Nonlinear System Analysis
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Module - 05
Lecture – 01
Equilibrium Points

Hello everybody welcome to this 5th week of lectures on linear systems theory. So, this week will be a little shorter module, but we will focus on some nice qualitative behavior of systems and these are essentially to do with the Equilibrium Points of the system. So, we would have encountered this definitions sometime during our previous courses. But we will give it a general setting here of the kind of systems that we will be dealing with.

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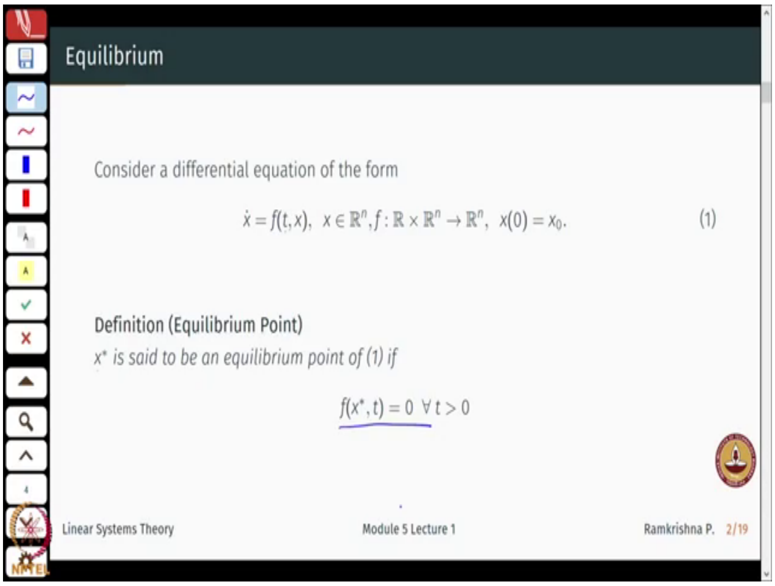

So, if I consider a differential equation of the form. So, this could be non-linear and also I put the t in just to also take into account the time varying nature of it. So, as usual x comes is an n dimensional vector f is a vector field from R cross R into R n with some initial conditions ok.

So, the basic definition of an equilibrium points is the following that x star is an equilibrium point of this system one, if it satisfies this equation so f x star comma t equal to 0 for all t greater than 0 ok.

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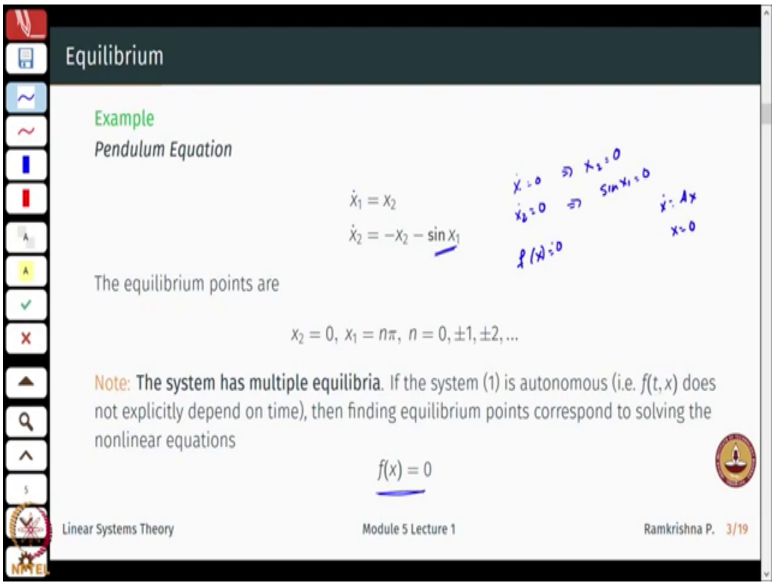

So, what does this mean? So, we will go through few examples to see different kinds of equilibrium that we will encounter. So, in the case of a simple pendulum where everything all the parameters are normalized to 1, I have an equation which looks like this ok. So, equilibrium points are one where you know I just say x 1 dot is 0 x 2 dot is 0 and so on.

So, what do I get from the first equation is that, so this will imply that x 2 is 0 which is kind of to check that the velocity would be 0 at the equilibrium and this would mean that sin of x 1 is 0. Which means it will have a variety of solutions right. So, starting from say pi to 2 pi and so on, so all these multiples of pi. So, what is different or unique about this system is that it has multiple equilibrium.

So, if I were just to look at the linear systems, we were essentially looking at x dot equal to Ax and if A was full rank or invertible then the origin was the equilibrium. So, this to begin with is a non-linear system, their non-linearity appears in this term here and this system is seen to have a multiple equilibria.

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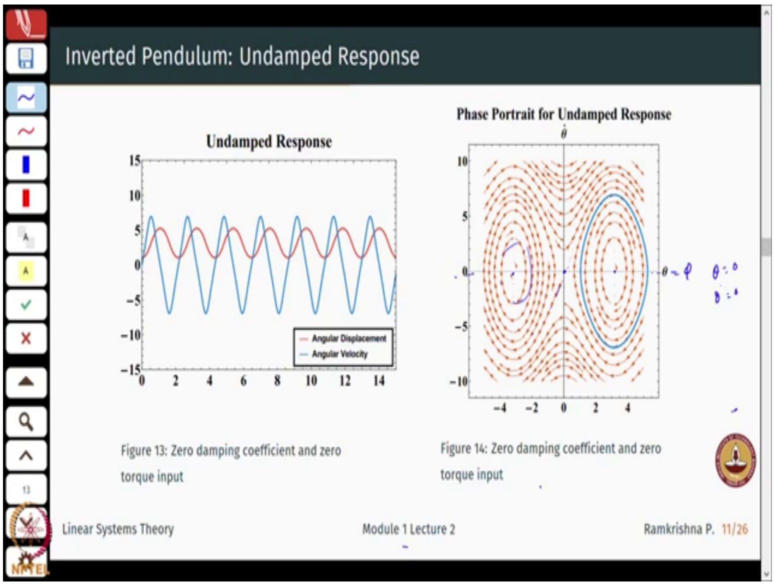

So, if I relate to the phase space that I had drawn earlier for the for the case of un undamped response. So, you can see here right, so this point here is a equilibrium sorry. So, you can see a couple of equilibriums here and one at the origin and so on sorry this is behaving weird ok.

So, if I go back to the phase space which I had drawn in one of our earlier lectures, you can see that this actually corresponds to set of different equilibrium point starting from here, you have an equilibrium point here, here and so on, if you keep on progressing to the right and the left. So, this is a typical case of a system or a non-linear systems which has multiple equilibrium points. What are the nature of this equilibrium points, do each of the equilibrium points exhibit the same behavior or not that we will see in the due course of this lecture ok.

So, in general, so if so, the we started off with the system f t x, but if the system is autonomous that which means that the that the system does not explicitly depend on time then finding equilibrium you corresponds to solving just a non-linear equation f of x equal to 0. Same in the case of a pendulum right. So, this was an autonomous system and I was just solving for f of x being equal to 0 ok.

