right.

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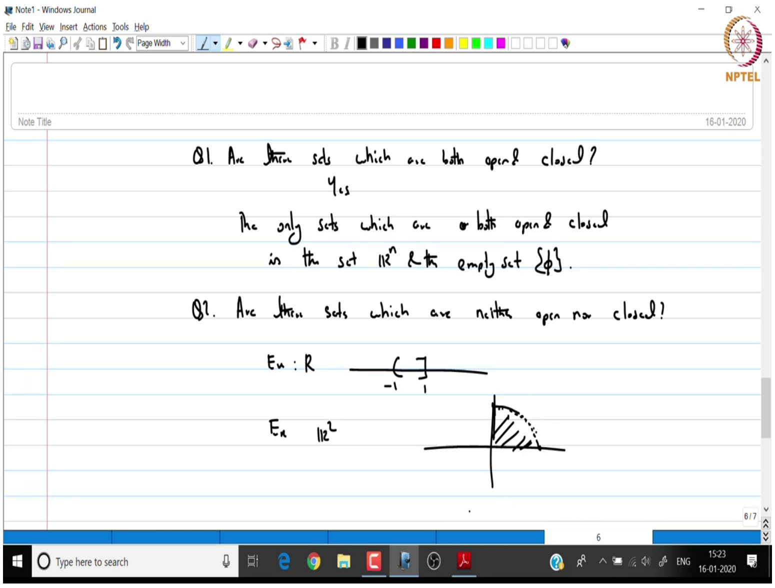


So, are there sets which are both open and closed? The answer is yes and what are those sets? In fact, the only sets which are both open and closed are the whole space and the empty space. So, the only sets which are both open and closed is the is the set R n and the empty set ok, we will denote the empty set by this symbol. So, the question 2 was are there sets which are neither open nor closed? Well, on the on the real line right so, you can have sets which are like this say this is minus 1 and 1. So, this includes all points excluding minus 1 right up to the point 1.

So, this is an example of a set which is neither open nor close. So, you can either call it as half open, half closed or half. So, this is an example where in in a real line and what about the in R 2 ok. So, this is in R say in R 2 ok. So, maybe you can have something a set something like this right. So, it includes all points inside this, but it does not include this boundary points right. So, this is an example of a set in R 2 which is neither open nor closed. So, we have seen now the notion of open set, close set, compact set bounded set right; there is another notion called, the notion of dense set.

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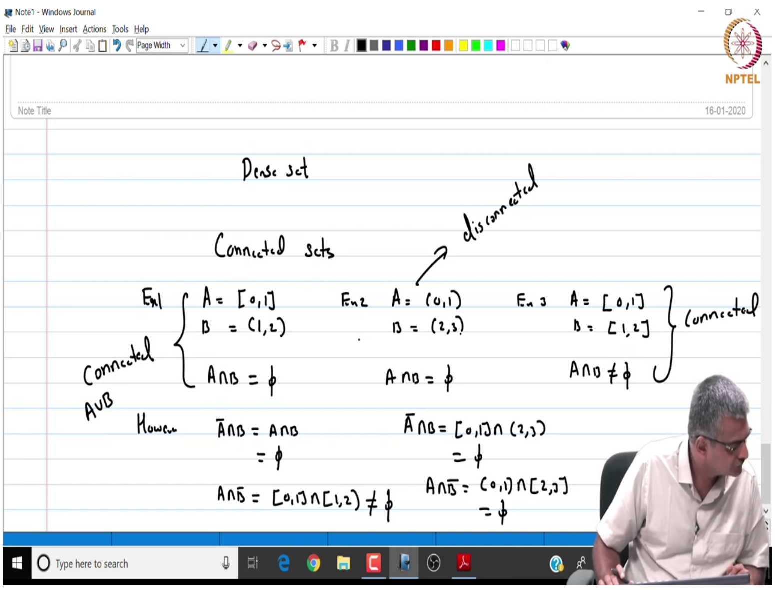

