Nonlinear System Analysis
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Lecture - 02
Math Preliminaries Part 01

Hello everyone. This is Arun Mahindrakar. I will be in this lecture you will do some we learn some tools from real analysis right. So, basically these are the mathematical preliminaries required to understand this course on Nonlinear Analysis. So, we will touch upon just the bare essential mathematical tools that are required to understand this course right.

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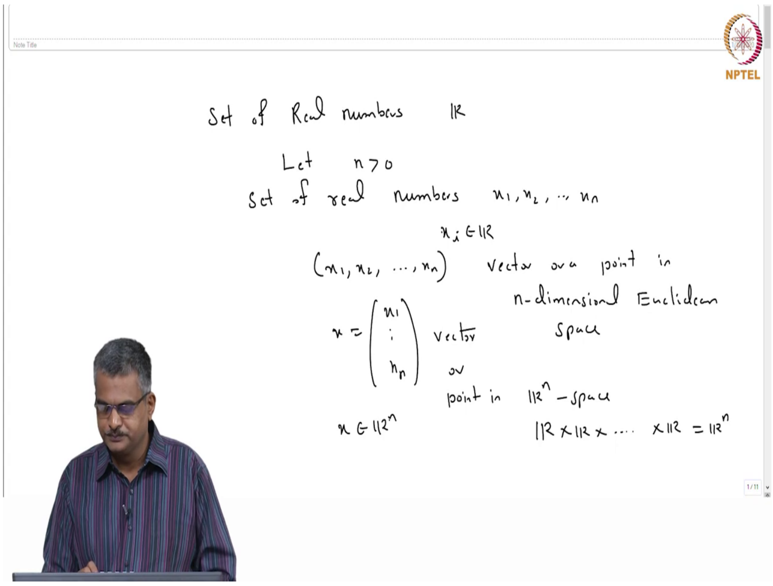

So, we will start with the real number right, set of real numbers. We will denote it by script R right. Now, let n be any integer right greater than 0 and now, what you do is you form a set of you take a set of real numbers; x 1, x 2 so on up to x n, where each xi is a real number right and then, you arrange them in an order right. x 1, x 2 so on up to x n.

So, what essentially, we have done is we have started with a real line right. We pick the real number; we picked n real numbers x 1 to xn and then, arrange them in a orderly fashion right and then, you are arrange this in this order right. We call this call it as a vector or a point right, in n dimension, n dimensional Euclidean space right. So, we have constructed a vector using real numbers right and in fact, the a the definition of Euclidean space follows by collecting all such points right; the collection of all such points or vectors is called a the Euclidean space right.

So, either we call x as a vector, we will denote it like a column vector right or we can say this is a point in R n space; the n dimensional Euclidean space. So, what is this R n space it is the cartesian product of n real spaces. We will denote it by n dimensional Euclidean space ok. So, we have a vector now right in an n dimensional space. This was the idea given by Riemann, who until Riemann Euclid, we had Euclid's postulates right, we have we had the work of this characters. So, all their work was confined to the plane; plane geometry, Riemann was first to think of imagining numbers beyond a two dimensional plane ok.

So, this is how you generalize a notion of a point or a vector in an n dimensional space. So, this is a real vector right. So, we will we will use the notation x belongs to R n to say that x is a vector in an n dimensional space ok. Now, once we have this vector, we can do some operations on this vector.

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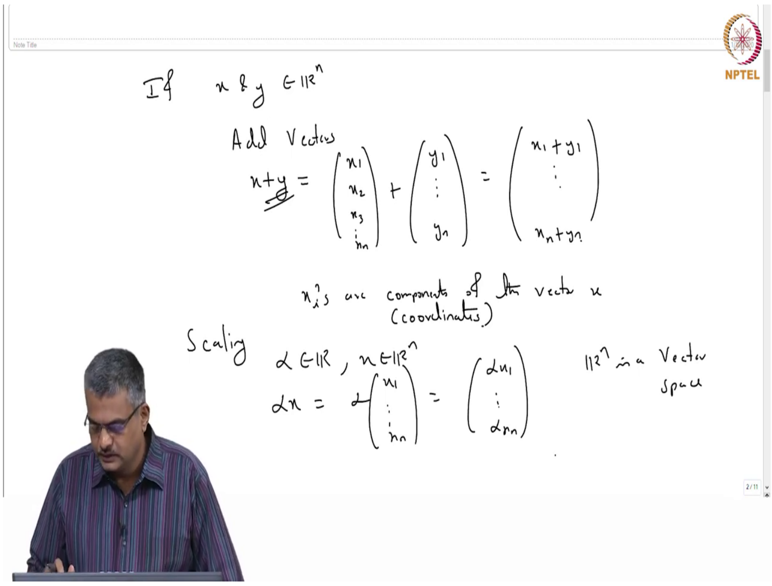

So, if x and y are vectors in R n right, then we can add this vectors, you can add vectors and how do you do that? So, this is what we want on the left hand side and the way you get is you do the you write down the components right correspondingly for the y s and then, you add them component wise.