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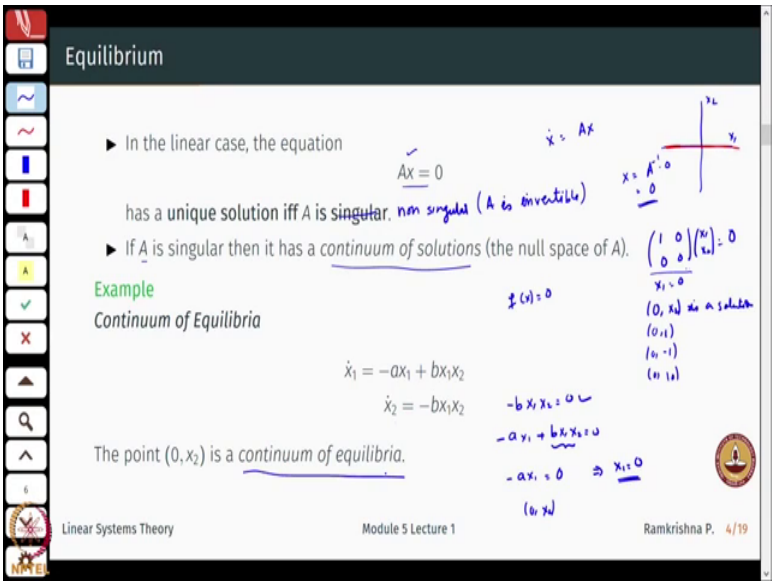

So, is it all very obvious in the linear case that the that the origin is always the equilibrium point? Well the answer turns out to be not very obvious, but we will look at a couple of examples.

So, in the linear case, the equation Ax equal to 0. So, my dynamics are the form x dot equal to ax I set Ax equal to 0, I am essentially solving for the situation. If Ax equal to 0, so this system has a unique solution if and only if. So, it is a typo error A is non singular or A is invertible. So, in case when A is invertible I can just write that x is A inverse times 0 that is 0. So, the origin is the unique solution of this equation if and only if the matrix A is invertible.

So, on the other hand if say A is singular what happens if A is singular. So, let us say I take an example like this sets right x 1 comma x 2 and looking for this to be 0 if I solve for this what do I get from the form that x 1 equal to 0 and I do not get any expression for x 2 which means that 0 comma x 2 is a solution to this.

So, what do I mean by this at any value write it is to be 0 comma 1 is a solution 0 comma minus 1, 0 comma 10 and so on are the solutions to this equation. Any value of x 2 with x 1 equal to 0 is a solution if I will just want to draw it here x 1, x 2. So, x 1 is 0 and x 2 being any value, this is the entire horizontal line is a solution to Ax equal to 0 in this case or it also means that if a is singular then it has a continuum of solutions right. So, this is sorry.

So, this entire line here is the continuum of solutions for this for this set of equations or the or for the system which is represented by A of this form ok. So, this is also the null space of A. So, here in the in the case when A was invertible or non singular, then the null space is just the trivial point that is at x equal to 0 is the null space well this phenomena can also occur in the non-linear case.

So, if I have a system which looks like this x dot is minus A 1 plus b x 1, x 2, x 2 dot is so on. So, I just look at the solution of what is the equilibrium, just look at the solutions of f of x equal to 0, the second equation will give me minus b x 1, x 2, is 0, first equation is minus a x 1 plus b x 1 x 2 is 0.

From here I already know that b x 1 x 2 is 0 therefore, I am left with just this equation which means at x 1 equal to 0. So, this is the only thing that I can derive from this equation and therefore, 0 and any x 2 is the solution to this equation is an equilibrium point for this system and therefore, this system also exhibits a continuum of equilibria. An interesting case that throws to us lots of insights into understanding equilibrium points is essentially with second order systems.

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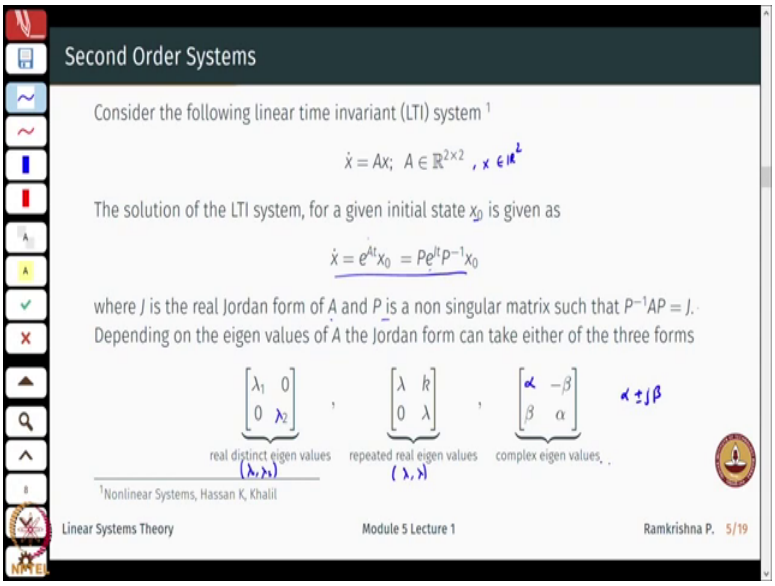

Let us just say when dealing with second order linear systems and this linear system could just be linear by itself or it could come as a process of linearization of a non-linear systems that we will do in the next lecture proceed succeeding this lecture.

So, let me just make things simple here and say I am just dealing with a second order linear system choose x dot is Ax A is r 2 cross 2 x is a two dimensional vector ok. So, for any given some initial state x 0, the solution will always take this form and now we know how to compute e power At we can also do it via its Jordan form and so on right. So, e power At and here j is the real Jordan form of A p is a non singular matrix which takes it from a given form to its appropriate Jordan form right then p is of course, non singular and we also know how to derive this matrix p ok.

So, depending on the nature of eigenvalues, the Jordan form can take several forms. In this case essentially it will take three forms. So, first is when the eigenvalues are real and distinct. So, this will just be the Jordan form will just be a diagonal this is should be a lambda 2 here which means the eigenvalues are just lambda 1 comma lambda 2. In this case, I will just realize a nice natural diagonal form.

In case the eigenvalues are repeated like lambda and lambda r my eigenvalues, then the Jordan form can take can be something like this where k can either be 0 or 1 depending on the multiplicity of the geometric multiplicity of the eigenvalues.

Third thing is when I have complex eigenvalues we have alpha plus minus J beta. So, this is an alpha missing here. So, we will have in this case complex eigenvalues ok. What do each of this eigenvalues signify? These are just information on stability or there is a little more of information than that and when do these cases actually occur ok.

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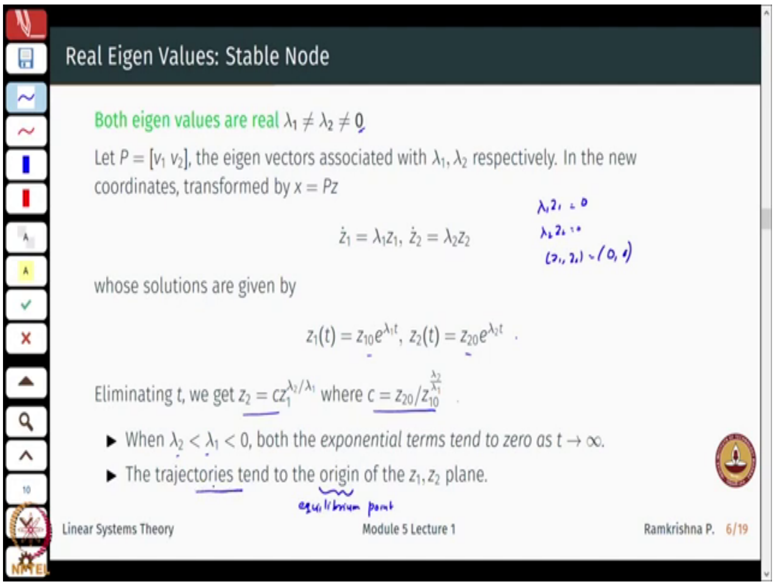

So, first let us start with the case of real eigenvalues. What does it mean? Both eigenvalues are real, they are both nonzero. We will come to the case of 0 eigenvalues a little later ok. So, let us say I have this set of eigenvalues, let me just derive this and I come and come back to the slide.

