**Nonlinear System Analysis**
**Prof. Madhu Belur**
**Department of Electrical Engineering**
**Indian Institute of Technology, Madras**

**Lecture – 27**
**Stability Notions: Lyapunov and LaSalle's theorem Part 04**

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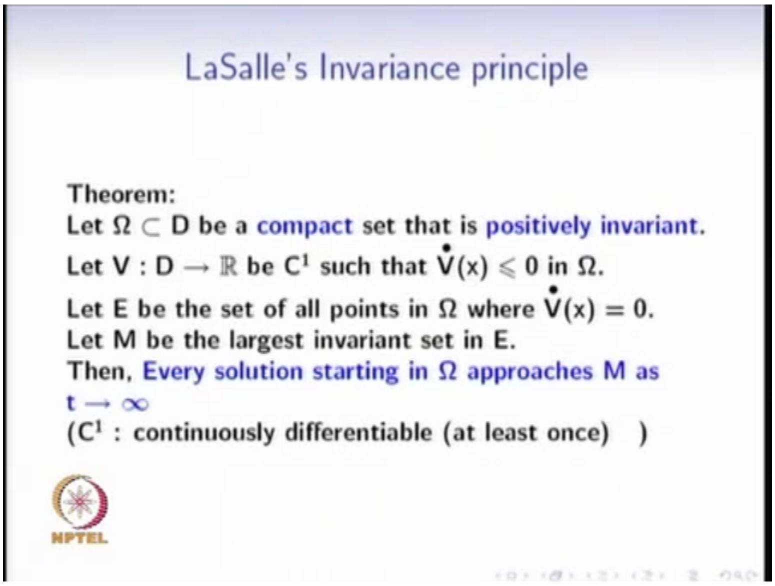


So we had started with LaSalle's invariance principle, the last lecture. So, let us just quickly review this. So, suppose omega is a compact set that is positively invariant and suppose we have found a function V that is C1; C1 means differentiable and the derivatives continues such that this V satisfies V dot is less than or equal to 0 on the set omega.

For this V, we will now find a set E such that V dot equal to 0 on the set E. And let M be the largest invariant set in E; largest invariant set in E means it is invariant under the dynamics of this dynamical system, it is contained in E and it is the largest such set. In other words, any other subset of E that has satisfies these properties is also contained in M. If these conditions are satisfied then, every solution starting in omega approaches this set M as t tends to infinity.

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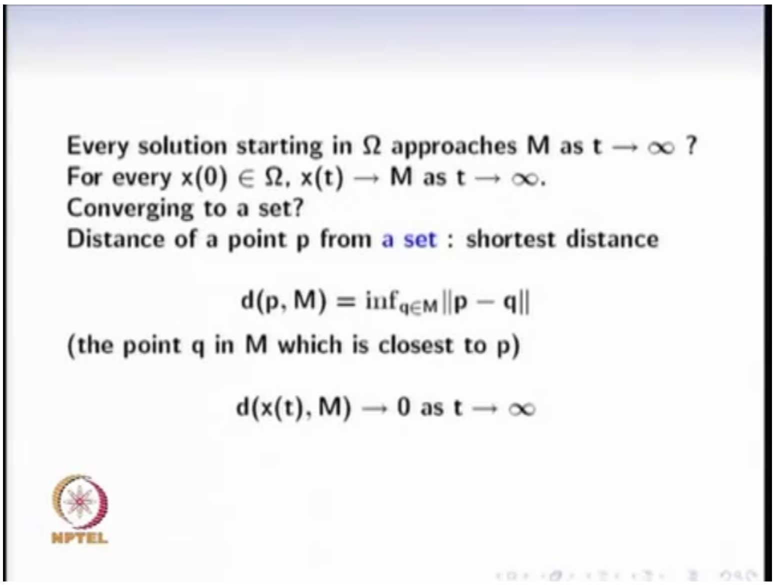


So, approach a set we had seen this as a definition. So, what the LaSalle's invariance principle says that, every solution starting in the set approaches M, in other words, for every initial condition that trajectory converges to M. What is the meaning of converging to a set?.

So the distance of a point p from a set is defined as the distance of p to different different points in M and the shortest such distance. So, this is a definition of distance of a point p from the set M. Now, as x evolves as a function of time, x of t is a point, and we look at the distance of x of t from the set M, and this distance decreases as t tends to infinity. That is the statement of the LaSalle's invariance principle.

So, we had already encountered the situation of the pendulum example in which, the natural energy function to take, did not satisfy strictly less than 0. So, let us do this example again.

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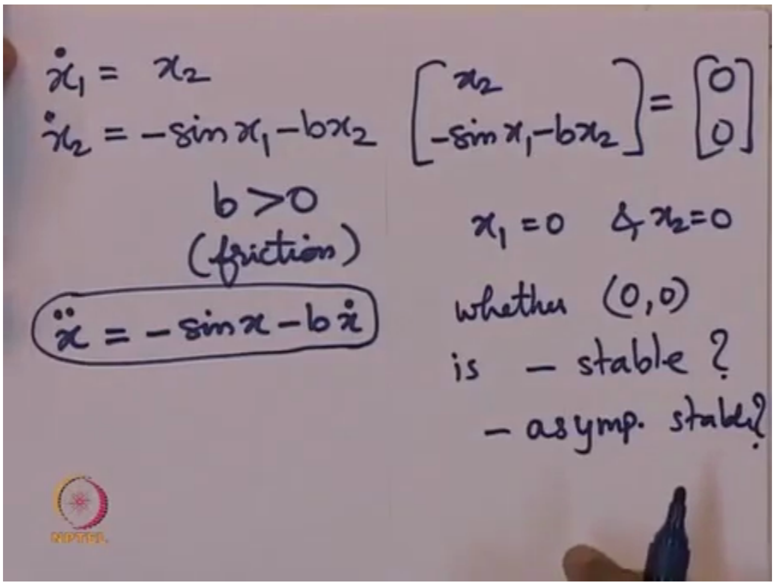


So we this is what we call as friction, this is the situation of a pendulum. The original differential equation was of second order which was equal to like this. This differential equation of second order be converted to a first order differential equation to first order differential equations by introducing x 2.