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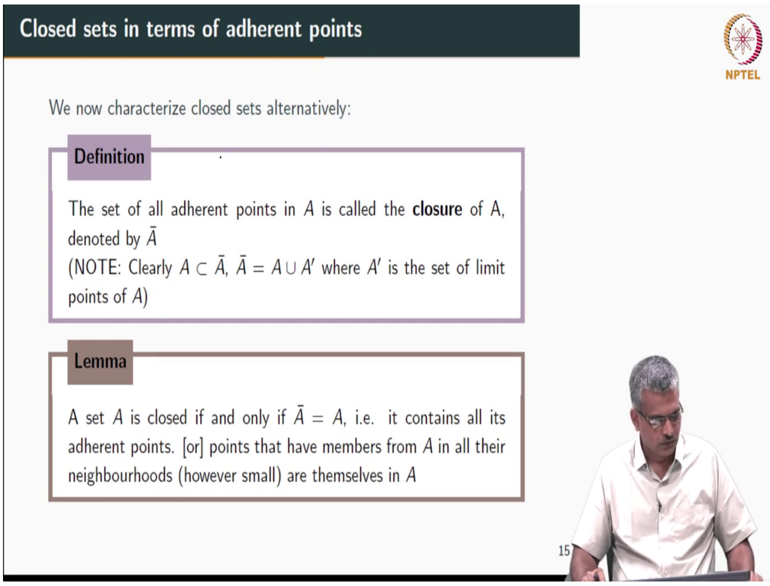


So, this is motivated by well known result in analysis which says that the rationals are dense in real numbers. That means, you give me an any real number and you give me an arbitrarily or you give me any neighbourhood or; however, arbitrary small you can always find a rational number sitting inside that ball right or in this case the interval, the open interval.

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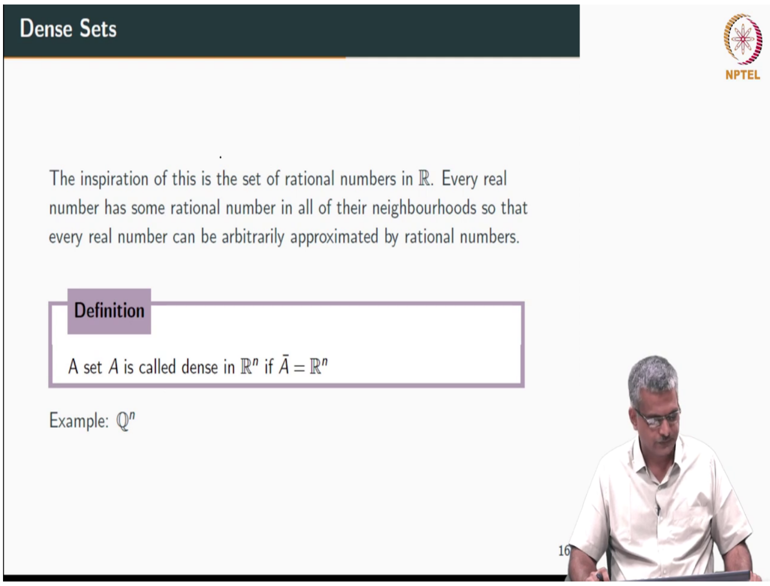


So, let us see the definition. So, a set is A is called dense in R n if A closure is the whole of R n ok. So, as I said the well known example is the set of rationals, having defined the notion of dense set we are left now with one more notion that is the notion of a connected set.

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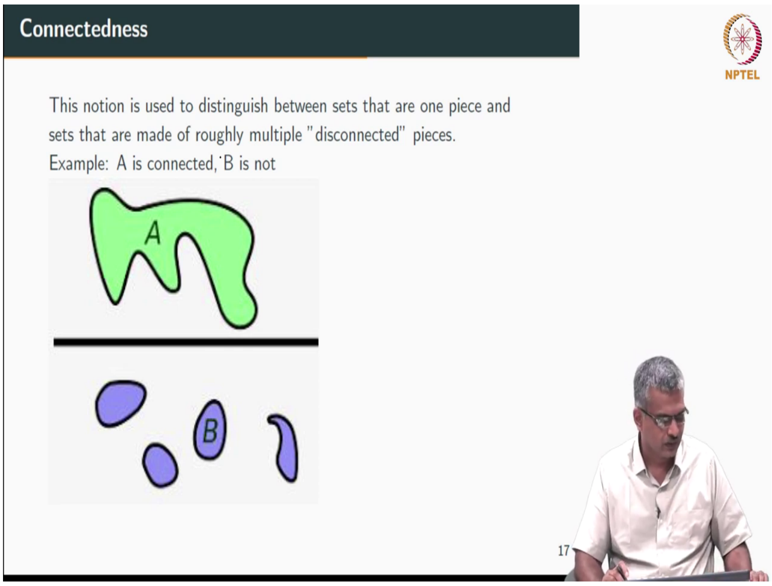


So, let us understand what we mean by disconnected sets so, it will be easy to understand what is meant by connected set right. So, the picture below here is an example of a disconnected sets; that means, they are not in one piece right. So, intuitively it means that a set is said to be connected if it is in one piece or it is said to be disconnected if it is not in one piece right. But, this is a very intuitive idea and let us see what does it what is the mathematical definition of a the connected set right.