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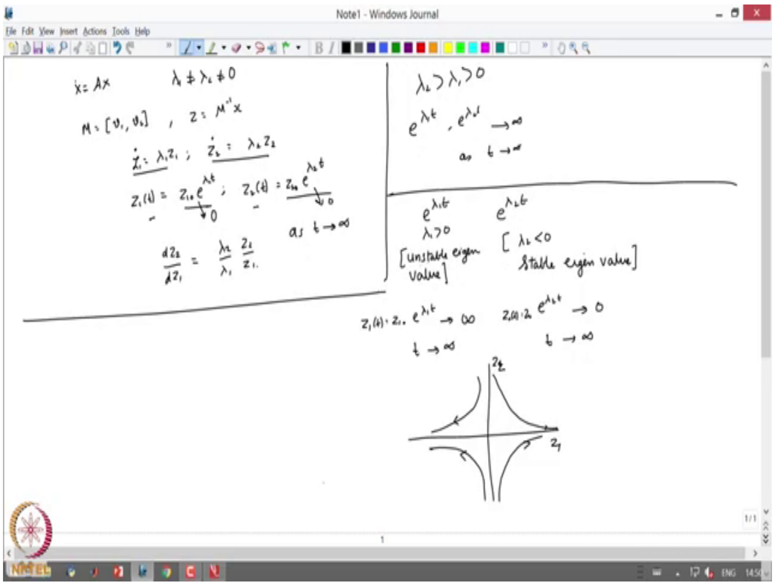

So, I have lambda 1 and lambda 2 which are not equal to each other and which are also not equal to 0. And again I am looking at system with x dot equal to Ax. Now, I know in this case that by taking its eigenvectors v 1 and v 2 and a coordinate transformation which looks like this m inverse x I can transform this system into a diagonal form and the diagonal form looks like this z 1 dot is lambda 1 z 1.

And z 2 dot is lambda 2 z 2 and this will have solutions z 1 of t is z 1 0 e power lambda 1 t, similarly z 2 of t is z 2 0 e power lambda 2 t ok. So, this is just how the solutions would look like. Now, depending on the values of lambda whether they are greater or less than 0 these solutions will either increase exponentially or will decrease exponentially and tend to the origin ok.

So, back to here. So, this is what we have now right the solutions in the diagonal form are just given by this. Now, I can just eliminate t to write my equations of this form this is a little straight forward to check also how do I eliminate t. So, I have these two equations right z 1 dot is lambda 1 z 1, z 2 dot is lambda 2 z 2. So, I just have d Z 2 by d Z 1 can be written of the form lambda 2 by lambda 1 z 2 over z 1.

And I can do all the all the calculus that I know of solving integral equations and I just end up with the with the solutions like this with c depending on the initial conditions and so on ok. So, this is this is a like easy to check ok. The first case which you would be interested is when lambda 1 is less than lambda 2 is less than lambda 1 and both are negative right.

The first observation is since both are negative. So, I will see it is easy to check that this will go to 0, this term will also goes to 0 as sorry as t goes to infinity this is; this kind of obvious ok. And if I look at in the z 1 z 2 plane, the trajectories tend to the origin like. So, here z 1 also goes to 0 and z 2 also goes to 0. And of course, in this case the 0 turns out to be the equilibrium point, so the origin is the equilibrium point. This just you can substitute z 1 dot is z 2 dot is 0 and end up with equations lambda 1 z 1 is 0, lambda 2 z 2 is 0. And therefore, z 1 z 2 is 0 comma 0 is the equilibrium point ok.

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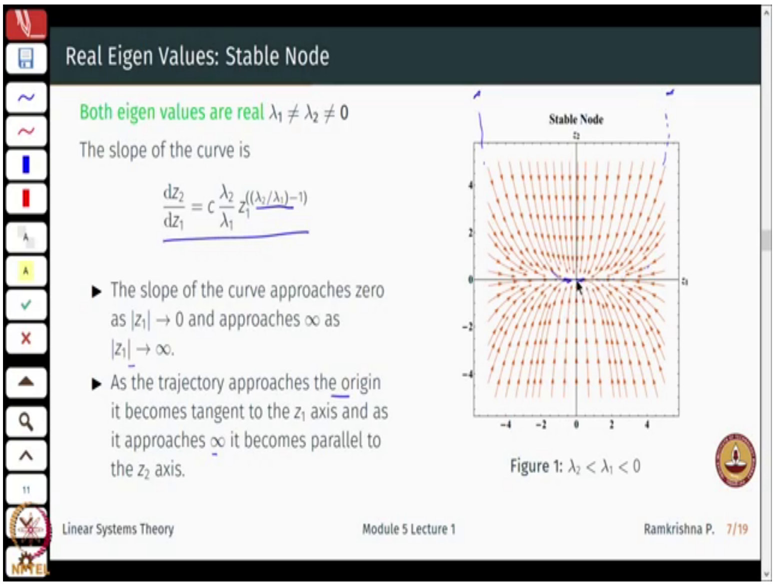

So, what happens in this case. So, if I look at so, what I know now is that these in the z 1 z 2 plane, the trajectories tend to 0 as time progresses at or exponents or asymptotically.

So, what how do they actually do that? So, from my equation relating z 1 and z 2, I can compute the slope of this line right. So, how does z 2 change with respect to z 1 and that is what we that that the derivative is essentially the slope of it. So, this is a positive number ok. The slope of the curve so, it is easy to check from here that the slope of the curve approaches 0 as z 1 goes to 0 ok.

Second is the slope approaches infinity as z 1 goes to infinity right. That is what is happening here right. So, as the trajectory approaches the origin, it becomes tangent to the z 1 axis. So, if you look at these things as they approach the origin they are becoming tangent to the z 1 axis and away from the origin they will be parallel to the z 2 axis right at really at infinity right.

And the second observation is as it approaches infinity it becomes parallel to the z 2 axis and you can you can just plot this for yourself and check. So, we have put up already the code to help you draw phase portraits of this form.

If you have already done this in our week 1, lectures ah bit of its starting with phase space. So, this is also a continuation to that there we really did not talk of equilibrium points and the nature of them, but slowly we will get to understand that the kind of things that we are doing today will essentially relate to stable equilibrium points, unstable equilibrium points if I am talking on stable I am looking at an under damped situation, I am looking at an over damped situation critically damped and so on right ok.

So, this is nice here right. So, I just see that all the trajectories are actually converging to the origin. So, it means that if I am at the at the origin I will always be at the at the origin right. So, but if I am slightly perturbed here say. So, if I say end up at this point here, then I will slowly come back to the origin right.

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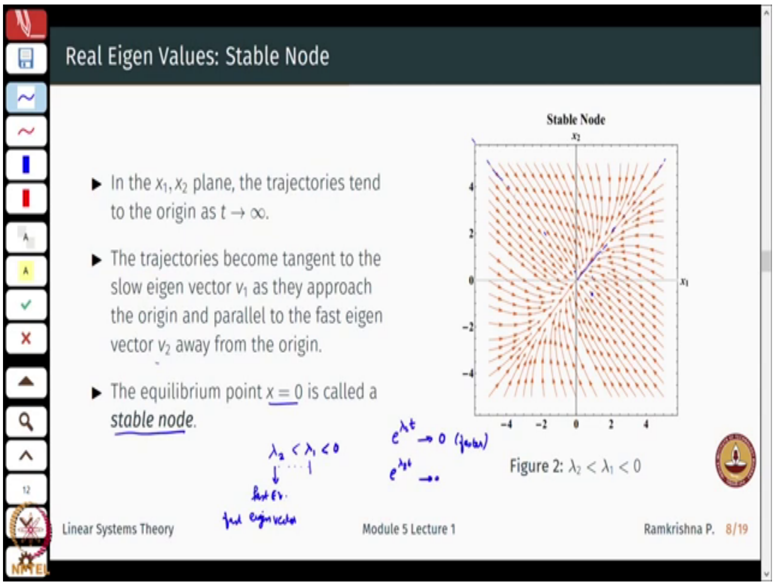

So, let us come go back to the x 1 x 2 plane and check what is happening here ok. In the x 2, x 1 x 2 plane, the trajectories tend to the origin again as t tends to infinity ok. Now, we have here eigenvalues which are lambda 2 less than lambda 1 and both are less than 0.