So, x 1 is the same as x and x 2 is a derivative of x 1 and these 2 first order differential equations we will now study the equilibrium point and the stability properties of the equilibrium point.

So for this dynamical system, we will now see x 2 and minus sin x 1 minus b x 2. This equal to 0 0, this gives us x 1 equal to 0 and x 2 equal to 0 as one of the equilibrium points. Of course, we could have x 1 equal to pi also that corresponds to the pendulum standing upwards which we know is unstable, we can also obtain that as a conclusion by linearizing about that point and checking that the eigen values are at least one of them is in the open right of plane.

That we will keep as an exercise, what we will now check is whether this equilibrium point whether 0 comma 0 is stable; asymptotically stable. This is what we will check now. For this purpose, we will take the Lyapunov function coming as the energy of the system.

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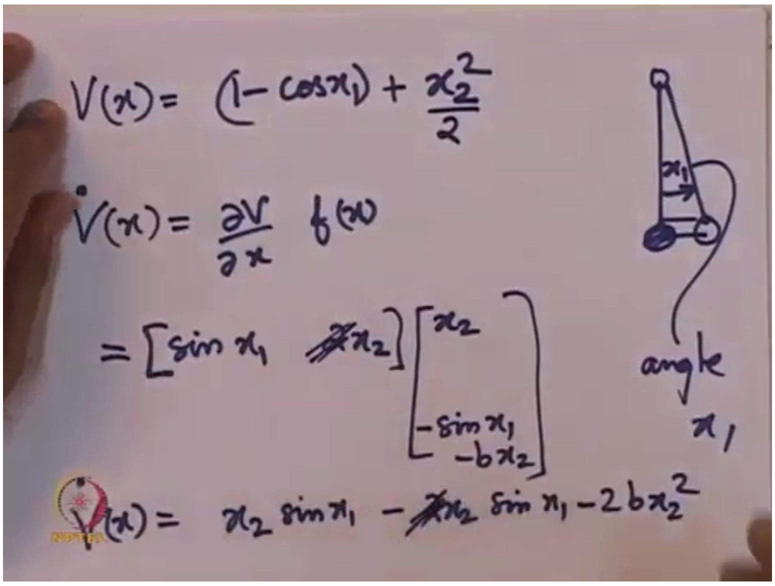


So, take V of x equal to 1 minus cos of x 1 why, because this is the potential energy. So, what was our x 1 variable? This is our pendulum when it undergoes a deviation of angle x 1.

So, this is the angle that time how much does it get raised? The amount by which it gets raised is the potential energy accumulated into the system and that turns out to be 1 minus cos x 1; of course, multiplied by the mass and the gravitational acceleration g. But, we have a considered a model where those parameters are not arising. This can be considered as normalization of the equations or as normalization of the mass 2, 1.

There is only the potential energy, the other energy term is actually x 2 square by 2. Let us check what happens if we take just x 2 square. So, this is not really energy because this is not kinetic energy second term is not kinetic energy, but twice the kinetic energy.

So, let us check what happens to V dot of x. So, this turns out to be del V by del x times f of x. And then we evaluate this, del V by del x is a row vector in which the first component here is a derivative of this with respect to x 1, which is exactly sin of x 1 and the derivative of this with respect to x 2, that is 2 x 2 times f of x. What was the f of x?.

The first component was x 2, second component was minus sin x 1 minus b x 2 and when we multiply this product, those are like inner product we get x 2 sin x 1 minus 2 times x 2 sin x 1 minus 2 b x 2 square. So, this is what we get as V dot of x. So, is this quantity positive or negative? That is the next thing we will investigate.

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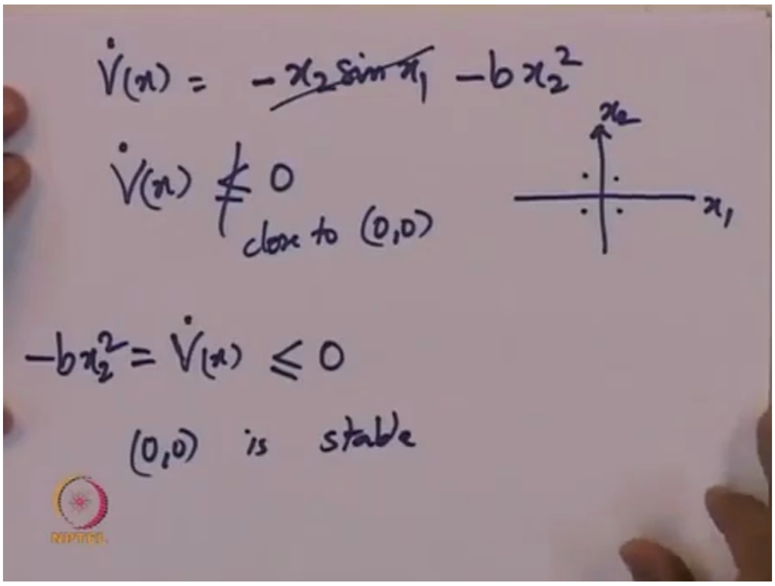


So, we got V dot of x was equal to this term is well behaved, but the other term is this is our x 1, this is our x 2. So, one term of course, does not change sign its always negative or equal to 0, but the other term x 1 and x 2 sin of x 1 has a same sign as x 1 close to x 1 equal to 0, but x 2 times this can change its sign depending on which quadrant it is. And hence, for small values of x 1 x 2; in other words, close to the origin we are not able to say that V dot of x is less than or equal to 0.

This is not satisfied close to 0 comma 0. This one can check oneself. In order to check oneself, one could first ignore this particular term. Why we can ignore this term? Because this is it to put b equal to 0 just means that we have a pendulum without friction and for the pendulum without friction, we know that the system is stable.

By intuition, we want to obtain that as a conclusion from the Lyapunov's theorem of stability and for that purpose, this particular quantity certainly changes sign , it will have different signs depending on x 1, x 2 on in on each of these 4 quadrants and hence this V dot does not satisfy less than or equal to 0. So, this is not a valid Lyapunov function, why? Because it is a Lyapunov candidate. It is positive definite, but it is not decreasing it is not non-increasing around the origin.

