Multirate Digital Signal Processing
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Lecture – 04 (Part-1)
Reconstruction Filter – Part1

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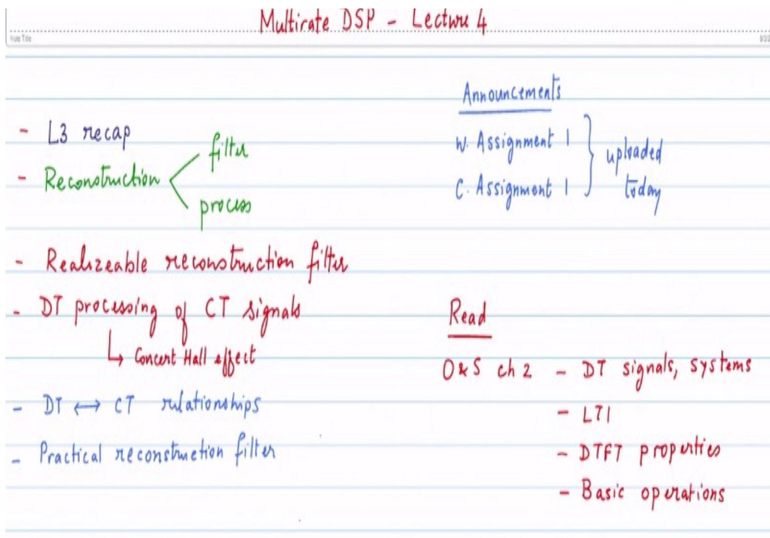

So, lecture 04, the topics for, to be covered are the following. As always we will do a quick summary of what we have done in the last class but with a little additional information, so that it is not just a repetition, some additional perspectives, today's lecture we will look at the process of reconstruction in terms of the; last time we talked about the filter that we will use for reconstruction.

So, today we will talk about the process of reconstruction, then from an ideal reconstruction filter, how do we work, or way towards realisable reconstruction filter, so it is a 2 stage process by which we get the realisable filter and we will be talking several times about discrete time processing of continuous time signals, so multiple times we will visit this concept because I want you to feel very comfortable about going from continuous time to the discrete time where the discrete time sampling is different.

So, basically in order to get a good understanding of the multi rate systems, we keep referring back to the continuous time and this is a very good example of what we could do, the relationships between continuous time and discrete time in the frequency domain, what happens when you go, when you do the process of sampling, when you do the reconstruction, this is a very, very important framework that is covered in today's lecture.

We will also talk about a, some aspects of how do we develop a reconstruction process, what happens in a D/A for example if that for an understanding, what we are assuming is that all the students are familiar with the; with what contents of Oppenheim and Schafer chapter 2, primarily that would be the definitions of discrete time signals and systems, understanding of the LTI, the discrete time Fourier transform, it's properties and basic operations.

We will just refer to some of them in passing in today's lecture that is our underlying framework, we are focused primary is on Oppenheim and Schafer chapter 4 for the time being, written assignment number 1 and computer assignment number 1 both will be uploaded in a few hours' time please do take time to look at it and get started, the Tuesday hour is for doubt clearing, it is not an official lecture but that is the time that the TAs and I will be available to answer any doubts, okay.

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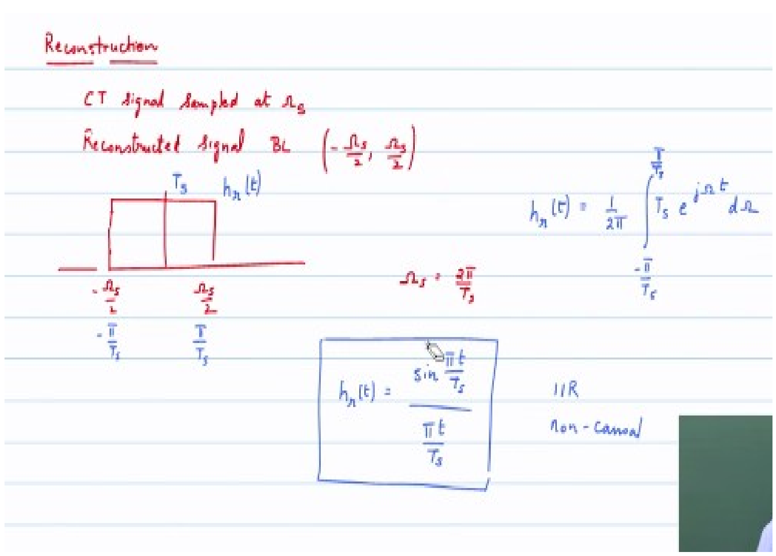