So, we call this as, so if this condition is true then the term e power lambda 2 t converges to the origin faster than e power lambda 1 t ok. So, this is faster ok. So, we call this the fast eigenvalue and then the corresponding eigenvector as the fast eigenvector. And similarly, with the slow eigenvalue and the slow eigenvector.

So, as time increases, the trajectories become tangent to this slow eigenvector in v 1 and as they approach the origin and parallel to the fast eigenvector away from the origin. So, you can see roughly here that they know the slow eigenvector should be somewhere around here and the fast eigenvector like somewhere around here right, I think you just quickly check for any example this should hold right.

So, we in the Z 1 Z 2 plane, actually looked quite nice of how the trajectories are going to the origin and here in this case this will be the slow eigenvector and this will be naturally the fast eigenvector and correspondingly in the x 1 x 2 plane when such a behavior is seen the equilibrium point x 0 is called a stable node ok.

Because, all any trajectory starting around the origin will actually come back to the origin. So, we will define the notion of stability formally in next weeks lectures, but for the moment, we can just observe this and call this a stable node right ok.

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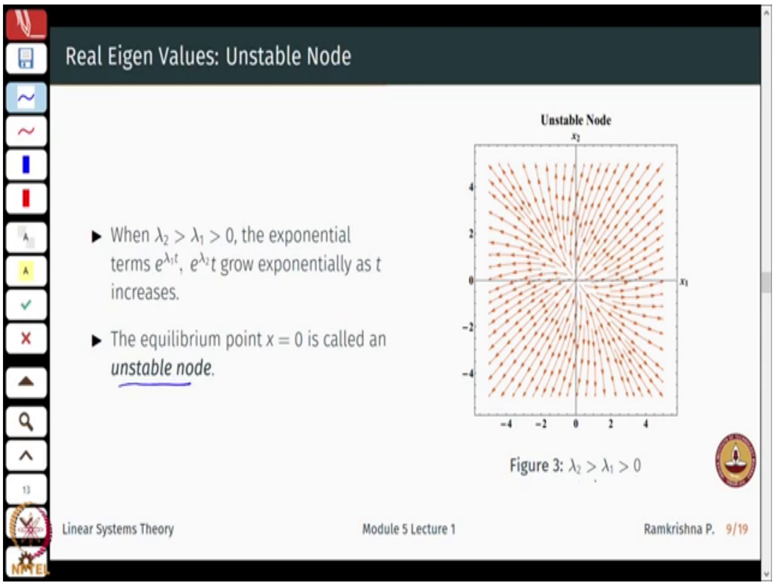

Unstable node its everything is still the same that your lambda 1 and lambda 2 are right. So, both are positive. So, e power lambda 1 t and e power lambda 2 t will both go to infinity as t goes to infinity ok.

So, when lambda 1 lambda 2 greater than 0 the exponential terms grow exponentially as time increases it will happen same. So, if I were to just plot it in the in the Z 1 Z 2 plane, it will just be the same except the arrows being reversed right.

So, all trajectory starting from the origin will or near the origin will tend to go away from the origin whereas, in the stable node case all trajectory starting around the origin will tend to come back to the origin. If your initial condition is the origin you will always be at the origin ok. So, in this case the equilibrium point is called an unstable node for eigenvalues which are which are greater than 0 ok.

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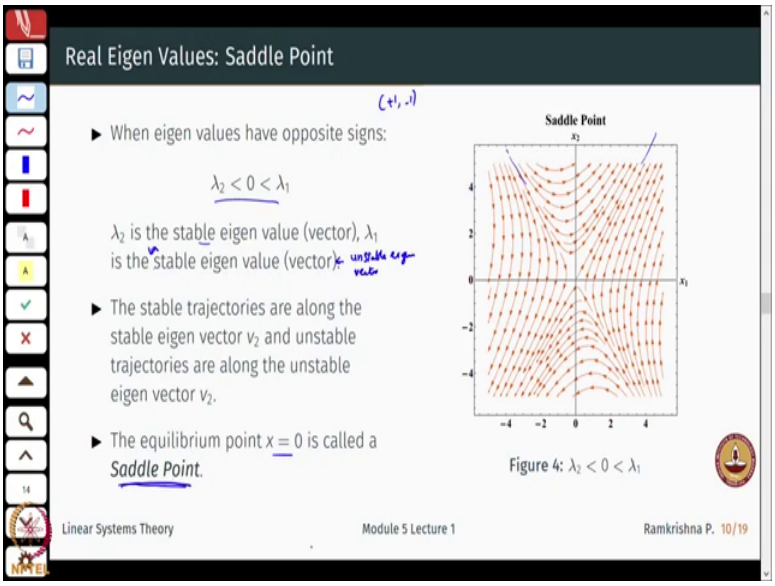

Interesting thing happens when we have eigenvalues which are real, but which have opposite sign say plus 1 and minus 1. Of course, this would corresponding to a correspond to an un unstable system. So, we call lambda 2 which is less than 0, the stable eigenvalue and of course, correspondingly the stable eigenvector and lambda 1 this would be unstable lambda 1 is the unstable eigenvalue and hence the unstable eigenvector.

So, lambda 1 is stable and lambda 2 unstable, this is a little type of here ok. So, how to understand the behavior of this? Let us again analyze these two terms here. So, I have e power lambda 1 t and e power lambda 2 t. So, lambda 1 is greater than 0 it is the stable eigenvalue and lambda 2 is less than 0, so this is the unstable eigenvalue sorry lambda 1 is greater than 0. So, this is my unstable eigenvalue and of course, the corresponding eigenvector will be called the unstable eigenvector, lambda 2 is less than 0 and I will call this the stable eigenvalue ok.

So, what would we expect as time progresses that e power lambda t, lambda 2 t will tend to 0 as t goes to infinity whereas, e power lambda 1 t will go to 0 sorry, will tend to infinity as t goes to infinity ok. So, just come back to, so e power lambda t corresponds to z 1. So, z 1 t is z 1 naught, z 2 t is z 2 naught e power lambda two t.

So, if I were to plot this in my z 1 z 2 plane ok. So, this plots would look something like this right at infinity well the z 2 will tend to 0 and z 1 will go to infinity. From starting from any initial condition this way, this way and this way ok.

This is how the plot will look in the z 1 z 2 plane when we have eigenvalues one of real eigenvalues, one of which one of which is unstable and the other one is stable ok. So, in back to the x 1 x 2 plane this would look something like this. So, the stable trajectories are along the stable eigenvector and the unstable trajectories are along the unstable eigen eigenvector.

So, this could this could. So, you can just differentiate the things here or more naturally over here and so on ok. So, in this case the equilibrium point x equal to 0 is called a saddle point ok. So, this was about real eigenvalues, what about the case when we have complex eigenvalues? Well that case also turns out to be interesting.

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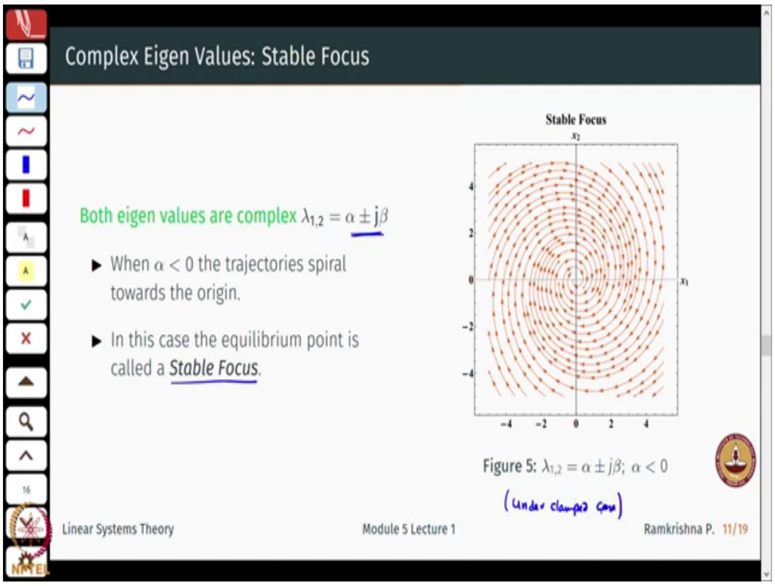

So, when both eigenvalues are complex, I have some eigenvalues; let me call they are alpha plus minus J beta ok.