So, let us go back to our Lyapunov function and make a small change here. We will now divide this by 2. This perhaps we have already verified once. So, this now is indeed the kinetic energy. So by doing this, we do not get this 2 term here and we remove this also here because of which this term now cancels out and we have V dot of x. Now it is indeed less than or equal to 0 why? Because V dot of x was equal to minus b x 2 square.

So, this at least proves that 0 comma 0 is stable; however, this had not helped us to prove that the origin is asymptotically stable even though, we by intuition we know that the equilibrium point is in fact, asymptotically stable because we have friction which continuously dissipates off the energy.

So, how do we obtain that? This particular function Lyapunov candidate does not help us. However, we can use LaSalle's invariance principle for the same Lyapunov function. So, construct the set E set of all x such that V dot of x was equal to 0. In other words, minus b x 2 square equal to 0, it gives us set of all points where x 2 is equal to 0 . So, this x 2 equal to 0 is nothing but the x 1 axis. The x 1 axis is the set of all points where rate of change of the Lyapunov function is equal to 0.

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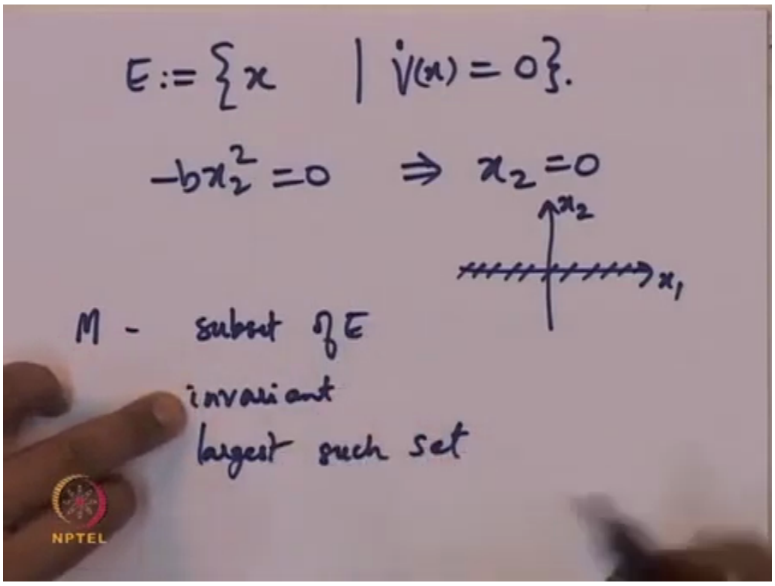


So this is our set E, now we want to look at the set M which is subset of E invariant under the dynamics of the system and largest such set largest such set. Largest such set which satisfies that it is a subset of E and it is invariant under the dynamics of the system.

So, how will you find the largest such set? We will look for what values of x 1 and x 2 are subset of E and are also invariant. When we try to do this, we will automatically get the set of all x 1 x 2 points that are invariant and containing E and hence, it will be the largest such set.

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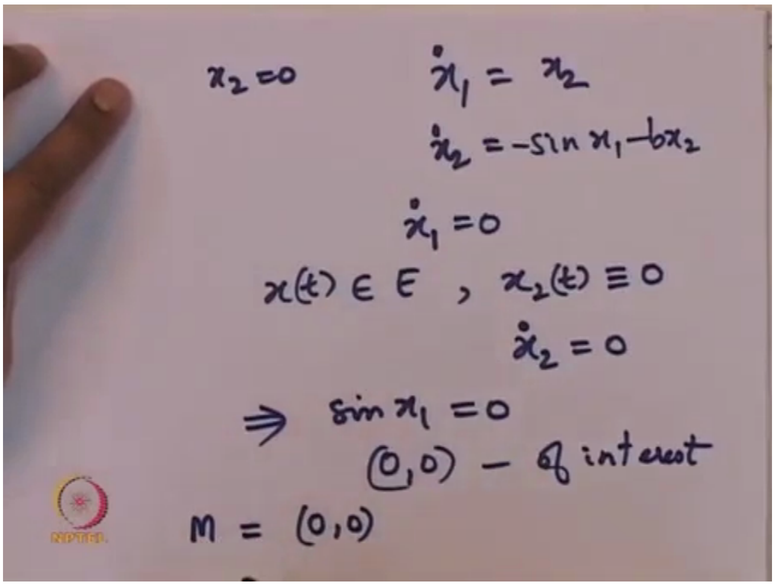


So, x 2 equal to 0 is requirement that the set M is contained in E. Now, we will put this in x 1 dot equal to x 2 and x 2 dot equal to minus sin x 1 minus b x 2. When we put x 2 equal to 0 we get x 1 dot equal to 0, we also put x 2.

So, x of t is contained in set E which means x 2 of t is equal to 0, always is uniformly equal to 0, equivalently equal to 0, identically equal to 0. These are the different ways we interpret this symbol, this equation. So, if some particular value of x has a function of time is always equal to 0, it is like a constant function which automatically means that x 2 dot is also equal to 0, identically.

So, when we put that x 2 dot equal to 0 then we also get sin of x 1 equal to 0 and this implies that sin of x 1 equal to 0 which of course, we know happens at either the vertically down position which is x 1 equal to 0 or the vertically up position which is x 1 equal to pi. Since we are interested about the stability properties of the position of the point 0 comma 0, we get if this is of interest. Since we are interested at this point, we get that M is just the point 0 comma 0.

When we ask the question inside the set E which are all those points which are invariant under the dynamics of f, the time we took the set E, which means you put the equation x 2 equal to 0 and we studied invariance by invariance for invariance we put the fact that x 2 is always equal to 0 its called which means x 2 dot also equal to 0, when we substituted that back here, we got sin of x 1 also equal to 0.

This term was already 0, this term we now got equal to 0, because of which we obtained the sin of x 1 equal to 0 and which means that we this can happen at the point pi comma 0 also. The first component is equal to 0 that is a 1 of interest which is vertically down position.

So, we have obtained that the set of all invariance points that satisfies the property that is invariant and subset of E gives us only this point and this is the largest such set, any other set would not satisfy the equations. We looked at looked for all the points that satisfy the equations and got this point.