So, just to pick up where we left of, the; we said in the last class, we made the statement that if you wanted to look at a continuous time signal sampled at omega s, we will get a band limited signal when you do the reconstruction process, it will be limited to omega s/2, from –omega s/2 to omega s/2, the ideal reconstruction filter is one that will get rid of the images, notice that the images will be located at multiples of omega s, this goes from –omega s/2 to omega s/2 and 0 everywhere else.

So, it basically removes all other images except the k = 0 version of the; or the primary image, now the other aspect is the scaling, so keep in mind that there is a scaling element which is part and I sure you would have been able to verify that the corresponding impulse response of an ideal reconstruction filter is obtained like, is basically a sinc function, okay we will come back to looking at it in a little bit more detail.

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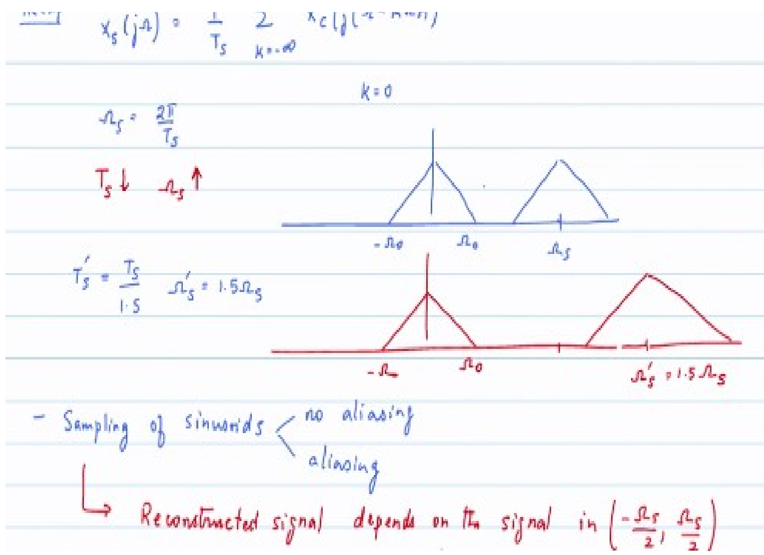

But let us just summarise the or recap all the key results that we have said so far, so the sampling process Xs of j omega, the spectrum of the sampled signal, we said has a scale factor 1 over Ts summation k = - infinity to infinity Xc of j omega – k omega s again, this is more sort are very familiar and comfortable with the expression, okay the scaling factor the multiple, so k = 0 is your underlying continuous time spectrum, multiples of k + -1, + -2, you get all the shifted versions and those are the images that occur during sampling.

Those are the ones that you need to remove when you do the reconstruction process, okay the relationship between the continuous time signal, the sampling frequency, omega s, sampling frequency = 2 pi/Ts and very important relationship as you reduce Ts that means, your sampling period is reducing, sampling frequency is increasing, okay that is a very important relationship as I suppose it is obvious from the fact omega = 2pi/Ts that they are inversely related.

What is more important is the corresponding frequency domain interpretation which I would like you to keep in mind so basically, if I have a signal that has got from –omega0 to omega0, it is the band limited signal, I sample it at some omega s that satisfies Nyquist frequency; Nyquist criterion, omega s, so there is no overlap of the spectrum and now if I specify Ts dash = Ts/ 1.5, just for interest, just to see what happens.