So, this xi's are called the components are components of the vector x. Components or sometimes we also call them as coordinates right or coordinate xi is a coordinate of the vector x. So, you do a component wise addition right that is how this left hand side is defined. One can also do scaling right. So, alpha is a scalar belong to real number and x is a vector, you can do a scaling of the vector by scaling each of the components. This is how we do the how the scaling is defined right.

So, note that the R n space is a vector space right, to the you add vectors right, it is closed under the operation of addition. When you add vectors, the resulting vector is also a vector in the n dimensional Euclidean space. What other operations we can do? Well, we also need the notion of the origin.

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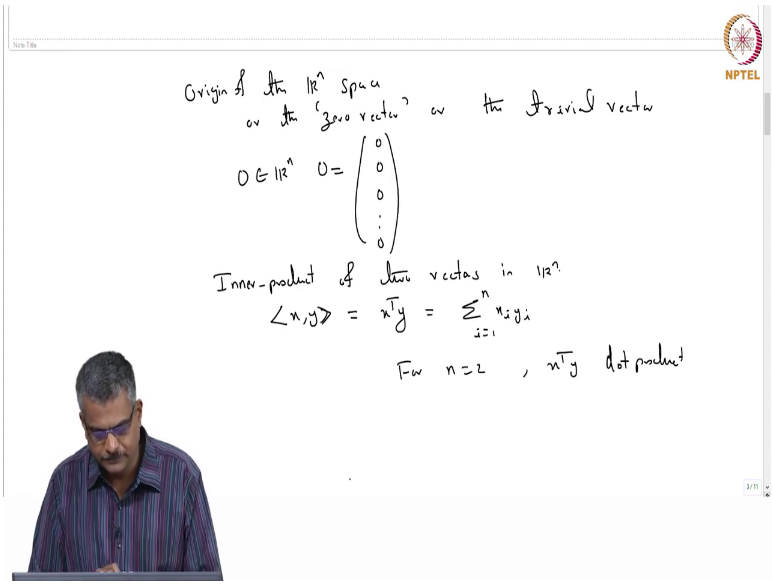

The origin of the R n space or the what is the zero vector, the trivial vector or the trivial vector right. So, the trivial vector 0 is nothing but the vectors, the vector whose all components are all 0 ok. Now, as I said we can do some more operations using this vector and we can think of what we are already familiar with and that is the motion of the dot product. So, one can extend the notion of dot product on a n dimensional Euclidean space. So, we use a special name for that. We call it as inner product of two vectors. So, the rotation is angular brackets.

So, this is a this would be read as inner product of x and y is nothing but x transpose y or in terms of components, it is the summation i equal to 1 to n xi y i. So, for n equal to 2, you have this is the usual dot product or for n equal to 3 you have the usual dot product that you we are already familiar with right. But this is a it captures the notion of inner of dot product in an n dimensional space.

Now, once you have the notion of inner product right and also you have the we have defined the notion of the zero vector, we can talk about length of the vector.

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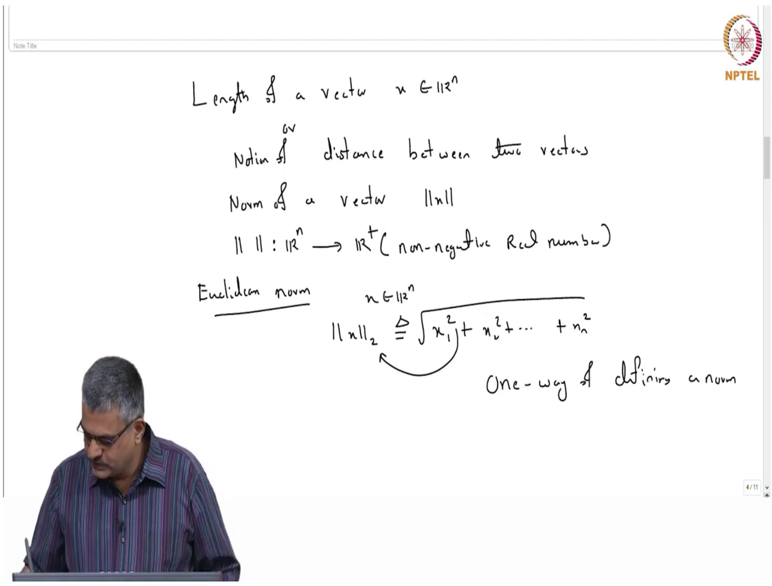

We want to talk length of a vector, x in order or we want to capture the notion of distance between two vectors right. So, both these notions are the notion of length of a vector or the distance between two vectors or distance between two points right is captured by the mathematical term called norm of a vector right.

