Nonlinear System Analysis
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Lecture - 01
Examples of Nonlinear Physical Systems

Hello everyone welcome to this course on Non-linear System Analysis this is the introductory lecture on the course. So, in this lecture we will look upon Examples of non-linear systems drawn from various disciplines such as electrical engineering, chemical process, biological systems, mechanical systems and so on. So, what we are going to see is ah, first we look at the linear system right. We all know that linear system is one which obeys the principle of superposition and homogeneity.

Now what is the advantage if you have the linear system, of course in nature no system is linear everything is non-linear, but suppose for our for our convenience let us say that you are given a linear system and what are the advantages of using a linear system. Now, as we know that all systems in nature are non-linear so, we will have to understand how to model non-linear systems that is the first step. Of course, the course is not about modeling the systems we will derive we will take models from existing models from system, but for explanation I will for to get a feel of modeling I will take the example of a simple pendulum system and explain it is modeling.

So, today we will see how to. So, as I said first we will see the advantages of linear systems then the what is the notion of a linearization right and why linearization is not a good tool to analyze a non-linear system. Then we will see one particular example of a mechanical system that is a pendulum system, we will derive it is model.

Then we will write down the state space representation of this system then once we have written the state space representation, then we will put it in some standard form and then we will identify what is known as the equilibrium points and then the operating point of the system. So, this will be helpful for us when we want later on to do linearization about an equilibrium point and operating point.

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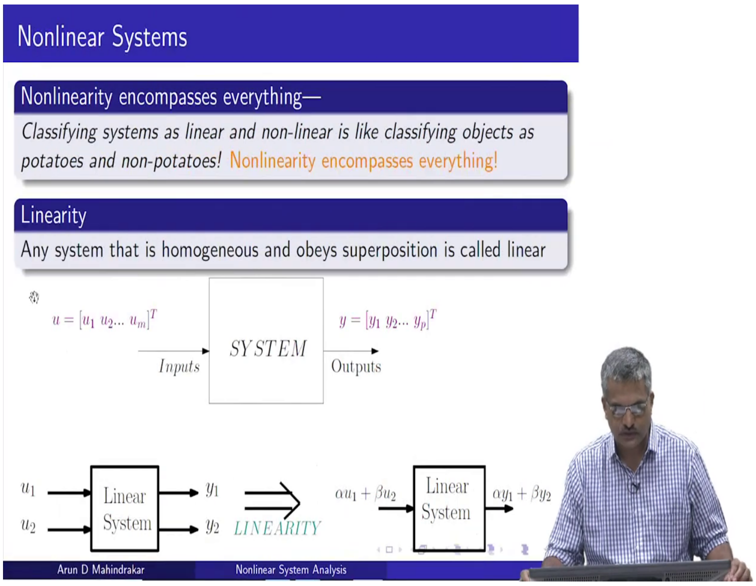

Just go through the linear system theory that we are already familiar. So, we have a system with m inputs and p outputs. So, we denote the inputs by u and outputs by y right and we say the system is linear if it satisfies homogeneity and superposition principle that is if I scale the input right say alpha u 1 and beta u 2 I scale it and then add it up right I get output which is what would have got individually if you had applied u 1 u 2 and u you would have scale it. So, let us see, what are the advantages of using a linear system?

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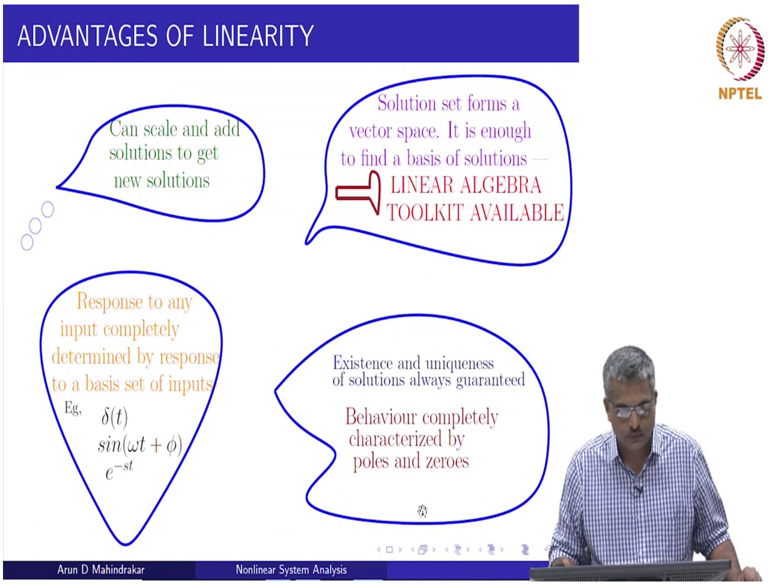

One the first advantage is one can scale up the solutions to get new solutions right. So, if you have two solutions you add up and you get a new solution and if you know the response of the system to existing set of inputs we call it as a basis set of inputs, then you can generate the solution to any other set of input by a linear combination of the responses to this basis inputs right.