So, this basically means that my corresponding sampling frequency, I have reduced my sampling period, so increase my sampling frequency by 1.5 omega s, the original, so the new version that we will get us in terms of the spectral representation, the original spectrum – omega 0 to omega 0, so now instead of omega s, it will go to 1.5 times omega s, so basically the spectrum; the copies of the spectrum now become shifted apart.

So, this is omega s dash that is = 1.5 times omega s, okay, so this is the underlying process again, being it to quickly relate to what is happening in the time domain to what is happening in the frequency domain, very important. Now, in the last lecture how we did of some amount of discussion on the sampling of sinusoids, okay, sampling of sinusoids, we looked at a case where there was no aliasing and we did a reconstruction.

We got; looked at the case where there was aliasing and also looked at the reconstruction, so the statement that we can make as far as the sampling of sinusoids is concerned and the reconstruction, is that the reconstructed signal; the reconstructed signal depends on the spectrum that lies between – omega s/2 to omega s/2, it does not matter whether that was part of the original spectrum or one of the shifted spectrum that is what we saw in the last lecture.

So, depends on the signal in the range – omega s/2 to omega s/2, in the case of sinusoids with the aliasing, we found that what lies within the spectrum is also a sinusoid but of a different frequencies, so that was the observation okay. Now, in terms of the ideal reconstruction filter, we had looked at this information, so we would not repeat that.

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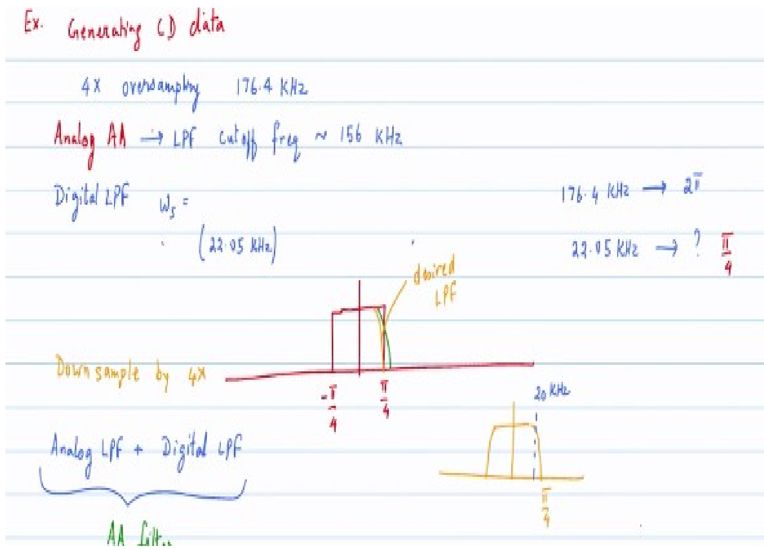

Now, let me just sort of mention the key observation from the example that we looked at that was the example of generating CD data, compact disc, generating CD data, I will not redraw the figure, I am sure you can refer to it, the key principles that we said was we were doing 4x over sampling, okay, so 4x over sampling was the strategy, so the sampling frequency is 44.1 went to 176.4 kilohertz, okay.

Now, the reason for doing that because we had an analogue anti-aliasing filter; anti-aliasing filter really was not an anti-aliasing filter because it allowed aliasing to occur, this was actually turned out to be low pass filter with very relaxed criteria, the cut-off frequency as we mentioned in the last class, cut-off frequency was actually not omega s/2 but something well beyond omega s/2, in fact we said you could go as high as 156 kilohertz, okay.

Now that is interesting, I hope you had a chance to look at that and sort of feel comfortable with it that this is a analog filter that will allow aliasing to occur as part of the sampling process, does not affect us in the final result because there is a digital filter that follows; there is a digital low pass filter that is after sampling, digital low pass filter, what was these stop band cut-off for this digital low pass filter? It was in terms of actual frequency it will be 44.1 divided by 2.