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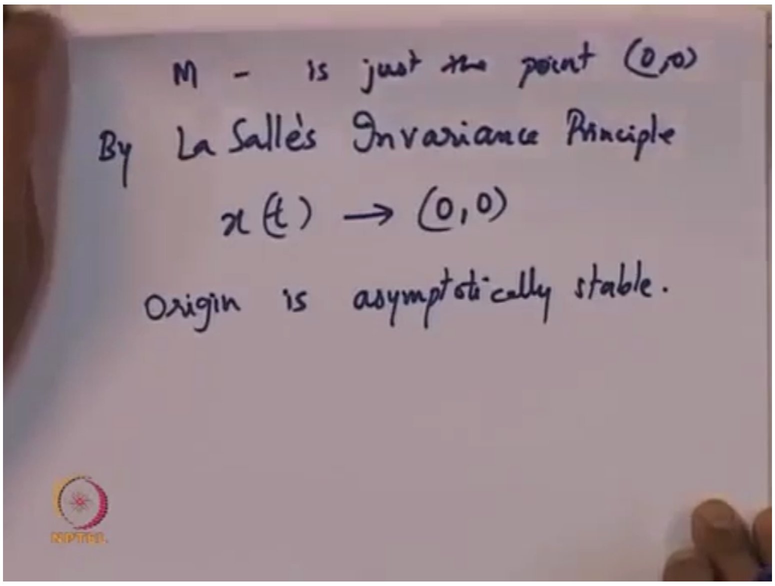


So in other words, we have obtained that M is just the set, just the point 0 comma 0. So, what is the meaning? So, by so by LaSalle's invariance principle by LaSalle's invariance principle x of t converges to the set M which is just 0 comma 0. So, this in other words proves that origin is not just stable, which we already concluded from the Lyapunov's theorem of stability. But we in fact, got that the origin is asymptomatically stable. So, this concludes proof of the statement that the origin is asymptotically stable; by using what principle?.

Not by using Lyapunov's theorem of asymptotic stability, but by using LaSalle's invariance principle which we used to conclude that the set M has just 1 point the origin. And by LaSalle's invariance principle the trajectory is x of t converge to M and hence, the origin is asymptotically stable. Now we will investigate whether the linearized system at this point is also asymptotically stable.

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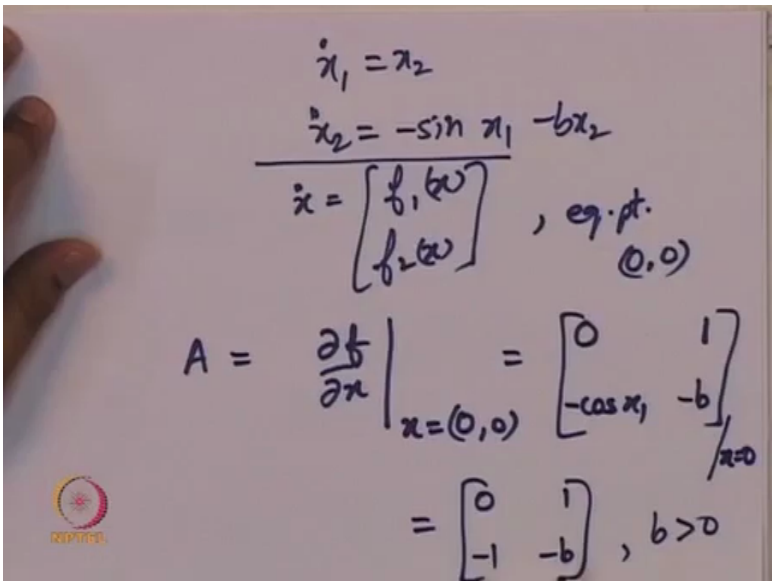

So consider again, x 1 dot equal to x 2 and x 2 dot equal to minus sin x 1 minus b x 2. So, x dot is equal to this can be written as f 1 of x and f 2 of x as a vector. So, what is the linearization? We have already checked that the equilibrium point of interest is 0 comma 0, we have checked that this is an equilibrium point. Now what is this linearized system? It is this particular matrix evaluated at x equal to 0 comma 0. So, let us find what this matrix is.

The term that comes here is the derivative of f 1 with respect to x 1. So, in f 1 which is this equation x 1 does not come at all. In other words, the derivative of f 1 with respect to x 1 is 0. What is the derivative of f 1 with respect to x 2? That is the term that comes here that is precisely equal to 1 why? Because f 1 of x is equal to x 2. So, derivative of f 1 of x with respect to x 2 is equal to 1. What is the derivative of f 2 with respect to x 1?. So, where all does x 1 appear in this equation.

It appears only here. In other words, derivative of minus sin x 1 with respect to x 1 that is minus cos x 1 what is the derivative of this term with respect to x 2? X 2 does not come here, it come it appears only here and so we put minus b here. This as expected is a matrix, but it depends on x 1 and x 2, it depends only on x 2 in this case. So, we are now required to evaluate this at x equal to 0 at the origin.

So, which means that in the in the first 2 entries x 1 x 2 do not appear, it appears only here when you put x 1 equal to 0 and we get minus 1. And of course, b is greater than 0. So, let us check how the eigen values of this matrix look.