What is this? The norm of a vector is denoted by the vector with two parallel lines right. What does it do? So, it is a mapping right. So, we denote the norm of vector by norm of x and what is this norm do to the vector? It takes a vector in R n and gives you a real number. In fact, it gives you a non negative real number because it does not make any sense to talk about distance being negative right. So, we will denote it by the symbol R plus for the collection the set of all non negative real numbers right.

So, we are familiar with the some properties of a distance right. So, let us see whether they hold for this object that we are defined called the norm of a vector right. So, before I list out the properties of the norm of vector, let me give an example of what is known as an Euclidean norm. So, this is one way of defining a norm, Euclidean norm right ok. So, x is a vector in R n. So, the symbol used is you put a subscript 2 to say that you are in you are using an Euclidian norm. The two comes because of this square terms.

So, you take the components each of the components, you square them and add them up and then, take the square root sign right. So, this two comes from the taking the square of the components right. So, this is one way of defining one way of defining norm. So, it is called the Euclidean norm. So, this is a this is I; use this symbol to define an object right.

So, you can, one can define you one can define a norm right which may not necessarily look like this right. However, whatever norm we come up with it should satisfy certain properties, it should be consistent with the idea that the norm captures the notion of distance between two points or the length of a vector right.

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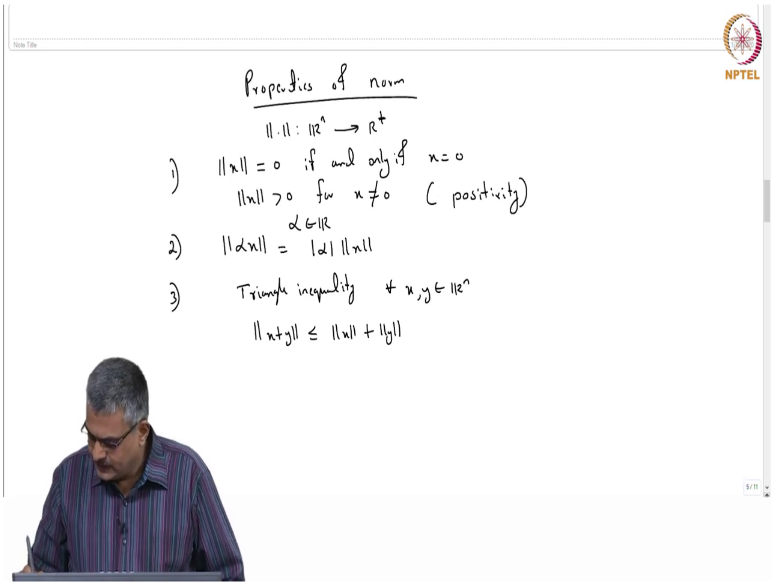

So, let us see the properties of the norm. So, this properties of the norm it applies to any norm right. So, far we have just defined the Euclidean norm, but shortly I will define some other types of norm. So, as we have seen the norm is a mapping from R n to R plus. So, the first properties right the positivity property; that means, the norm of x right is equal to 0, if and only if the vector is the trivial vector right.

So that means, if you show norm of some vector is 0; that means, the vector has to be identically equal to 0 and conversely, if you take a 0 vector, its norm is 0 ok. And norm of x is positive whenever x is not equal to 0 right.

So, this is what we understand the length of a vector is always a positive non negative quantity right. It is 0 only when the vector is a zero vector. So, this is known as positivity property.

The second property is about scaling; norm of alpha times x right, where alpha is a scalar is nothing but the absolute value of the scalar times the norm of x ok.

So, this is the scaling property. And the third property, which is the most important property is the triangle inequality. And what does it says? For every x and y in R n norm of x plus y is less than norm of x plus norm of y right. So, these are the three properties for this function to qualify as a norm.

So, as I said one can come up with your own definition of norm right, as long as it satisfies this three properties right. So, as an exercise you can check that the Euclidean norm which we defined earlier satisfies all these properties. So, as I said I will give more examples of norm right.