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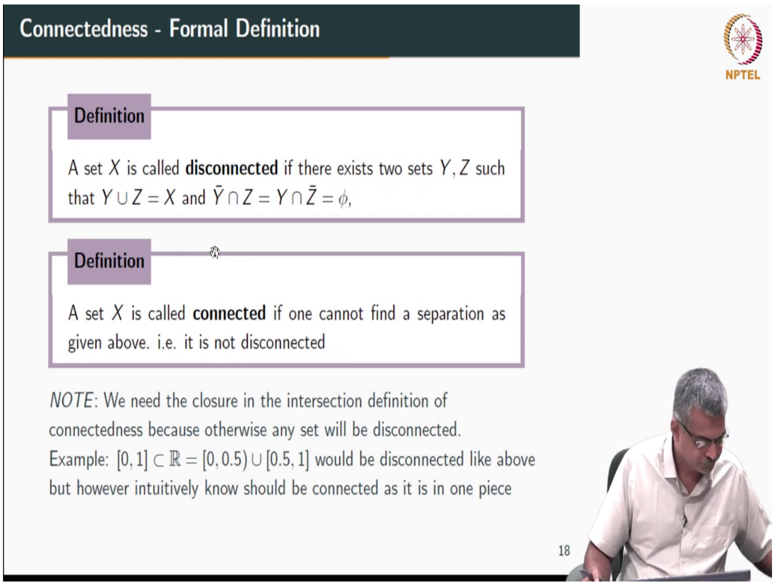


So, there is a notion called separable separated sets or disconnected sets. So, two sets Y and Z are said to be disconnected if their union is the whole of the set X and Y closure intersection Z and Y intersection Z closure both are empty sets right. So that means, they do not have the limit points, the set Y does not have the limit points of Z and so, also the case with the set Z. So, once you have the notion of disconnected set we say the set X is connected if you cannot find a separation right.

So, let us make this little more formal. So, we are now talking about connected sets so, I have an example. So, let me take A as 0 1, these are sets in R and B has 1 2 right. Now, what can you say about A intersection B? It is a null set because this 1 does not belong to the set B whereas, 1 belongs to A right. So, their intersection is the empty set. Now, let us take one more example, this is example 1 we will call it as example 1.

So, A is 0 1 and B is 2 and 3, well here to A intersection B is the empty set right and let me take another example where A is 0 1 and B is 1 and 2 certainly here A intersection B is not empty right. So, if you apply the definition of two sets being not disconnected so, this example 3 fails. So, in that sense that they have an non-empty intersection so, they cannot be separated right; so, they are connected.

Now, come back to example 1 and 2. What is the difference? Well, the difference is both have empty intersection; however, if you compute A closure intersection B right so, A closure is nothing, but here A itself. So, it is A intersection B and A intersection B is null set and A intersection B closure is 0 1, intersection 1 2 which is non-empty right. So, in that sense this A and B right are not So, that let us go back to the definition. So, are A and B disconnected? Well, for them to be disconnected right their intersections with the closures right should be empty.

And, that is not true in this case because you have a non-empty intersection with one of the one of the two cases right. So, A and B are not disconnected so that means, they are connected. So, here is an example. So, this example it is a connected set right. What is the connected set? A union B is a connected set, it is in 1 piece whereas, here if you do A closure intersection B right so, it is 0 1 intersection 2 3 which is empty. And so, also A intersection B closure which is 0 1 intersection 2 3 which is empty right.

So, it satisfies the definition of two sets being separated right; so, they are disconnected. So, this is an example which is not connected or disconnected or we also use the word separated right whereas, here obviously, they are not separated so, they are connected. Here things are little obvious because both have a common point 1, but in the example 1 things were not very obvious because, 1 is not common to both. Because, in one of them 1 is a is a boundary point whereas, in the other it is an interior point right.

But, by our notion of connected means that, one of the set should contain the closure of the other set and so also the other way right. So, we have the we have now more or less dealt with all the notions of sets right, we have few more notions remaining; we will continue it further.