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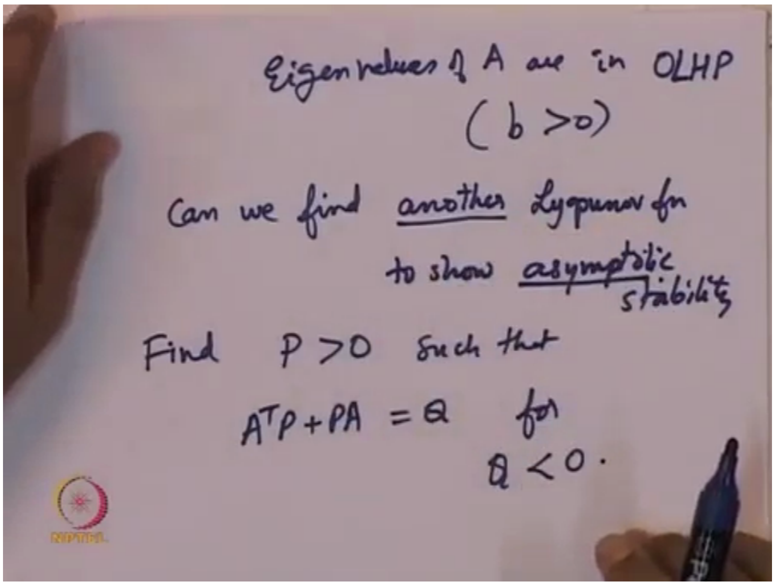


So, upon checking one can do the calculations and check that eigen values of A are in the open left half complex plane. One can check that by using the fact that b is greater than 0, the eigen values of that particular matrix we wrote are both in the open left half plane, which means that the origin is asymptotically stable.

And for a linearized system if A is Hurwitz, if the origin of the linearized system has all eigen values in the open left half plane; then we know that the non-linear systems equilibrium point is also asymptotically stable.

However, that Lyapunov function could not help us with that. So, can we find another Lyapunov function? After all, the Lyapunov theorem was only a sufficient condition for stability and asymptotic stability. Since, we already know that the equilibrium point is asymptotically stable; can you find another Lyapunov function? To prove, to show asymptotic stability.

The energy function already helped us to prove stability, but we want to prove asymptotic stability. So, we will consider finding a Lyapunov function for a linearized system. In other words, find P greater than 0 such that A transpose P plus P A is equal to Q; for Q a negative definite matrix.

This is a problem that we will solve now. Why? Because this particular Lyapunov function for the linearized system will also help as Lyapunov function for the non-linear system. So, we can in fact, choose for linear systems because A is Hurwitz, for any Q we will be able to find such a P.

So, take Q equal to minus 1 0 0 minus 1. In other words, this Q will correspond to our V dot of x. So, the corresponding V dot of x will turn out to be equal to minus x 1 square minus x 2 square. Why? Because V dot of x is nothing but x transpose Q x for linear systems. So, when we put this particular Q we will get precisely this, and this we know is negative definite, it is strictly less than 0 for all x 1 x 2 except of course, x 1 equal to 0 and x 2 equal to 0.

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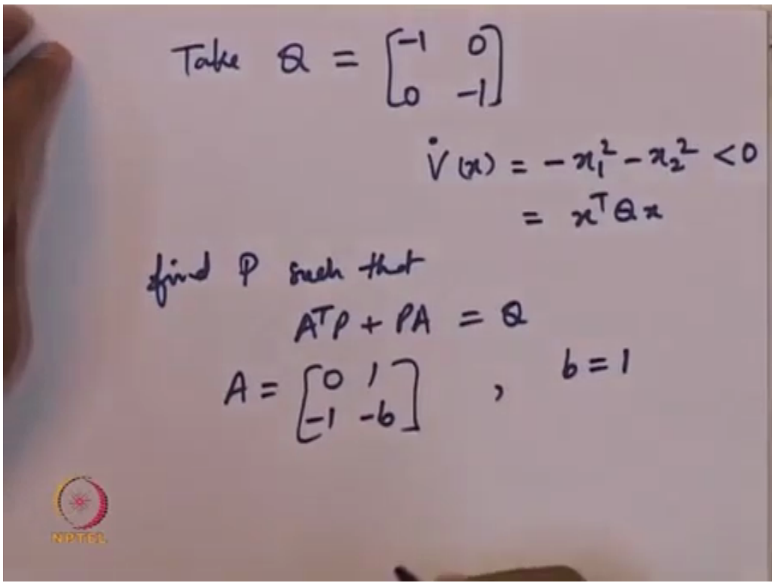


So, for this particular Q we will now look for a P such that, find P such that A transpose P plus P A is equal to this Q because, that particular A is Hurwitz, the P that we will obtain from this equation will or will turn out to be positive definite matrix.

This is a equation that you will solve now. So, notice that A was equal to 0 1 minus 1 minus b. For the purpose of solving, we could take b equal to 1. This is the rate at which energy decreases due to friction and this is required to be positive. So, we have taken b equal to 1. What do we get by solving for P?

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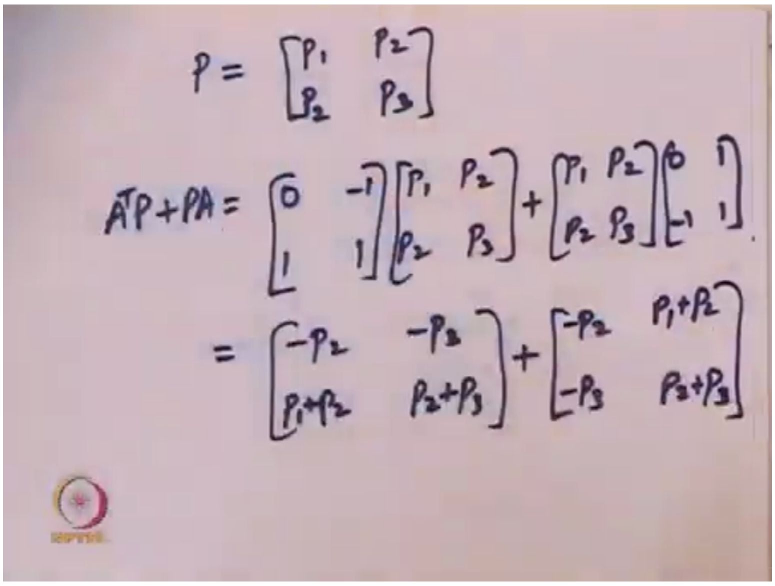


We can assume P has these entries. P 1 P 2 P is a symmetric matrix hence, this entry is also equal to P 2 and P 3. So, when we do A transpose P plus P A, that time we get this to be equal to this is A transpose, what is written here is a transpose hence, P 1 P 2 P 2 P 3 plus the same matrix P 1 P 2 P 2 P 3 times A which was equal to 0 1 minus 1 1.