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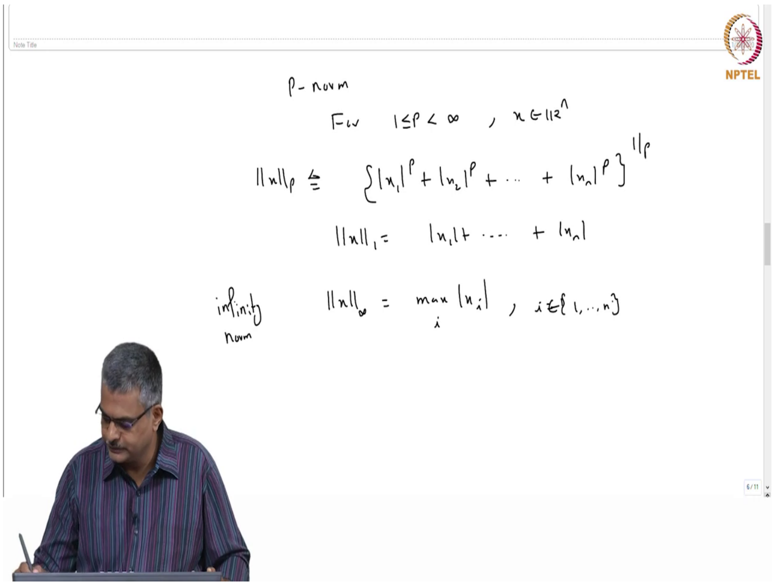

So, there is something known as the P norm which sort of generalizes the class of norms that one can talk and as a special case when P equal to 2, you have the Euclidean norm right. So, for P greater than or 1, but less than infinity you have the P norm. So, x is again a vector in R N is defined as; so, take the components, raise it, take the absolute value, raise it to the power P right so on for all the components and then, take pth root of the resulting expression right. So, this is known as the P norm.

So, one can talk about one norm. So, one norm becomes just the sum of the absolute values right. So, to norm you have already seen, we get what is known as the Euclidean norm. So, for P equal to infinity we defined; this is known as the infinity norm. So, it is defined as the max or i of the absolute value of xi ok, where i varies from one to n.

So, we have seen the notion of a norm of a vector right. We have seen the properties of the norm. So, when does a norm when does this function right that we have defined qualifies to be a norm and as I said one can come up with your own set of norms as long as it satisfies properties 1, 2, 3 ok. Now, we earlier define the notion of inner product right. So, let us see what is the relation between the inner product and the norm of a vector right; norm of two vectors.

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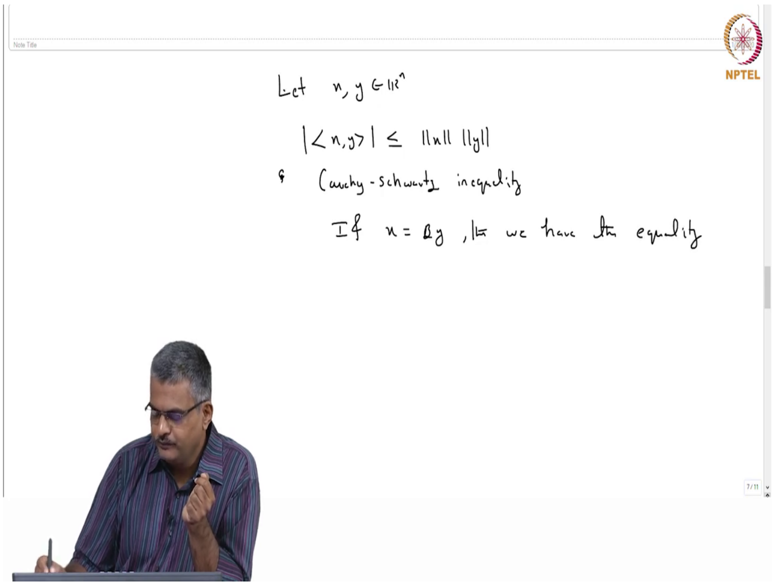

So, we have. So, let x, y belong to R n right. So, we have the inner product between x and y right. Now, this is a scalar. So, we can take the absolute value. When you take the absolute value it is related to the norm of x and y through the following inequality, that is the absolute value of the norm of the inner product of two vectors x and y is less than the norm of x times norm of y right. So, this is known as the Cauchy Schwarz Inequality, very useful equality which we will use when we do the stability analysis.

So, in this inequality becomes an equality if x and y are linearly dependent right. So, if x is some beta times y; then, you have the then we have the equality. So, we dealt with the notion of a vector in an Euclidean space and n dimensional Euclidean space right. We talked about adding a vectors scaling up the vector the notion of zero vector, then we do we also introduced the notion of inner product which captures the familiar notion of a dot product between two vectors.