So, this should be 22.05 kilohertz, okay but digital frequency; when you represent it in terms of the discrete time filter that there is no kilohertz notation, it has to be only radians, so how do I convert it to radiance, my sampling frequency is 176.4 kilohertz that corresponds to 2pi, okay that is my sampling frequency. So, now the question is if I have 22.05 kilohertz, what does that correspond to; pi/4, okay.

So, this is a low pass filter; the digital filter is a low pass filter with cut-off pi/4, -pi/4 okay that is if you could design and ideal low pass filter but in we are basically remember this is a practical system, so do I design a low pass filter that looks like this or do I design a low pass filter that looks like this? Which one is the one that I am allowed to do, so if I want to strictly satisfy the anti-aliasing property, I have to go with the orange one that is the desired or the desired low pass filter that digital low pass filter.

Remember pi/4 corresponds to 22.05, if anything outside of that is allowed then I will run into problems with the effect of aliasing because there is some unwanted signals that are present because my analogue filter allowed the things to creep in, okay now having designed a low pass filter which looks like this, practical low pass filter, so the edge, this is 20 kilohertz, you have to translate it to discrete time; discrete frequencies.

This is; this corresponds to pi/4 that is the filter that we use and the last step that we did after the digital filtering was down sample by a factor of 4; down sample by 4x, okay, so the combination of the analog low pass filter; low pass filter plus the digital low pass filter effectively this was my anti-aliasing filter, okay this is my anti-aliasing filter, the advantage that I have gained through this mechanism is that the complexity of the analog filter was shifted over to the discrete time domain.

So, the digital filter could be the sharp filter and effectively the aliasing is not present in the signal band of interest, okay so again this is an example but it sorts of highlights the important results that we want to emphasise okay. We also need to make sure that we are formally stating the relationship, so let me just mention that very quickly before we move on to the rest of the lecture.

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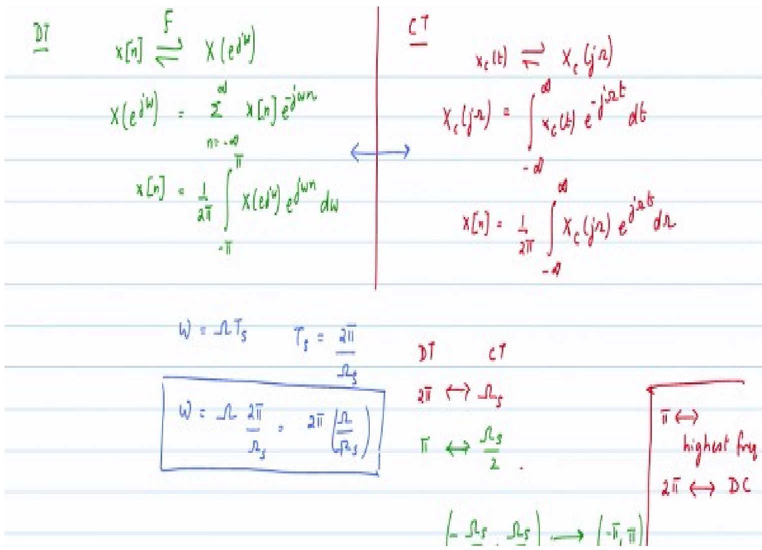

So, the discrete time relationship, if I have a sequence; discrete time sequence x of n, I talk about a discrete time Fourier transform which is Xe of j omega and the relationships Xe of j omega is continuous in omega periodic with period 2pi summation n = - infinity to infinity x of n e power – j omega n, okay and x of n can be obtained by the inverse transform 1/2 pi integral –pi to pi because anything outside of a –pi to pi range is a repetition, Xe of j omega e power j omega n d omega.

Again, this is something that you are familiar with but I needed to write it down, so that we are comfortable with what is the basic definitions and very often in DSP, we do not really worry too much about what happened in the continuous time, but we need to be fully aware because we are going to be going back and forth, so the underlying continuous time signal, so if it is xc of t with a Fourier transform xc of j omega, let me just write that down for completeness.