For example if you know the response to a step input then you know the response to a impulse input by just differentiating the output right and the one can also show that the set of solutions of a linear system forms a vector space right. So, vector space has a structure when you define the notion of the so called addition and the so called scalar the so called vector addition and the so, called scalar multiplication right.

So, and if you one can show that the linear system forms a vector space and then we can use all the tools of a linear algebra to do further analysis right. The nice thing about one of the nice things about linear system is that one need not worry about the existence and uniqueness of the solutions. So, given a differential equation in general nothing can be said about it is existence and uniqueness if it is a non-linear system right.

So, of course, you need to put some conditions on the right hand sort of side of the differential equation for example, if the right hand side of the differential equation is locally Lipchitz then one is ensured of existence and uniqueness, but the linear system satisfies all those necessary the sufficient conditions for the existence and uniqueness. So, one you not bother about these two properties that is existence and uniqueness.

So, for all our purposes we can assume in fact, the solution exists and this unique and we have also studied the linear systems from a transfer function point of view so, characterized by poles and zeros. So, the linear system is completely characterized by poles and zeros.

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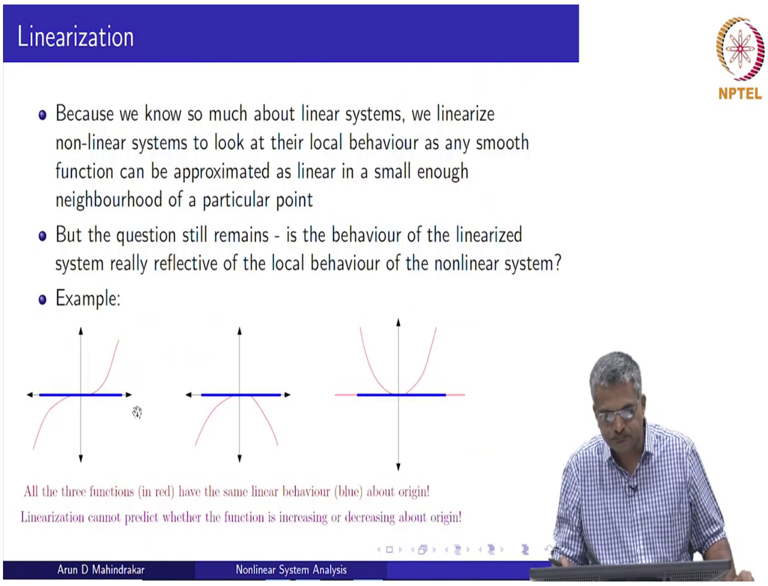

Now, since linearization is a very powerful tool because once you linearize you have a linear system and it is a well studied notion you know how to solve a linear system. So, the first step typically would be to linearize the system to linearize a non-linear system about some point. In fact, we will let us say about what are the kinds of points that we are interested were we would like to linearize right.

Now, but the problem also there is a problem associated with linearization as you can see in these 3 examples that I have a non-linear system and I have a linearized it about the origin. So, the linearized system is given by this lines the line segment the blue line segment right. So, in all the 3 examples even though the non-linear systems are quite different the linearization throws up the same behavior right.

So, this is what the what is lost when you do a linearization. So, it kills the non-linear behavior and the only thing you can comment is the local behavior whereas, you one loses the global picture one when one does the linearization so.