We also saw the how to capture the length of a vector by the notion of norm or distance between two vectors right. So, we also saw the relation between the inner product and the norm through the Cauchy Schwarz inequality right. So, with this, we are now in a position to define topological concepts that we will require during this course.

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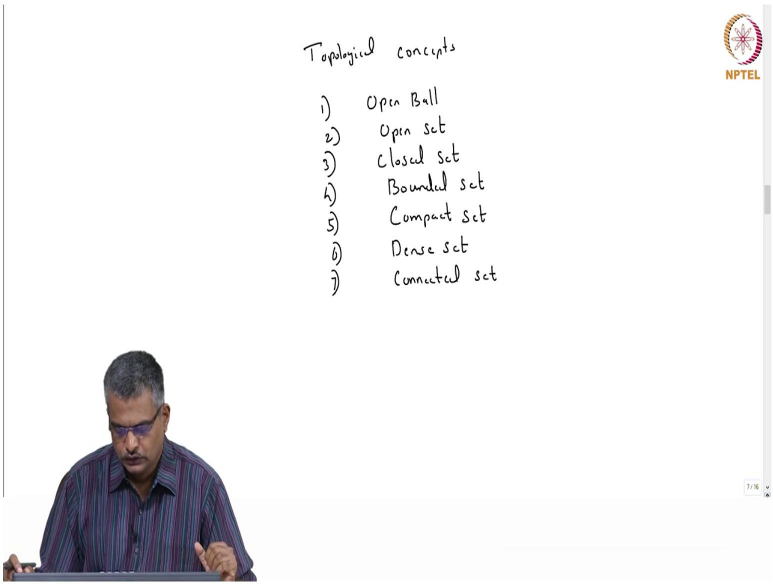

So, the notions that we need to understand are as follows right. So, these are topological concepts drawn from point set topology. So, to begin with, we need the notion of open ball which leads us to the definition of open set. So, once we have the notion of open set, we can talk about closed set and then, bounded set when is a set said to be bounded. This also leads us to the notion of compact set, very important notion in our analysis right; very powerful property. It has compact set, then we also need the notion of dense set and we will also need the notion of connected set ok.

So, these are building blocks in the new study topology point, set topology right and with the whatever notions that we have already studied, the norm of a vector and the inner product right, we should be able to define all these sets right. So, the objective is that we should know the definitions of this set; how the definitions are built? We should also be able to identify when is a set open set; when is it closed right; when is it compact right and so on. This is the objective of understanding the map prelim is behind this course right.

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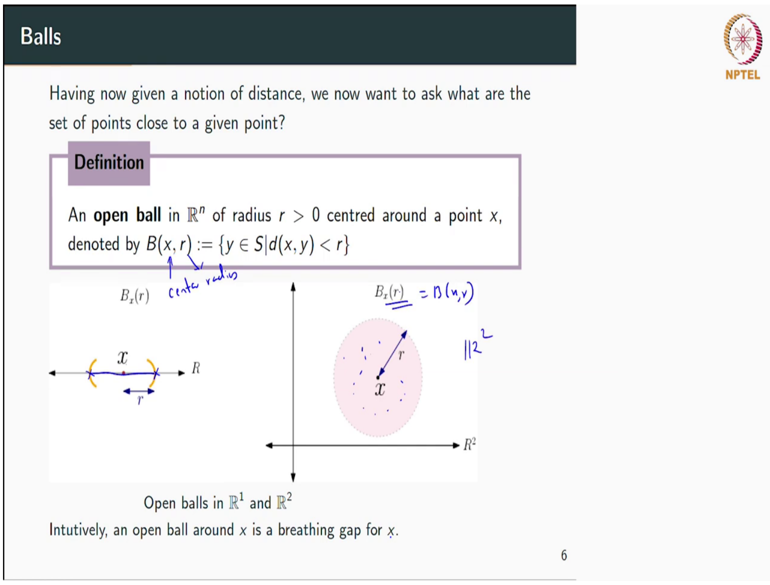

So, we will begin with the notion of open set. So, to do that we will require the notion of open ball right. So, the an open ball in R n right is a is again a set, it consists of its a ball of radius r strictly positive. So, and centered around a point x right and we will denote it by B x comma r.

So, x is the center of the ball and r is the radius of the ball right. So, it consists of all points in the R n space such that the distance between x and y is strictly less than r ok. So, you see that the points to be noted is so we talk about a ball of radius r centered at x right; x is some point given to us and we collect all points such that the distance between x and y is strictly less than r right. So, we call such a set the open ball with center x and radius r.