Xc of omega is an infinite integral - infinity to infinity xc of t e power -j omega t dt, again you can see a close similarities, the inverse transform in this case also turns out to be an infinite integral, it is 1/2pi -infinity to infinity xc of j omega e power j omega t d omega okay so that is more or less sets the framework of what we are trying to do, then what is left is that the link between the continuous time and the discrete time.

Remember the link that goes back and forth, one in one way it is sampling, the other way is the reconstruction process, so the discrete time frequency omega is related to the continuous time frequency through the sampling period Ts; Ts is 2pi divided by omega s, so basically any frequency omega, continuous time can be mapped to the corresponding discrete time frequency using this following 2pi/omega s or 2 pi times omega/ omega s, okay it is as if you took the continuous time frequency, normalised it by omega s, and then mapped it in the range 0 to 2pi, omega s (()) (16:22) to 2 pi, so again this is the useful relationship simple one but it helps us be able to map from one domain to the other, now comes an important, it is more of an observation, so omega s is the sampling frequency, that is the frequency with which the spectrum repeats now in the discrete time, this corresponds to 2 pi, so 2 pi in the discrete time, discrete, I will just put it as discrete time, this is continuous time, frequencies 2 pi maps to omega s, okay.

And very often we are interested in omega s/2 because that is the portion of the spectrum that will get reconstructed so that means anything from minus, from in the discrete time in the discrete time represented (()) (17:17) of interest, okay, so typically the reconstruction process is in the range –omega s/2 to omega s/2, this would correspond to in the discrete domain, this would correspond to –pi to pi, we look basically, looking at one period of the discrete time fourier transform, the spectral properties, okay.

And another sort of related observation now, when I go from 0 to 2 pi, where does my highest frequency occur in the discrete when I am looking at the discrete time representation? It occurs at pi, so basically an observation that we would note down here maybe to the side is that the pi corresponds to the highest frequency; highest frequency which basically also says 2pi will wrap around to DC, okay.

So, again that is just by way of all the different relationships that we have, okay, so I think we are now ready to take on a new topic in our discussion that is the process of reconstruction.

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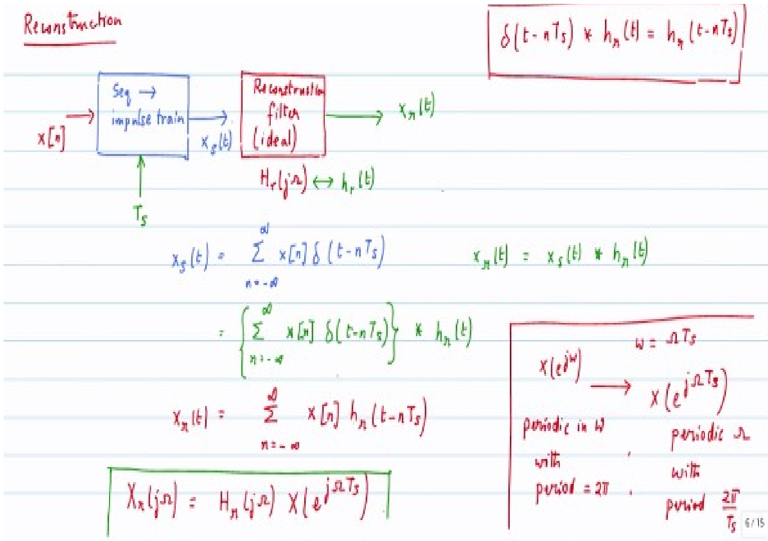

Again, it is a subject that you would have studied in basic DSP but our interest is to (()) (18:54) of this because we will be looking at it multiple times in the context of multi rate system, so the process of reconstruction so the basic underlying system is that I have a sequence x of n, I must reconstruct it into a continuous time signal, xr of t and the steps that we would need to take, the first step is going to be that I need to go from the discrete domain, where I am representing in terms of numbers, it is a sequence of numbers to something in the continuous time.