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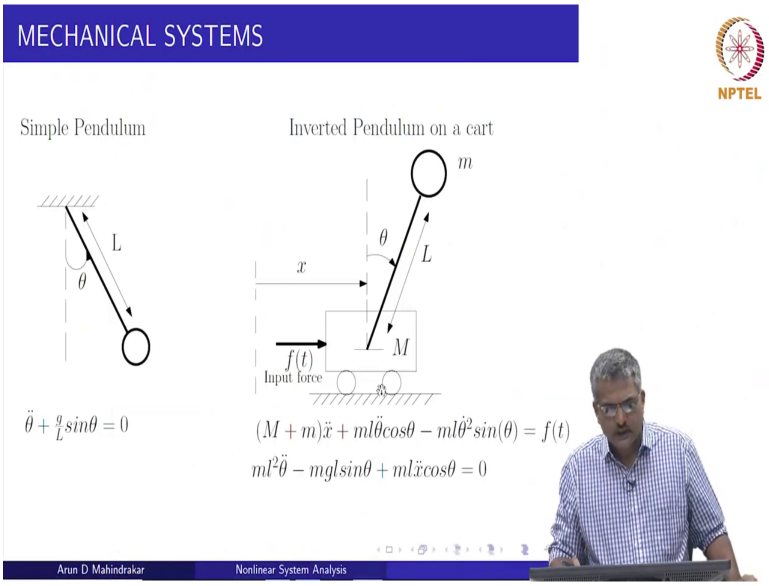

Let us go back go to a simple mechanical system the pendulum the simple pendulum. So, the simple pendulum consists of a massless rod right with the L length L and then there is a the mass is assumed to be a point mass all right with mass m here and the pendulum is it makes an angle theta with d vertical right and the whole thing can move under the influence of gravity.

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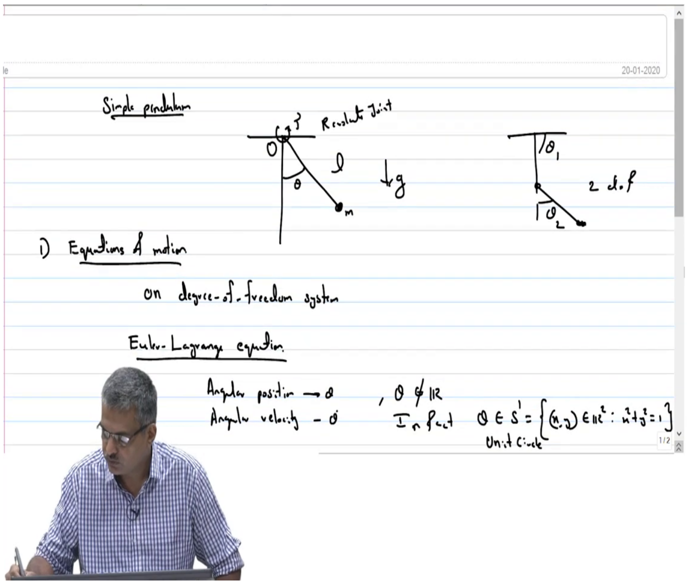

Now, we will assume that there is a torque that can be applied at this joint right of course, we can talk about unforced pendulum or a forced pendulum. So, we will talk about a forced pendulum think of a motor which is which actuates this pendulum right and we will also assume that the pendulum is free to move in the vertical plane it can make complete rotations in this plane right.

So, what are we going to study with this system? Right. So, we have a simple pendulum as a mechanical system. So, for the first thing is we will derive the equations of motions; equations of motion. So, to do that note that this system is a one degree of freedom system; that means, to specify the position of the any point of this pendulum I just require one variable in this case theta and that is why the system has one degree of freedom right.

For example one can think of 2 link manipulator right so, something of this form. So, I require one angle here theta 1 and the other angle may be with respect to the first one theta 2. So, to specify say the location of this point right I require a minimum number of 2 variables to describe the pose of the system. So, this system has 2 degree of freedom. So, let us not go too much into the degree of freedom right, but we now understand at least what is meant by 1 degree of freedom system or a 2 degree of freedom system. Now, the there are various ways to derive the equations of motion and we will use what is known as the Euler Lagrange equation ok.

So, first we identify what is the degree of freedom. So, the theta is the degree of freedom here I mean theta captures the parametrizes the degree of freedom here. So, the angular positions; angular position here is the theta and the angular velocity is theta dot, note that this theta does not belong to the real line.

In fact, theta belongs to what is known as a unit circle and what is this unit circle? This is a set of all points in R 2 such that x square plus y square is equal to 1 right, all the points on the unit circle because this is because we identify the 0 angle with 2 pi right. So, 2 pi and 0 are same whereas, if you were to represent it on the real line the 0 and 2 pi are different points right. So, the space in which the angle evolves is S 1. So, we call this as unit circle.