So, to picturize to visualize this in n equal to one dimensional right that is in the real line. So, you have a point x, you have a radius r. So, it consists of all points which are distance less than or equal to r. So, all these points on this line constitute the elements of the open ball right. So, here the open ball is denoted by this limiters right, the round brackets right. So, it is.

So, the the interpretation is that it consists of all points except the point, the endpoints right. So, this points are excluded from this concept. You can visualize this same open ball in r two right in two dimensional, where x is on the center and r is the radius right. So, this picture when we talk about ball is very clear to us right; but we should be able to extend this the idea of open ball in an n dimensional space.

For example, in n equal to r three you have the 3D ball right. So, here I have talked about this round brackets. So, here instead of the brackets, I have a dotted line. So, it consists of the n ball, the ball with center x and radius r right, it consists of all points right except the points which are marked dotted right. So, all these points belong to the ball of center x and radius r. Sometimes, I use the notation in this or this is the same as this notation right. So, once we know the notion of an open ball in an n dimensional space right, we can define the notion of a open set right.

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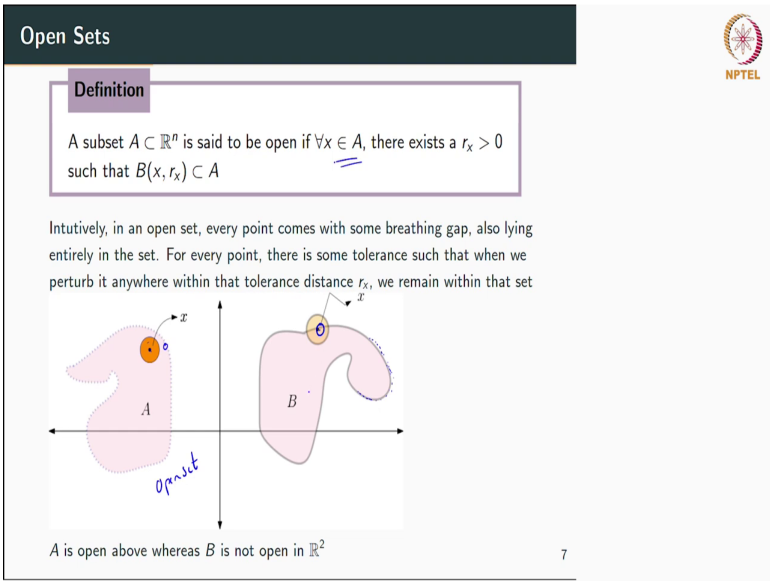

So, a subset A. So, somebody gives you a candidate set a set A right. So, a set A is said to be an open if for all x in A right, for every point in A, there exists a radius which could depend on that point x right such that the ball with center x and radius r right is contained in the strictly contained in the set A right.

So, let us understand this. For example, this is the set A right and the where if you see this dotted line it means those it means the set does not include those points right. So, is this set open. Well, let us take a candidate point we took a point x right and then, we are able to come up with the some radius right, centered around this point such that all of these points is contained in the set A right.

Now, you go very close to the dotted points, you can always find a small ball of arbitrarily small radius such that all its points right are contained in the set A. So, this set a is an open set.

Now, come to the picture on the right the set B, where the set also includes points this unlike the set A the all the boundary points are also include. Note that, there is a mathematical definition of boundary, we have not still introduced that. But we understand, we understand what is meant by the boundary through the picture right.

So, we will later on give a precise definition of a what is meant by a boundary of a set right. So, for the time being, we will take that these are the boundary points right and if you pick a point x on the boundary point, no matter how small the radius is right. If you can if you draw a ball around that right, I can take an even a smaller ball right not all points in that ball right are elements of the set B. So, the set B is not an open set right. So, to understand this notion of taking a point right and coming up with a ball all of whose points should belong to the set right is better captured through the notion of the interior point.