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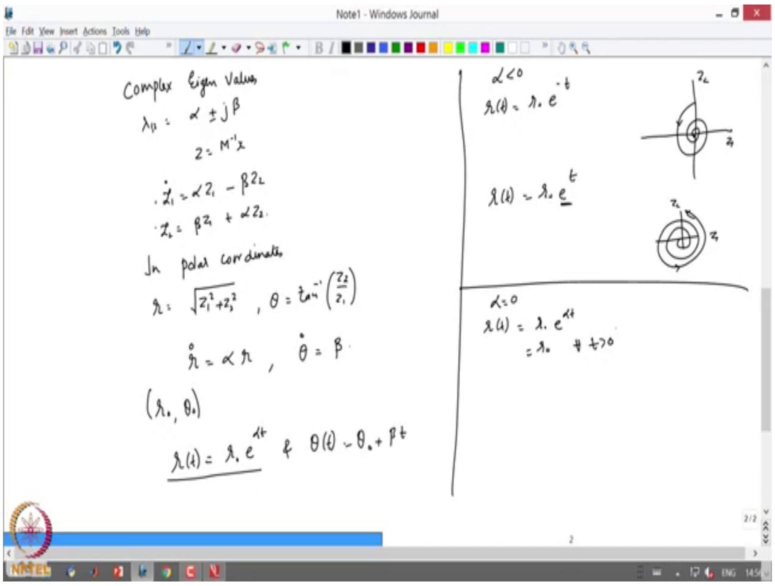

So, let us again do this over here ok. So, when I have complex eigenvalues which means; my lambda 1 2, 1 comma 2 are of the form alpha plus minus J beta and of course, I do the usual change of coordinates from z equal to M inverse x to get my system of the following form z 1 is alpha z 1 minus beta z 2; z 2 is beta z 1 plus alpha times z 2.

So, I just do a little change of coordinates let us say change to polar coordinates. What do I have in polar coordinates? r is square root of z 1 square plus z 2 square and the angle theta is the tan inverse of z 2 over z 1. Now, if I write these equations in polar coordinates I will have two coupled first order differential equations that is r dot is alpha times r, in the second equation given by theta dot is beta ok.

So, for a given initial state r 0 and theta 0, the solutions are the form r of t is r 0 e power alpha t and theta of t is theta naught plus beta t ok. So, this is interesting here right. So, I see that I have some, if I am looking in the polar coordinates I am looking at the radius of it, so to speak which is which has an exponential term depending on t ok. So, if I just say what happens to the radius when alpha is less than 0.

So, I have r of t is r 0 e power say alpha is minus 1 0, e power minus t. So, this will mean that in my z 1 z 2 plane, my trajectories will this spiral to the origin say maybe in this way the z 1 z 2 plane right. It is also obvious from here I start with initial radius and it will just go spiraling to the origin now back to here.

So, when alpha is less than 0, the trajectory spiral to the origin. In this case of complex eigenvalue I call my equilibrium point to be a stable focus ok.

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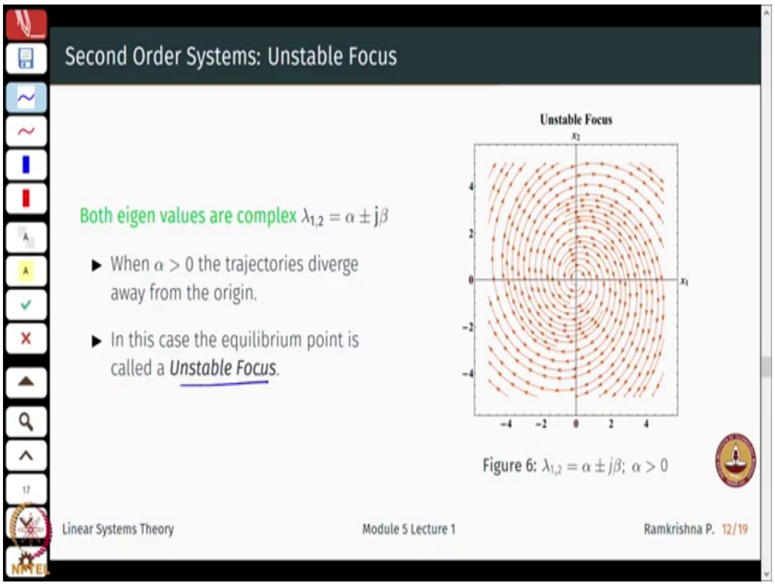

Now, the same thing when alpha is greater than 0, well I just see from here that when alpha is greater than 0, r t which is r 0 let us say alpha is 1. You see that the radius actually starting from some initial radius the radius actually increases with time or it in other words there it just spirals away from the origin ok.

So, this is z 1 z 2, the direction of arrows will be away from the origin in this case they will be towards the origin ok. In such a case all the trajectories are going away from the origin, the equilibrium point is called an unstable focus ok.

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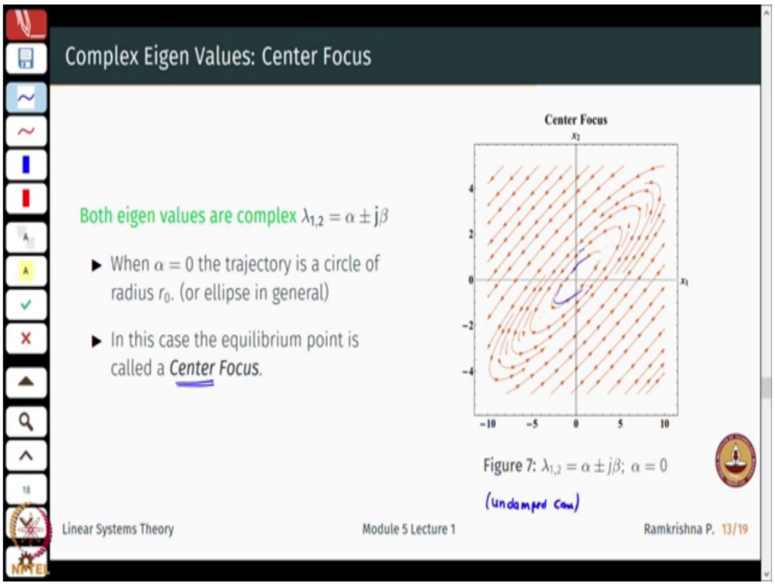

Interesting thing happens when r equal to 0; when r equal to 0 in this case sorry when alpha equal to 0 alpha is 0 r of t is r 0 e power alpha t when alpha is 0 this just is just r ok. r r 0 it just be the radius where it actually started from for all times t this will just be for all times t greater than 0. So, essentially I am looking at a circle of constant radius r for all times t greater than 0.

So, if I come back to my x 1 x 2 plane. So, this could also in general be ellipses right. So, when alpha equal to 0, the trajectory is, so they just they just are in some periodic orbits around the origin like here right or this one they either the circle of radius r in this case or more generally they will look like an like an ellipse. So, in this case the equilibrium point is called a called a center ok.