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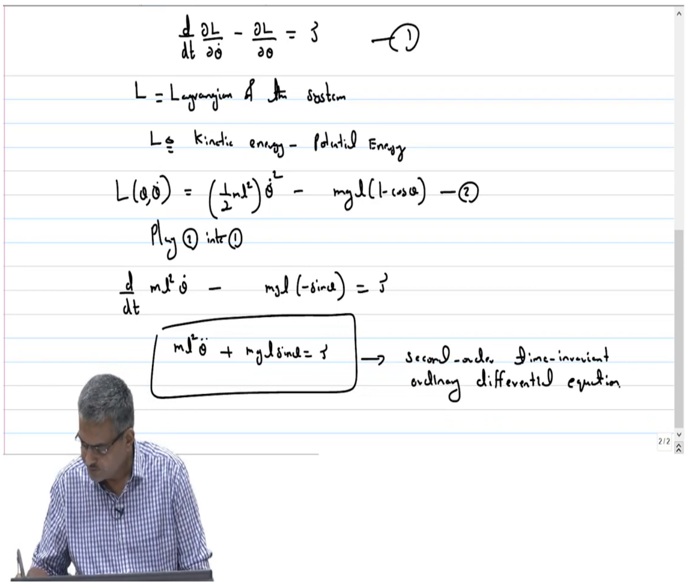


So, let us write down the Euler Lagrange equation for this. So, the Euler Lagrange equation is given by d by dt of dou L by dou theta dot minus dou L by dou theta is equal to all the external forces acting on the joint right. So, what are the external forces acting on the joint, there is only one joint here this is the joint right. So, let us call it as o and what is the external forces we will assume that there are no frictional forces. So, the only external forces is the torque applied to the joint right. So, we will assume that this is a revolute joint. So, there is a torque applied to the, if there was no torque then we will set this to 0 right, but we because we want to model a control system let us bring in the torque here right.

Now, what is the L in this, L is known as the Lagrangian of the system and how is it defined? L is defined as kinetic energy minus the potential energy. So, what is the kinetic energy here in this case? So, the kinetic energy is half ml square because we have assumed to be point mass into theta dot square right. So, ml square is the inertia of the point mass right.

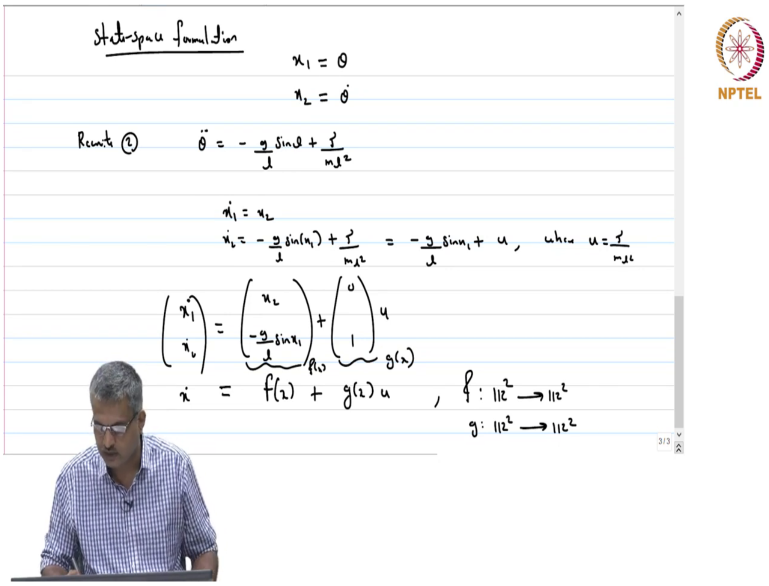
So, kinetic energy is half I the velocity square minus we will take the potential energy as mgl 1 minus cosine theta right well. Why did we take this? We took this because at theta equal to 0 we have 0 potential energy and at theta equal to pi we have the maximum potential energy which is equal to 2 mgl right. So, one can always add a constant to the potential energy and it will not change the equations of motion.

So, you can reset your 0 potential line the way you want right, but in our case we have taken the downward position as the 0 potential line 0 potential reference and the pendulum at the upward as the maximum potential reference. So, now, we have constructed the Lagrangian right.

So, the Lagrangian is a function of theta and theta dot right. So, let us plug this into 1 right. So, what we get? So, we have d by dt of dou L by dou theta dot all right. So, there is no theta dot in the contributed by the potential energy whereas, the kinetic energy contributes theta dot. So, you have ml square theta dot right minus dou by dou theta of the Lagrangian right.