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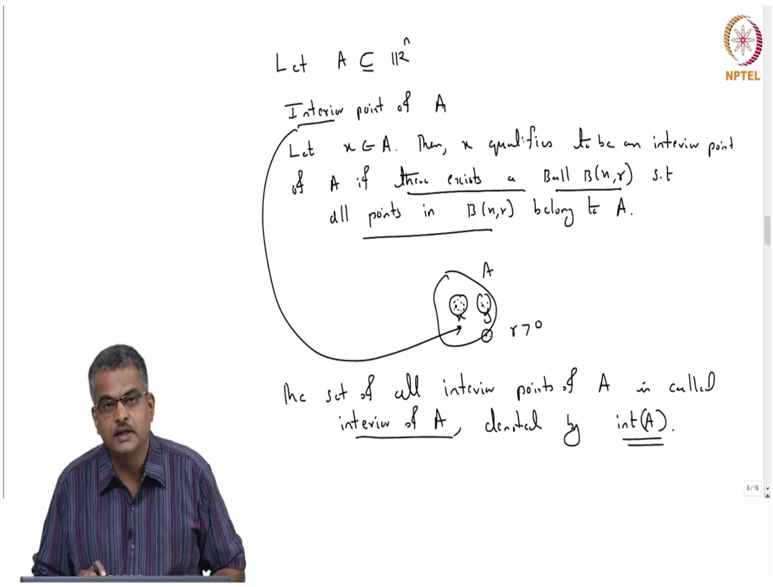

So, let us define what is meant by an interior point. So, we have a set let A be a subset of R n ok. For simplicity, when I am drawing pictures either n is 1 or 2 right, but the definitions holds for any dimensional space right. Now, I want to define the notion of interior point of A. Let x belong to A ok, then x qualifies to be an interior point of A, if there exists a ball. It's center x and some radius right; that means, if there exists radius r positive right. So, with that you build up this ball B x, r. So, there exists a ball B x, r such that all points in B x, r belong to the set A.

So, I have a set A ok. Suppose, I take a point x right and I can always find a ball of some radius all of whose points belong to the set A right. So, the challenge really lies in taking a point very close to the boundary right and then, can you draw can you find a radius r such that all of this points in this ball belong to the set A? And the answer is no right.

So, this word interior sort of captures that things will go bad right when you are on the boundary of the set. However, things are nice inside the set. Nice in the sense that you will be able to come up with the ball says that all points in this ball belong to A right. So, the set of all such points right constitutes the interior of a set right. So, the set of all interior points of A is called interior of A denoted by i n t and int A ok. So, here in this definition we took a point x, we saw that we could come up with the ball right all of whose points belong to the set A. You take another point y right, you will be able to come up with a ball all of whose points belong to the A right.

So, collect all those points which satisfy that condition and that collection of all that points is called the interior of A right. Now, how is this related to the open set, the definition of open set right. So, a word of caution. So, if you look at this definition, the key phrase is if there exists A right; that means, you have to come up with just 1 ball right such that all points in that ball belong to set A. Then, that point x qualifies to be an interior point right. It is not required that for every ball around that point, this property holds right. So, we will see how we one has to be careful with this phrases right in interpreting the definitions.

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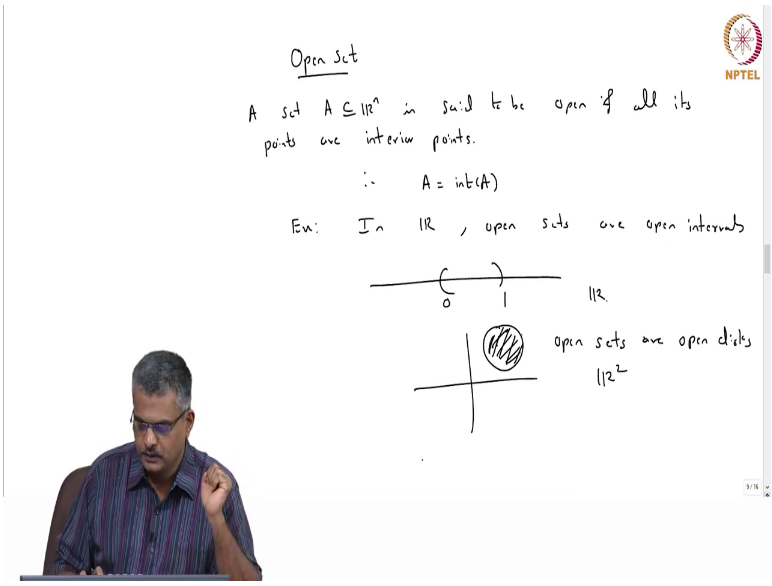

So, now we are in a position to define the notion of a open set. You earlier give a definition of open set as if you can find an open ball which is contained in the set a right. So, a set A is said to be open, if all its points are interior points. That is, A is something but the interior of A is nothing but the set A.

So, examples in real line in R open sets are open intervals. We saw that earlier. For example, take this open interval 0,1 right. So, this is called the open interval because it excludes points 0 and 1 right. So, in R, this is an example of an open set. In R2, you can take any center right with a ball and some finite radius right. So, all the points in this except the boundary points right. So, it is an open disc. So, in R 2 the open set is nothing but the open disc; open set open sets are open discs right.