So, this also if I look at it correspond to few cases that that we learnt earlier right. So, this is usually the undamped case, these things would correspond to the under damped system. Of course, I am not really talking of what is the damping in the stable in an unstable system that they does not make sense.

So, this is, this are things that are corresponding to the under damped case or the under damped system where as I go back here, this would correspond to something like an over damped system or when lambda 1 lambda 2 are equal it will correspond to a critically damped system. And these are this kind of plots we drew earlier in our week 1s lectures.

Now, there we just talked about damping the damping properties of the system by this is a little more general way of looking at it. What happens when we have repeated eigenvalues or for the case when we look at a critically damped system?

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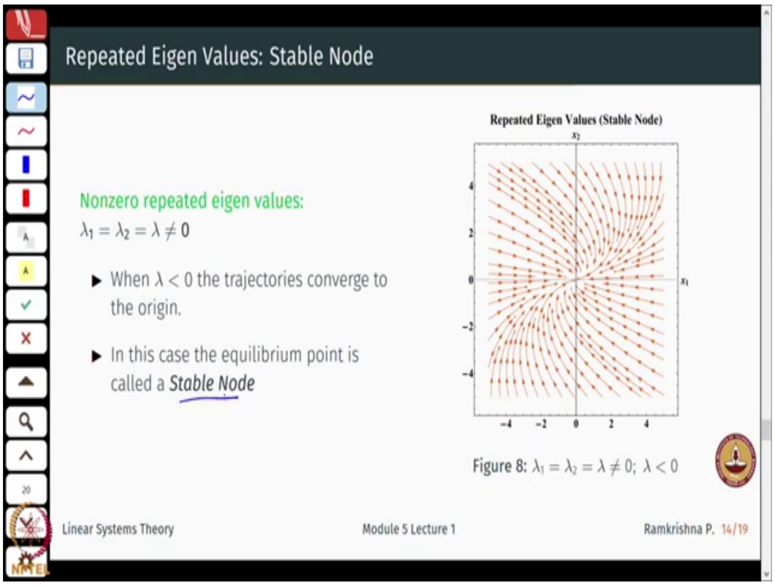

Well when we have non 0 repeated values which means lambda 1 and lambda 2 both are equal to lambda the trajectories, well as usual they converge to the origin and in this case again the equilibrium is called a stable node ok. So, let us check this in a bit more detail.

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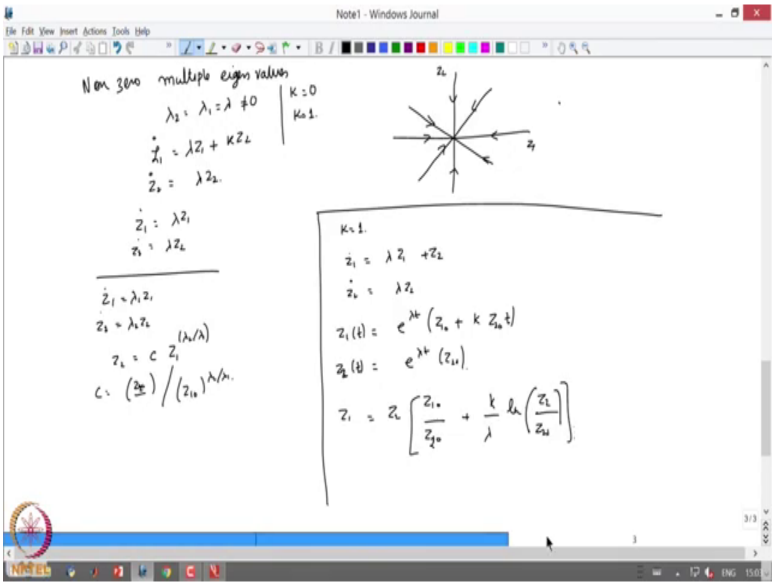

When we have non zero, multiple eigenvalues ok. Again, I will look at ok, so this means essentially that lambda 2 and lambda 1 are equal to lambda and this is actually not equal to 0, we will come to the case of for the 0 eigenvalue a little later. And with appropriate transformation I can write the system as z 1 dot is lambdas z 2 plus k z 1, z 2 dot is lambda z 2.

So, couple of cases can occur when k is 0 and when k equal to 1 ok. When k equal to to 0 I am just looking at these two equations right z 1 dot is lambda z 1 and z 2 dot is lambda z 2. So, if I just compare with the first case which I had of z 1 dot is lambda 1 z 1, z 2 dot is lambda 2 z 2 which had the relation between z 2 by z 2 was given by c times z one of lambda 2 by lambda 1 ok.

And then c was z 2 naught by z 2 naught by z 1 naught lambda 2 by lambda 1. So, if I just compare this to the situation where I will have lambda 1 equal to lambda 2, it should be easy to check the following right that in if I just draw this in my z 1 z 2 plane all the trajectories will be coming to the origin this way or this way or this way ok

So, this is like the little contrast with the slow and fast eigenvalues and eigenvectors and so on ok. So, what is also interesting is the case when K equal to 1 this should be easy to plot them you can just check by yourself. So, when K is equal to 1, then I have z 1 dot is lambda times; time z 1 plus z 2 and z 2 dot is lambda times z 2.

So, the solutions would be z 1 of t is e power lambda, t z 1 0 plus k z 2 0 t z 2 of t is e power lambda t z 2 of 0 of course, if I just want to write it in terms of z 1 and z 2 that will simply be z 1 is z 2 of z 1 0 by z 2 0 plus k over lambda log of 2 by z 2 0 ok.

So, I will just not plot this, but we will just check what this means in the in the x 1 x 2 case. So, it will turn out not surprisingly that that you are actually talking of a stable system because the eigenvalues are less than 0, we are talking of eigenvalues being equal to each other. So, we are talking of some critically damped situation where we would naturally expect the trajectories to come back to the origin starting from a neighborhood of the origin. So, this is how they will look like and we will already, I will also call this as a stable node.

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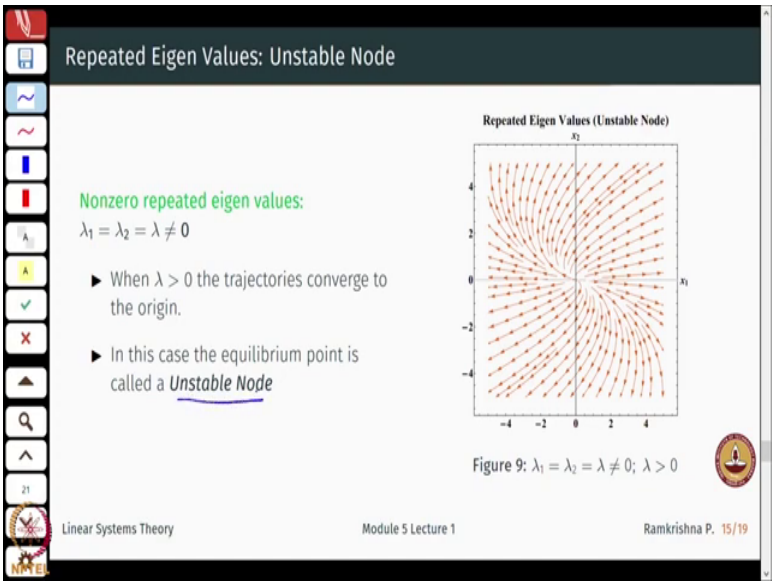

What happens when it is an unstable node? Well just the plots will be similar with just the eigenvalue, we just it the direction or of the arrows being reversed.

So, naturally I am looking at say eigenvalue of plus 1 plus 1, the system will naturally be unstable and this equilibrium I will call as an unstable node ok.