So, there is no theta here there is a theta here right. So, this a one has to be careful about the signs there are too many minus signs. So, minus minus plus right and the derivative the cosine will yield one more minus sign right. So, that will be minus sin theta equal to the torque. So, we get left hand side ml square theta dot plus sorry this is we already took the derivative with respect to theta. So, we have mgl sin theta equal to tau right.

So, this is a second order time invariant ordinary differential equation. So, the course is all about studying p or the analysis of an ordinary time invariant ordinary differential equation, it could be second order or third order that does not matter, but what is what we are going to restrict is to time invariant and ordinary differential equation right. (Refer Slide Time: 16:42)



Well now, we have the equations of motion of the simple pendulum right. So, what do we do with this? Well as control systems we will try to put it in state space formulation right. So, the next step is state space formulation. So, state space formulation is not just restricted to linear system one can write the state space even for a non-linear system. So, we have to choose a states for the system.

So, how many states? There will be 2 states because we have a second order system right. So, we have x 1 and x 2. So, typically for a mechanical system choosing the states is a very easy task because what all that one has to do is to identify the positions and velocities, right. So, the set of positions along with the velocity their velocities will form the complete state space.

So, here we have the position as theta and the velocity as theta dot right of course, one can take a position and moment also, but again the momenta is related to angular velocity. So, it is a choice that we made that we took the angle and the angular velocity as the states of the system right.

For example, if you had a mechanical system with no with only linear motion then you would have taken position and the linear velocities as the states right. And again if you have a; if you have a double pendulum, then you would have taken the states as x 1 as theta 1, x 2 as theta 2, x 3 as theta 1 dot, x 4 as theta 2 dot right the positions and the velocities right.

So, having chosen the states let us write down the state space for the simple pendulum system that we have derived right. So, I have x 1 so, let us rewrite this equation let me call this as equation 2 right rewrite 2. So, I can solve for theta dot as minus g by l sin theta plus tau by ml square right I just divided throughout by ml square.

Now, if you the usual way that we do we take the derivative of x 1. So, x 1 is theta dot and theta dot is nothing, but x 2 and x 2 dot is nothing, but theta double dot theta double dot will come from this equation. So, it will be minus g by l sin of x 1 plus tau by ml square. We will for notational convenience we will simply write this as u, where u is tau by ml square, I have just redefine the variable as u because in controls u is identified with sorry with the control input ok.

So, again let us write down the system x 1 dot as a in a vector notation is equal to x 2 minus g by l sin of x 1 plus 0 1 times u note that I have rewritten the state space in a little special form and what is the speciality of this structure, well let me name it and then we will see what is the speciality. So, I will call this vector as f of x and this vector as g of x right. So, and the left hand side I will call it as x dot.

So, I have called this vector as f of x this vector as g of x. So, if you see I have written the pendulum system as in the in the state space representation, but I have given some structure to it and I called it as x dot is f of x plus g of x right. So, note that f is a mapping from R 2 to R 2 it takes x 1 and x 2 and throws up a vector right, in this case the vector is the vector of this x 2 and minus g by l sin x 1 and g also is a mapping from R 2 to R 2 right because it takes 2 elements and gives you a vector in R 2.

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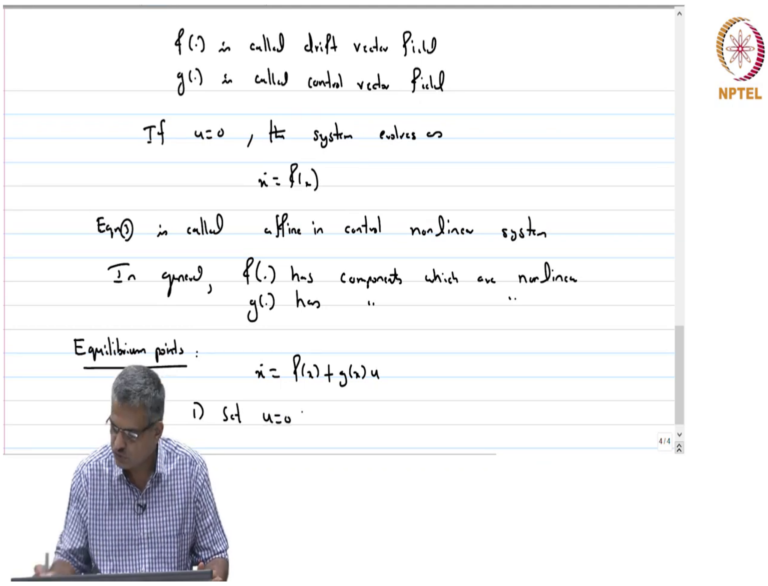

Well, if you note that notice that f the vector f in control the vector f is called drift vector field and the vector g is called it is associated with the control right note that the way I have split up the vectors is I have collected all terms which are not associated with the control right and I have collected all other terms associated with the control u right so, g is called control vector field.