So, it consists of all points except the boundary points. So, you saw that to define the notion of open set, I require the notion of interior points. So, once you identify the interior, once you understand the notion of interior points right. You collect all the interior points and that gives you the open set right and if you can show that the interior of the set A is nothing but the set A itself, then A is open.

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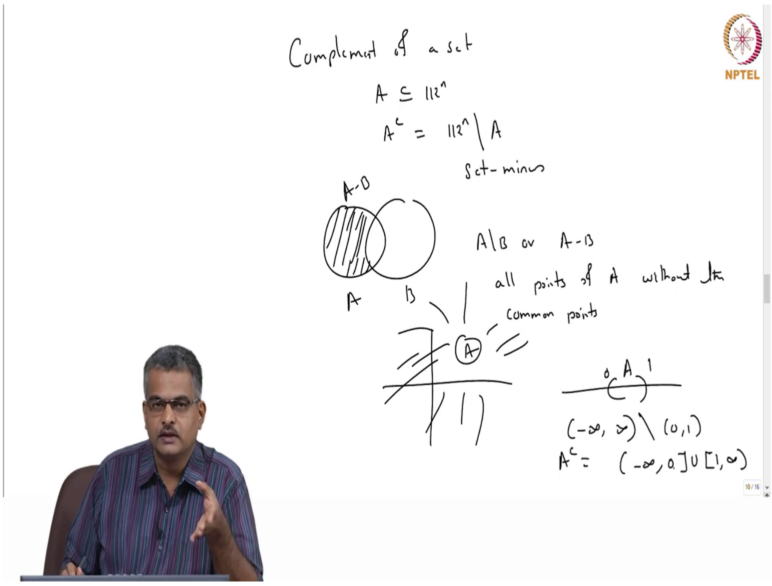

Now, we want to define the notion of closed sets for which we require the notion of complement of a set; complement of a set right. So, A is a subset of R n right. So, A complement right is all those points in R n except the points in set A right. So, one can also define this using the notion of set minus. So, what is this? This is the symbol for set minus. So, if I have two sets; A and B and if I have a non empty intersection, then A minus B are also denoted by A minus B is nothing but all points of A right, without the common points.

So, in this example this is the A minus B right. So, the complement of the set A is R n minus A right. So, if I have a disc here right. So, the complement is of this set A is all the points other than the set A right. So, in the real line the complement is; so, the real line is minus infinity to infinity right. So, from the real line you take out say if I have 0,1 as the open interval right as I said 0,1. Then, this is nothing but minus infinity right to 0. So, this square bracket implies that it is it includes the 0.0 union 1 infinity right.

So, this is the if I call this as A; this is the complement of A ok. So, once you understand the notion of complement of a set, it is easy to define the notion of closet set. The closet set is nothing but a complement of an open set right.

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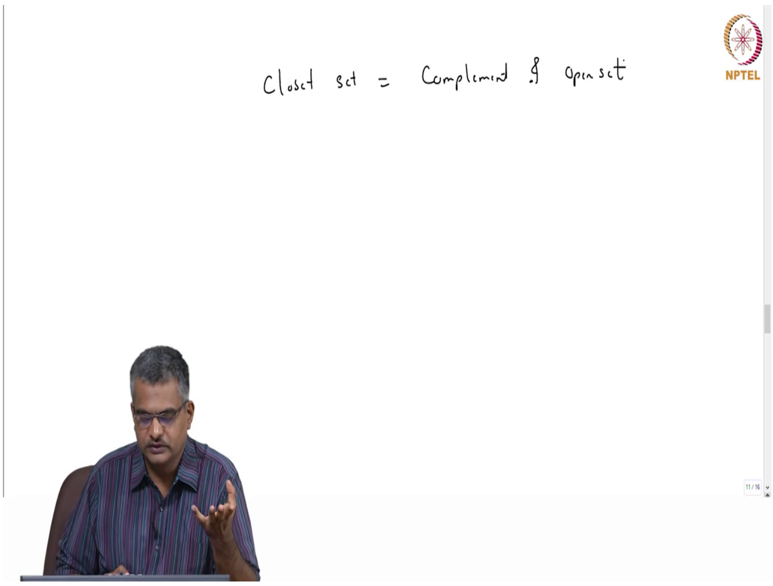

So, closet set is nothing but complement of open set. So, one may ask, can I defined the notion of closet set just like the way we define the notion of open set using the notion of interior points. So, can I define some other concept right, give it a property and then say that collection of low all those points is a closed point right? So, if one can do that and we will see that in the next lecture.

Thank you.