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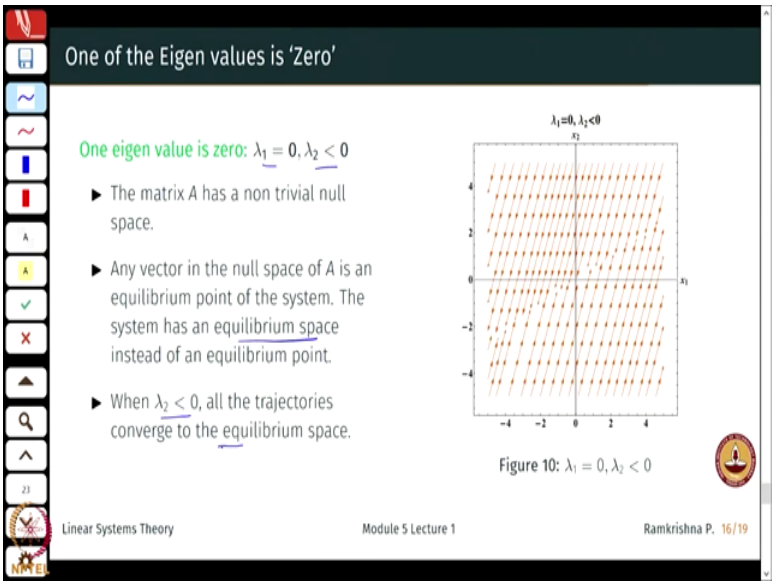

The next thing that we will discuss is an interesting case when there is a possibility of having 0 eigenvalue ok. So, let us start with just one of the eigenvalue being 0 let us say lambda 1 is 0 and then lambda 2 is less than 0 ok. So, how will it how will the system look at in how the transformed system look like.

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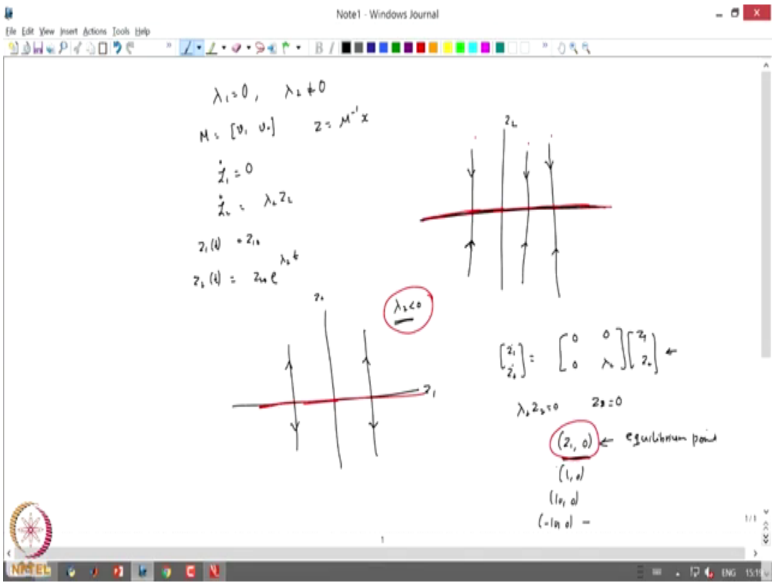

So, when I just write it in this Jordan form. So, lambda 1 is 0, lambda 2 is not equal to 0 and I just again use a transformation matrix v 1 v 2 and where the transformation z is M inverse x in my new coordinates, I have z 1 dot is 0 z 2 dot is lambda 2 z 2.

The solutions are pretty straightforward to compute z 1 of t we will just be whatever it began with its initial condition, z 2 of t will be z 2 naught e power lambda 2 t. So, if I were just to plot z 1 and z 2, say for initial condition over here z 1 will just be here and if lambda 2 is less than 0, the trajectories would just behave this way.

Say if this is the initial condition of z 1, then the trajectories will be here if this is the initial conditions the trajectories will go this way for all lambda 2 which is less than 0 ok. So, what does that mean? Right.

So first is the matrix A has a non trivial null space. And any vector in the null space of A is an equilibrium point of the system ok. How will we reduce the equilibrium point? So, if I have a system like this, z 1 dot z 2 dot is 0 0 0 lambda 2 z 1 z 2. What are the equilibrium points? If I just solve for this I will have lambda 2 z 2 equal to 0 which means z 2 equal to 0 ok.

Then there is this is the only solution I get from solving this. So, any point of the forms z 1 comma 0 is an equilibrium or is an, is an equilibrium point of the system or in general here I will have an equilibrium space instead of an of an equilibrium point right. So, any point take any z 1 with z 2 equal to 0, So 1 comma 0 is an equilibrium similarly is 10 comma 0 minus 10 comma 0 and So on.

Now, when lambda is less; lambda 2 is less than 0 all trajectories converge to the equilibrium space. So, what is the equilibrium space here? this is entire z 1 axis right. So, any point in z 1 with say 2 be equal to 0 is a equilibrium space. So, here let me just draw it in a red. This is my equilibrium space which is obtained by this one. This is my equilibrium space.

So, when lambda 2 is less than 0, any trajectory right. So, this let this trajectory, this trajectory, every each of this trajectory will converge to the equilibrium space ok.

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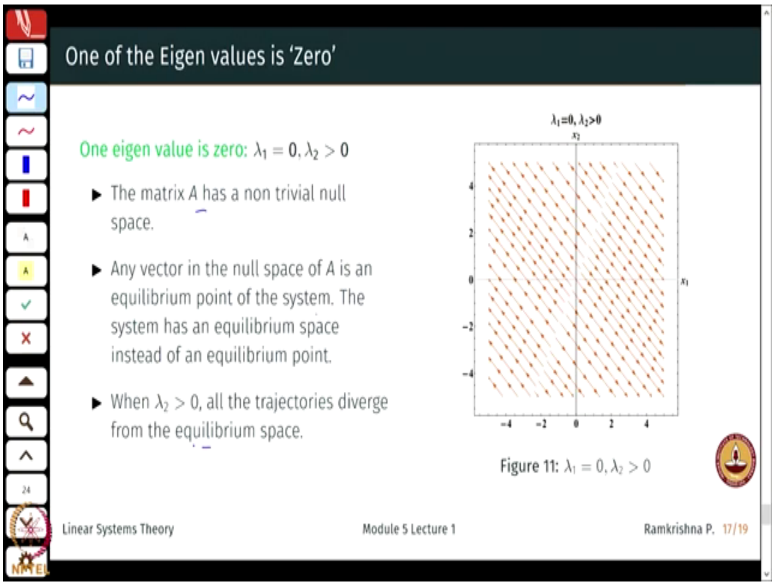

Next, what if lambda 2 is greater than 0, everything will be the same, the matrix a will still have a non trivial null space and any vector of in the null space of a is again the equilibrium point and the only thing that will change is all the trajectories diverge from the equilibrium space. So, let me just draw it here ok.

So, the trajectories, so this is z 1, this is z 2 and this will just be my trajectory. So, all trajectories will go away from the equilibrium space, the equilibrium space is the entire horizontal axis in this case ok.

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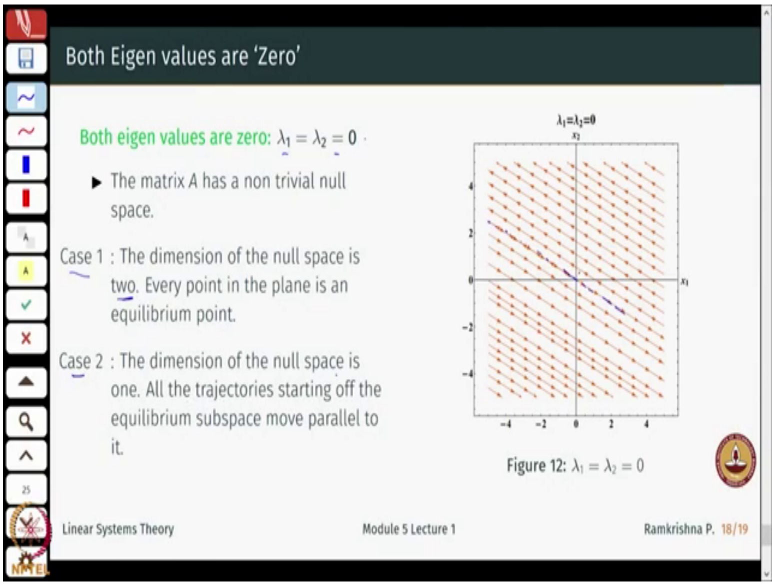

The last case that we will look at is the case when both the eigenvalues are 0 ok. Not surprising to note that the A matrix will still have a non-linear, sorry the A matrix will still have non trivial null space. So, in this case when both eigenvalues are 0, we can potentially look at two cases.