Now, the nomenclature for g is quite obvious it is called control vector field because g is associated with the control right, f is called drift vector because for the reason that in the absence of control input right the system. So, if u were to be 0 then the system evolves as x dot is equal to f of x. So, how the x based on an initial condition x 0 drift is entirely determined by the vector f the vector field f of x all right. The control input plays no role because you have set control to 0, right and that is the reason f is called the drift vector field right. So, equation 3 is called affine in control non-linear system.

So, in general f has components which are non-linear and so also with g, in this particular example of the simple pendulum g is a constant vector field it so happens that, but in general it may not it may not hold right. So, both f and g are non-linear functions of x, the only difference between these two vector fields is one of them is associated with the control the other is not associated with the control right.

So, we call the one which is associated with the control is the control vector field and the one with not associated with control is called the drift vector field most non-linear systems can be expressed in this form right. So, I make no claim that all non-linear systems can be put in this form, but most of the non-linear systems can be cast in this form right what is this form affine in control non-linear system.

So, we have a state space representation for a non-linear system right in particular we took the simple pendulum, but one can as I said write it for many of the mechanical systems, chemical processes or biological systems or electrical systems and so on right. So, what do we do next right? So, let us find out the node let us define what is meant by equilibrium points.

So, let us see what is meant by equilibrium points; equilibrium point or equilibrium points or continuum of equilibrium equilibria and so on all right. Now, in this state space representation that we have x dot is equal to f of x plus g of x u right. So, set u equal to 0 that is no control.

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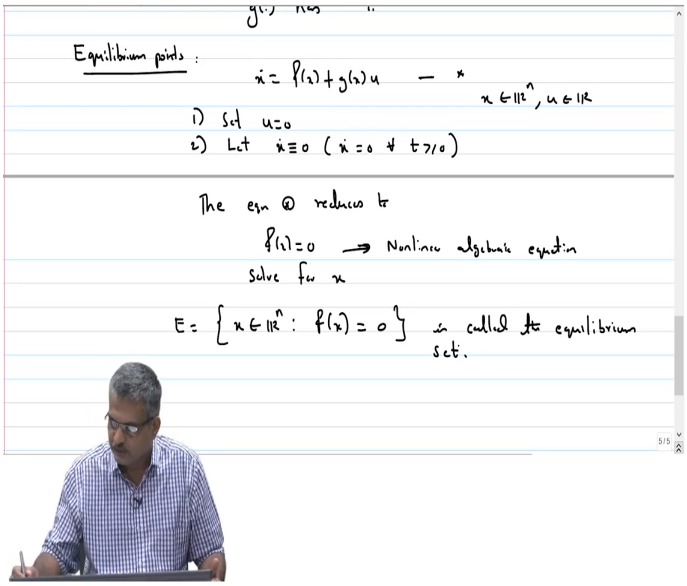

And second step analyze right or let x dot be identically equal to 0, identically equal to 0 means x dot is equal to 0 for all t greater than or equal to 0 right. So, what do we get? We this equation reduces to already we have numbered it whatever it is. So, reduces to f of x equal to 0. So, f is a non-linear function of x right. So, solve for x from this non-linear algebraic equation right. So, here what we have is a non-linear algebraic equation right. So, the x it may be unique x, it may be unique and isolated or it could be multiple valued right or it could be a continuum we will see that little later.

So, this x right so, let us collect all those x which satisfies this algebraic equation f of x equal to 0. So, let me call that set as E. So, it is all x in well we have taken a general system. So, let me take a here x as belonging to R n we will take a single input R. So, the set of points in R n such that f of x equal to 0 right is called; is called the equilibrium set. So, what does it mean? That means, there is no control input and the system. So, what are those states, at which the system can in the absence of any control input the system can stay identically at that position right for all time t.