So, we will now evaluate this is equal to minus P 2 minus P 3, this is P 1 plus P 2 and here we have P 2 plus P 3. This is minus P 2 minus P 3 and P 1 plus P 2 and P 2 plus P 3.

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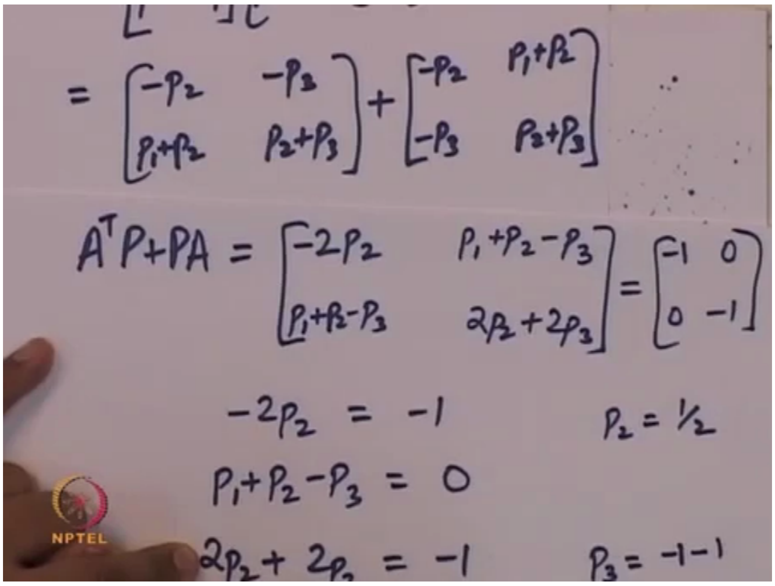


So now, we will equate this to Q and while doing that; so, we can add these 2 matrices to get finally, A transpose P plus P A is equal to minus 2 P 2 P 1 plus P 2 minus P 3. Here, we get the same thing P 1 plus P 2 minus P 3, and here we get 2 P 2 plus 2 P 3. So, since P was symmetric we have got this particular matrix to be symmetric and that is the reason that we should be choosing Q also to be symmetric and we have chosen Q to be equal to.

Let us find values P 1 P 2 P 3, a particular theorem we already saw claims that this system of equations is solvable. So, there are only 3 entries to 3 equations 1, 2 and 3. Why? Because this entry equal to this is the same equation as this entry equal to 0. So, let us put minus 2 P 2 equal to minus 1, P 1 plus P 2 minus P 3 is equal to 0. And finally, 2 P 2 plus 2 P 3 equal to minus 1.

So, the first equation just tells that P 2 is equal to 1 by 2 which when we substitute in the last equation we get P 3 was equal to minus 1 minus 2 P 2, minus 2 times P 2 is nothing, but minus 1 again which gives us P 3 equal to. So, we have taken A equal to 0 1, let us go back to this equation; we have got A equal to 0 1 minus 1 minus b and we are put b equal to 1.

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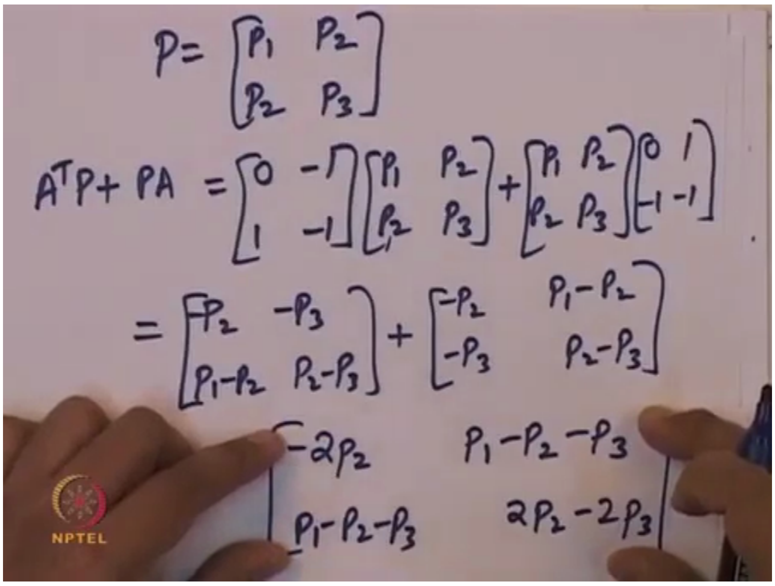


And let us now take P equal to P 1 P 2 P 2 P 3, we have taken P to be symmetric, that is why, we have taken a same entry here. So, let us solve for A transpose P plus P A this gives us for A transpose, we will write 0 minus 1, 1 minus 1 times P 1 P 2 P 2 P 3 plus the same matrix P 1 P 2 P 2 P 3 times A which is equal to 0 1 minus 1 minus 1 and then, we solve this.

So, this when we do, we get minus P 2 minus P 3 P 1 minus P 2 and P 2 minus P 3 plus this particular matrix product when we evaluate, we get minus P 2 minus P 3 P 1 minus P 2, P 2 minus P 3 and when we add these 2 matrices we get minus 2 P 2 P 1 minus P 2 minus P 3, P 1 minus P 2 minus P 3 and finally, 2 P 2 minus 2 P 3.

So, this matrix we finally got is nothing but A transpose P plus P times A. Now, we will equate this to Q. So, we had already taken. So, notice that this matrix is symmetric because P we have because we have taken P to be symmetric, this matrix has been obtained to be symmetric and hence, it is important that this matrix be equal to a Q which also should be assumed to be symmetric.