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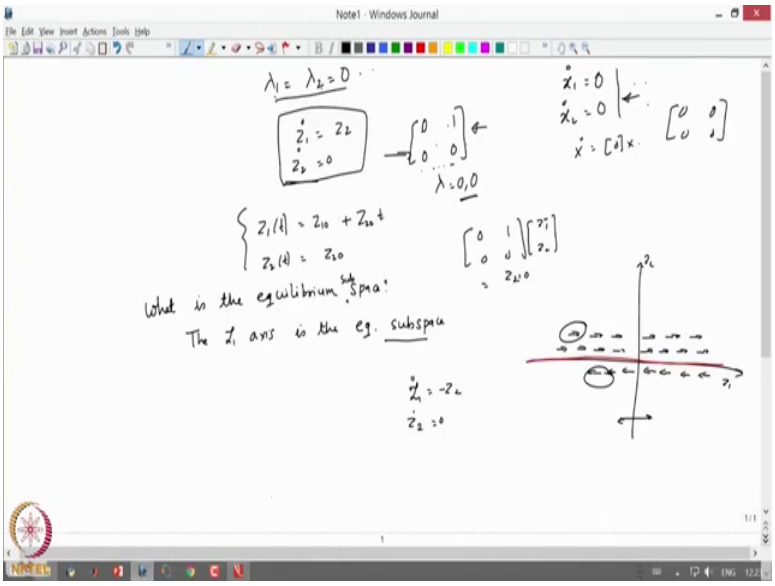

So, looking at both eigenvalues lambda 1 and lambda 2 being equal to 0. So, one case could be that I am looking at systems of the form x 1 dot is 0, x 2 dot is 0 or in other words x dot is the 0 matrix times x ok. So, this will have both eigenvalues to be 0 this would correspond to the case when the null space is of dimension 2 and not only that every point in the plane will correspond to an equilibrium point ok.

So, this case is it may not be too interesting for us to look at the phase plane, but what is interesting is the case when both eigenvalues are 0 and the dimension of the null space is 1 ok. In, how does that that happen? That could happen in cases when well I have systems of the form z 1 dot is z 2 and z 2 dot is 0.

So, in this case the A matrix is a from 0 1 0 0 and if you compute the eigenvalues they will turn out to be 0 comma 0. So, the A matrix is not completely 0 here it has an it has a nonzero entry here, but still both of the eigenvalues are 0 you can compute check this as a simple exercise ok.

So, it may not necessarily be in this form all the time, but via some appropriate transformation you can write the system to be in its form. So, when does this case happen? And when does this case happen? It again is the same that you are looking at a certain Jordan form. So, if you compute the Jordan form of this form; you will you can check its algebraic and geometric multiplicity and check it is all algebraic and geometric multiplicity and you will have the appropriate Jordan form.

So, this is the Jordan form when the algebraic multiplicity is 2 and the geometric multiplicity is 1. Whereas, here it will be a slightly different case. I will leave that as an as an exercise ok. So, this case is a little interesting to draw the phase space.

So, what can I see directly even without worrying about the solutions is that z 2 is a constant and z 1 dot various positively with z 2 or with the sign of z 2, if it is minus z 2, then it will be vary negatively with increasing z 2 and so on ok. So, how does the solution look like? So, I will have z 1 of t is some initial condition z 1 0 plus z 2 0 of t and z 2 of t is z 2 0 ok.

Now, what is the equilibrium space? So, in this case, well you can you can check easily right. So, I am just looking at the solutions to this one for z 1 and z 2. So, this will give me that z 2 equal to 0 and therefore, the entire z 1 axis is the equilibrium subspace, this is called the equilibrium subspace.

So, if I were to plot on the on the z 1 z 2 plane. So, this entire z 1 axis is my equilibrium subspace, the z 1, z 2 plane ok. What happens close to the equilibrium? It is easy to check that my phase curves will just be parallel to the equilibrium space or the equilibrium subspace. So, here they just be the reverse sign ok. Now, so there could be another cases when z 1 dot is negative of z 2, z 2 dot is 0; what will happen to the phase curve will be exactly the same just with the with the directions of arrows here and here being reversed ok.

So, that is like two different cases when lambda 1 and lambda 2, both are equal to 0 and the equilibrium subspace depends again on the Jordan form right. So, when the algebraic multiplicity is equal to the geometric multiplicity, the Jordan form simply has this form and the equilibrium subspace will have dimension 2. Whereas, in this case the Jordan form takes a form like this and you will have solutions for the phase space given by this set of equations. So, to summarize when both lambda 1, lambda 2 are equal to 0.

Case 1; the dimension of the null space is 2, again depending on the Jordan form. And the case 2; the dimension of null space is 1 and if I were to plot in general in an x 1, x 2 plane, so the equilibrium subspace could be somewhere like here passing through the origin. So, all trajectory starting of the equilibrium subspace either here or here. They will just move parallel to it similarly to what we saw in the of the of the plots in the in the z 1 and z 2 plane. And equivalently in the x 1, x 2 plane, they will look something like this right ok.

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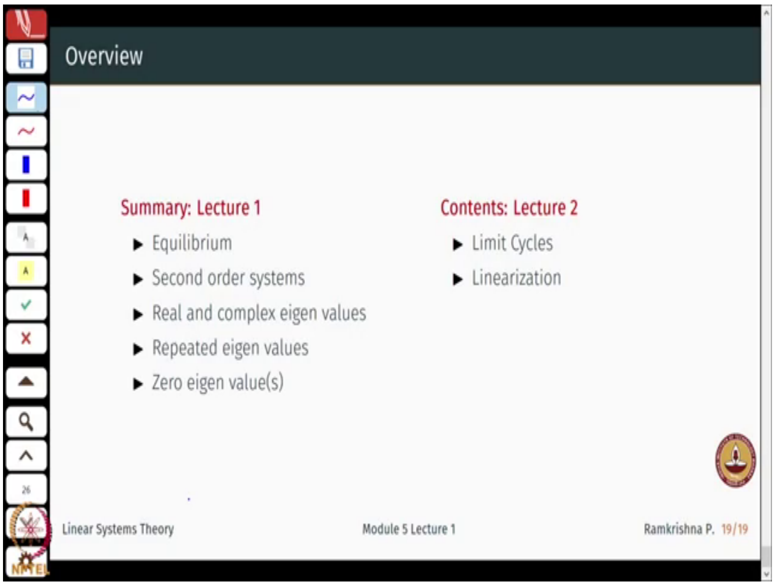

So, just to conclude, we have defined the notion of an of an equilibrium point, we did a lot of analysis, qualitative analysis for several equilibrium points of second order systems. We had real eigenvalues, complex eigenvalues, repeated eigenvalues, what if 1 or both of the eigenvalues go to 0. That is contains a bit of a rich information of what will be useful for us in stability analysis.

So, just to conclude this week's topics; we will deal with limit cycles which is a very interesting property of linear systems sorry of non-linear systems which not necessarily exists in linear systems. And then we will look at a couple of few methods of linearization of how do we start from a non-linear system and end up with a linear system. So, that will be in the next lectures.

Thank you.