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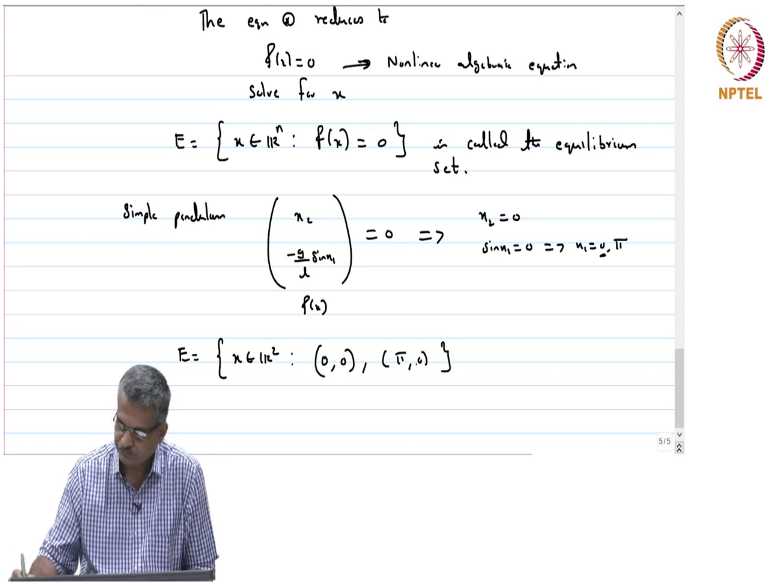

So, let us see what this equilibrium points mean for the simple pendulum right, of course, from the physics of the system we already have some notion of equilibrium points we know that no we identify the equilibrium position of the pendulum as the there are two equilibrium positions the lower and the upper equilibrium positions right. So, we know a little bit from the physics, but let us see with our definition of equilibrium points do we get the same points right.

So, for the pendulum system so, we had the vector field x 2 so, f of x for the pendulum simple pendulum so, we had x 2 and minus g by l sin of x 1 is equal to 0 right because this is a drift vector field. So, if you solve this it implies x 2 equal to 0; that means, the velocity has to be identically equal to 0 right and sin of x 1 equal to 0 implies x 1 can be 0 or pi you take 0 and pi because we identifies all the solutions in the on the unit circle right, we do not count the multiplicities of this solution.

So, for the simple pendulum the equilibrium points are all x in R 2 such that. So, you have 0 and 0 we have to take the order ordered pair right first comes x 1 second the x 2. So, 0 and 0 is one equilibrium point and the other one is pi and 0 right. So, the equilibrium points for the simple pendulum are there are two equilibrium points they are isolated and isolated in the sense that in the vicinity of the this equilibrium point there is no other equilibrium point right in a small neighborhood of that; that means, you can always find a small open set around this equilibrium point such that there is no other equilibrium point right.

So, these two are isolated equilibrium points and there are so we have two equilibrium points which is in tune with the physics of the system that we already know from the physics of the system that the pendulum has two equilibrium points right and we exactly got those 2 points by our definition of equilibrium points.

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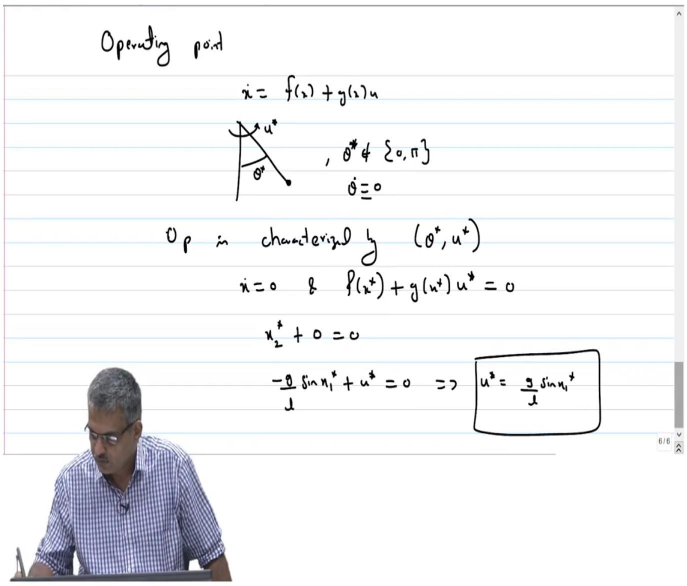

So to recap we model the system in state space representation we put it in the standard form which is called the control in a affine in control non-linear system right and then we once the system is in this standard form right we know how to write the equilibrium points of the system or how to solve for the equilibrium points right. Having identified equilibrium points we will also need to understand what is meant by operating point. Well, so earlier we talked about equilibrium points when there was no control, now let us bring in the control let us apply a torque to the pendulum suppose we were to hold the pendulum in certain angle not equal to 0 or pi; that means, other than the equilibrium points.