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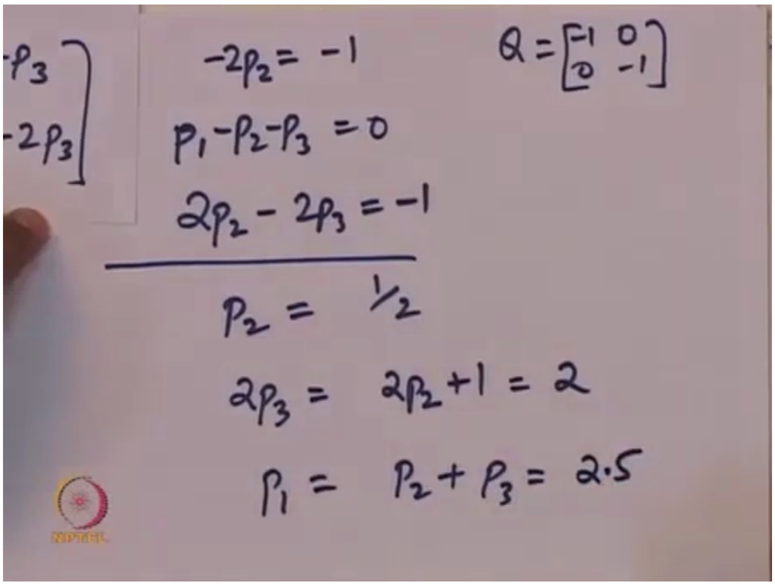


So, we have taken Q to be equal to minus 1 0 0 minus 1. So, when we equate this matrix to Q then we have it appears like 4 equations, this entry equal to minus 1, this entry equal to 0, this entry equal to 0 is again the same equation. So, it is not really 4 equations, but 3. What is the last equation? This entry equal to minus 1. So, these 4 equation now we will write here. So, we have minus 2 P 2 equal to minus 1 P 1 minus P 2 minus P 3 equal to 0 and 2 P 2 minus 2 P 3 equal to minus 1.

So, the first equation gives us 1 by 2 which, when we substitute into the last equation, we get 2 P 3 is equal to 2 P 2, plus 1 which was equal to 2, when we put P 2 equal to half, here we get 2. And these P 1 P 2 and P 3 when we substitute into the second equation, we get P 1 equal to P 2 plus P 3 which was equal to 2.5. .

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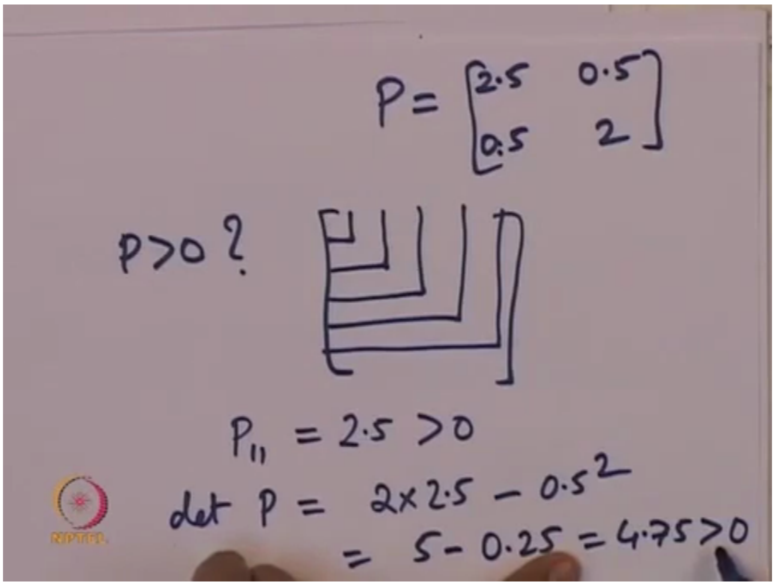


So, what is our matrix P? As a result of this, the matrix P was equal to P 1 P 2 P 2 P 3. So, P 1 has been obtained to be equal to 2.5, P 2 was equal to 0.5, which is the same entry here and finally, P 3 which was equal to 2. So, the claim is that this matrix P that we have obtained is positive definite because, A was Hurwitz and Q was negative definite.

We can check this. So, P greater than 0, how will you check this? One way to check that a matrix is positive definite is that all the principal minors, all the leading principal minors. So, for a square symmetric matrix this is a 1 by 1 minor, it is a principle minor because it is has a symmetric rows and columns taken to construct that submatrix and only the leading ones.

So, we take only this now, and every such matrix we take and look at the determinant and each of these determinants should be a positive number, that is a necessary and sufficient condition for this matrix P to be a positive definite matrix. So, let us do the check for this.

Here, we have to take only 2 determinants the first 1 by 1 determinant is nothing but 2.5, P 1 1, the first submatrix is equal to 2.5 that is greater than 0. What about the determinant of the whole matrix? The next leading principle minor is nothing but the whole matrix P. The determinant of the whole matrix is 2 into 2.5 minus 0.5 square, which is equal to 5 minus 0.25, it is equal to 4.75, that is positive.

So, because both the first leading 1 by 1 minor and the second leading 2 by 2 minor are both positive determinant, this means that the matrix P is positive definite. So, if we had taken the Lyapunov function coming from this P for the linearized system.

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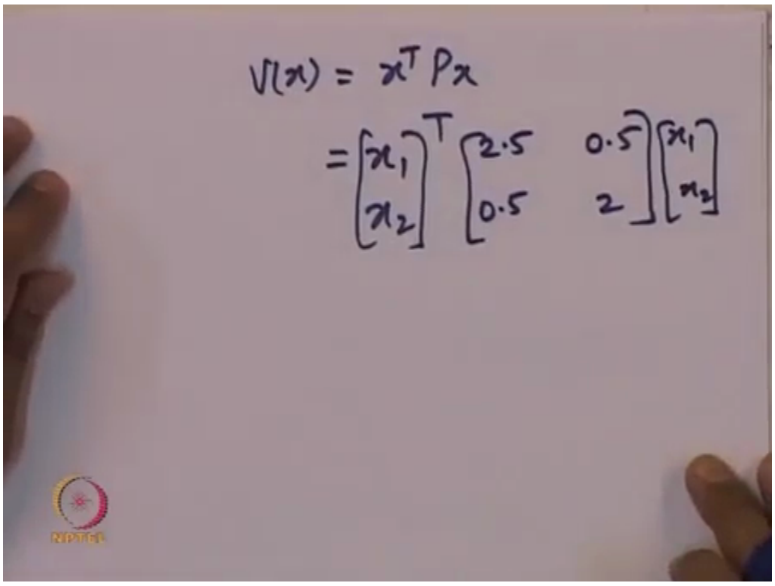

So, what is the Lyapunov function? It is x transpose P x in which P was in which the matrix P was the one that we just now obtained. If you take this as the Lyapunov function, and the origin turns out to be asymptotically stable by Lyapunov's theorem of asymptotic stability.