So, you want to say keep the pendulum in this position right so; obviously, we have to apply a control to compensate for the gravity right. So, this point this position is known as the operating point. So, the notion of operating point comes in various systems for example, in power system we will talk the operating point is frequency some say rated field in India we have the rated frequency as 50 Hertz the rated voltage for the consumer is 230 volts. So, that becomes the operating point right.

So, you want to why are we interested in these points well we would like to do some kind of linearization about that point and infer the system we want to do local analysis of the system about that point right. So, it is imperative that we first understand and extract what are the equilibrium points and what are the operating point right. So, again let us go back to our pendulum system. So, we had x dot is equal f of x plus g of x u.

So, assume that I want to keep the pendulum at some theta star right, where theta star is not equal to 0 and pi; that means, I am excluding the sorry this is star not theta dot theta star I want to exclude their equilibrium points ok. So, and I want to maintain the pendulum at this position. So, theta dot is 0 identically equal to 0.

So; obviously, I will have to apply some tau star right or in our language we have replaced tau by u. So, let us call it as u star. So, what is how do we get this u star? So, the goal is to the goal is to find this operating point right. So, already in the sense the operating point is already fixed somebody has given us this theta star. So, what should be the u star? Right.

So, the operating point is characterized operating point is characterized by both the position and the control right these two will completely specify what is the operating point. So, let us go back to this equation our pendulum standard equation. So, my x dot has to be 0 because I want to hold the pendulum at that point right; that means, my theta dot has to be 0 identically equal to 0; that means, theta double dot also has to be identically equal to 0, right.

So, x dot is 0 and then we have f of x star because when the system is at some operating point let us call it as x star plus g of x star to u star equal to 0. So, what does this boil down to for the pendulum? So, you had x 2 star right plus g had 0 and 1. So, there is 0 here. So, that tells us that the velocity has to be 0 and the second term second component is minus g by l sin of x 1 star x 1 star is nothing, but theta star right plus u star equal to 0. So, which implies I think it has to be yeah this is u fine.

So, u star is nothing, but g by l sin of x 1 star. So, this is precisely the effort required to hold the pendulum at an angle theta star right. So, this is this torque compensates for the gravity.

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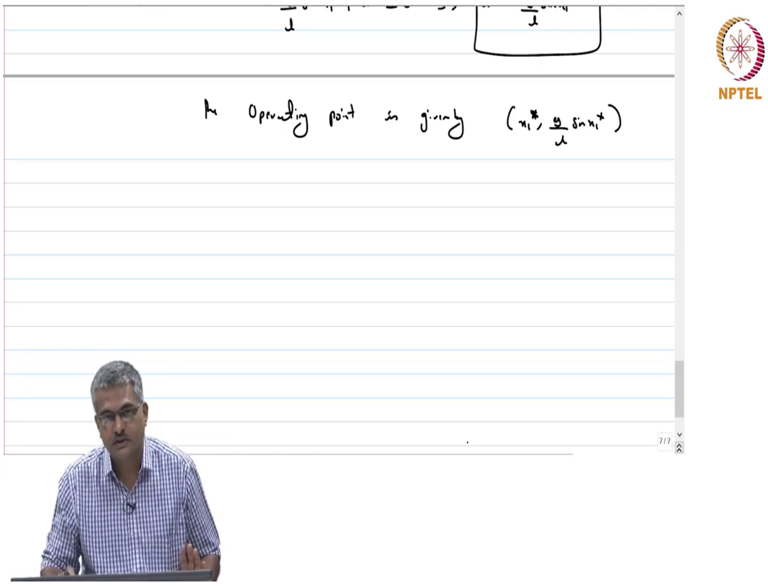

So, unlike an equilibrium point where u is 0 in an operating point u is not 0 there is some control that has to be applied to keep the position keep the system in that position right. So, the operating point is given by theta star right and g by l sin of x whole star or I can replace it by x 1. So, we are now we know how to find an equilibrium point and then and also the notion of operating point right. So, then final step in this analysis is how to do the linearization, right. So, that will be the topic of our next lecture.

Thank you.