And the same Lyapunov function will also help us improving asymptotic stability of the non-linear systems equilibrium point which happens to be the origin again, but if we had started with this Lyapunov function, then we would not have required LaSalle's invariance principle, because a Lyapunov function theorem itself would have claimed, stated that the equilibrium point is asymptotically stable.

Unlike the energy function, unlike the physical energy that we had taken, which helped us to prove only stability. By Lyapunov's theorem of stability. So, this completes Lyapunov analysis, we have seen some solved examples also. We can have another set of problems which we will use as exercises. We will now move on to the next topic which is about periodic orbits. Why? Because periodic orbits are an important part in the context of building oscillators.

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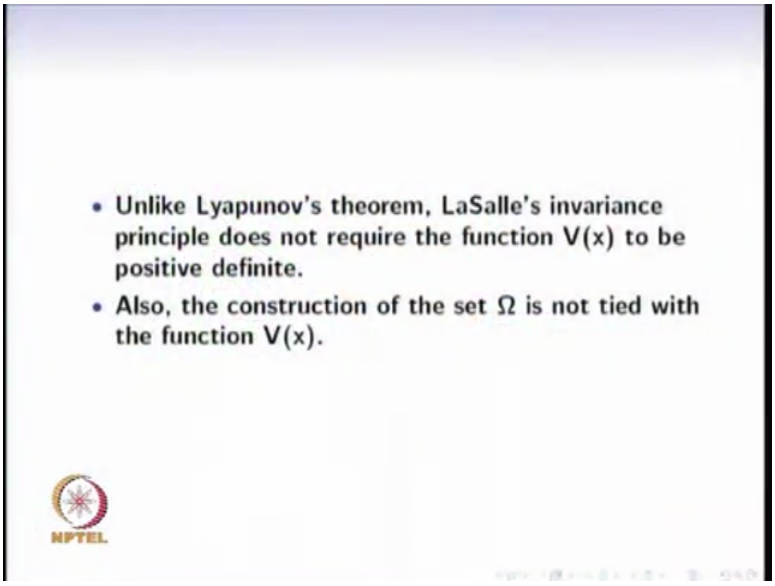


So, before we move to that topic there is one other slide that we had skipped, so, the LaSalle's invariance principle which we saw in detail and also saw in example is different from Lyapunov's theorem in 2 ways. The first way is, unlike the Lyapunov theorem, the LaSalle's invariance principle does not require the function V to be positive definite.

Notice that we did not assume that V was positive definite. Second, the positive invariant set that we had constructed in the proof of the Lyapunov theorem that set omega was constructed using the Lyapunov function V, here we are assuming that we already have a positive invariant set.

In fact, it is that is the reason that we are not assuming V as positive definite; because, on a compact set omega we are V always achieves its minimum and we can subtract that minimum from the function V here, by which we can always obtain another function V that indeed is positive definite.

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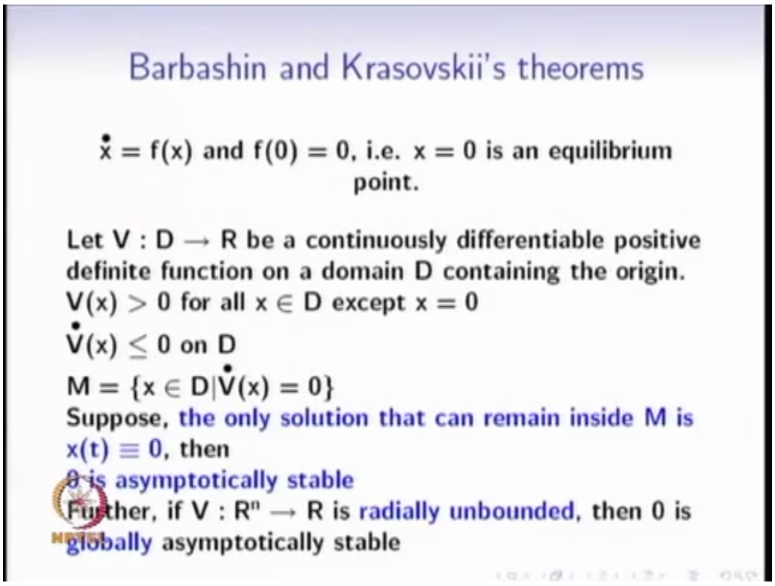


We will also see an application of LaSalle's invariance principle; so, there are well-known results that are, that turn out to be a special case of the LaSalle's invariance principle. So, one of them is Barbashin-Krasovskii's theorems, what is the statement of the theorem?.

So, suppose x dot is equal to f of x is a system, in which x can have many components, x is an element of RN and suppose the origin is an equilibrium point. Suppose, there exists a function

V from a domain D to R which is continuously differentiable, and suppose V is positive definite function; in other words, V of x is greater than 0 for all x except origin x equal to 0.

And also V satisfies that it is less than or equal to 0, on the domain D. Construct the set M that is made up of all the points where V dot is equal to 0. Suppose, this particular M has a property that the only solution that can remain inside M is x t identically equal to 0, then the origin is asymptotically stable.

Notice that this is precisely the situation that had happened for the pendulum example with friction. So, the Barbashin-Krasovskii's theorem is a more general statement to this effect. What can we speak say about global asymptotic stability of that equilibrium point we just now claim to be asymptomatically stable?.

If V is radially unbounded further, in addition to the above assumptions, if V is also radially unbounded, then the origin is in fact, globally asymptotically stable. So, this particular theorem we already saw for the case of a pendulum, as well as asymptotic stability is concerned. Of course, a pendulum example it is not globally asymptomatically stable simply because there are other equilibrium points.

However, the Barbashin-Krasovskii's theorem says that if V were radially unbounded, then the origin is in fact, globally asymptotically stable. One can check that the Lyapunov function V we had used for the case of the pendulum example with friction, that is not going to be radially unbounded. Otherwise, the origin there would have been globally asymptomatically stable account to this theorem.