So, from a sequence, we need to convert it into an impulse train, okay, then once you have it in the form of an impulse train, then it is a continuous time signal, let us we call it as xs of t, then this one has to be, will be reconstructed by removing the unwanted portions of the spectrum, so this is going to be the reconstruction filter and for the moment, we are going to assume that it is ideal we will remove the ideal assumptions very shortly.

But for now, we will assume that this has a spectrum Hr of j omega and then based on that we produce an output which is the reconstructed time domain signal, okay so the inputs that we need to give; one is reference point which tells what is the spacing between that so, there is a underlying Ts that needs to be specified, when I do the reconstruction process because otherwise, these are just samples, I could interpret them as one milli second spacing or I could implement; interpret them as 100 milli seconds between them.

So, we need to specify at what sampling period we are going to do the reconstruction, so writing it mathematically gives us the best insight, so let us write down the process that we have written down, so xs of T, the impulses, so it takes the sampled values x of n and generates the corresponding dirac deltas, t – nTs, notice that Ts sampling period has come in to play and the shift of the delta function based is depends on the sequence number.

And this is n = - infinity to infinity so and the next step which is the reconstruction step xr of t, if this is xs of t and this can be related to an impulse response Hr of t, it is an LTI system, so there is an impulse response, so xr of t will be xs of t convolved with Hr of t, very; Fourier transform relationship basically, I am looking at the input output of a LTI system, the impulse response convolved with the input gives me my output.

So, if you write down the expression now, you get a very interesting result, so it is summation n = - infinity to infinity x of n delta of t – nTs, this function convolved with Hr of t, okay, now I would like to invoke a result that I am sure you are familiar with, in the continuous time and in the discrete time but the continuous time property basically says that if I have a delta function which is; which occurs at a value nTs and I convolved this with Hr of t.

What do I get? Basically, the impulse response will shift to the centre will shift where the impulse occurs, so Hr of t – nTs, okay, so that is the result that is know when I convolve anything with an impulse function basically, the function is preserved but is only shifted to where the function occurs, so this then tells me that the reconstructed signal can be very nicely expressed as n = - infinity to infinity x of n times hr of t – nTs.

And this is a; I am sure this is a not an new equation but I want you to sort of look at it with a fresh pair of eyes so basically, what we are saying is that the reconstruction has gotten a certain impulse response, and we showed that the ideal impulse response is a sinc function. Now once we get to the point where we have written the reconstructed signal as shifted versions of the reconstruction filter impulse response.

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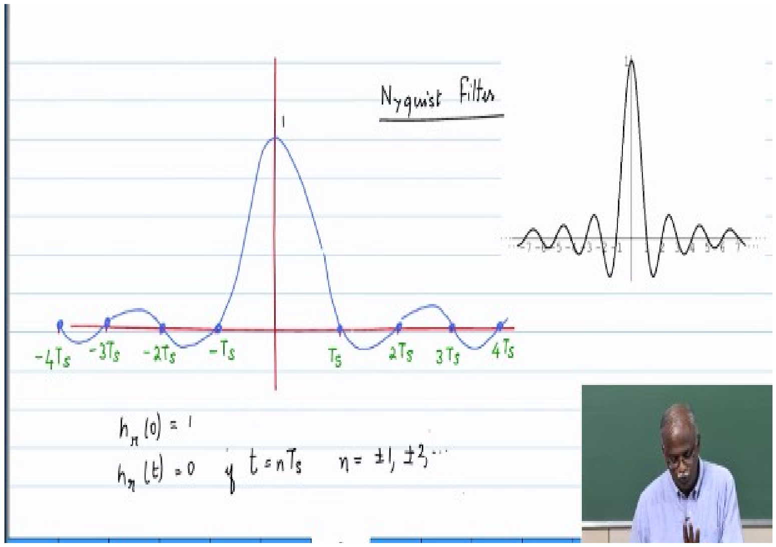

But each one appropriately scaled by the sample value, okay, so now here is the visualisation, need to look at it basically, if I were to sketch the hr of t ideal reconstruction filter, it is a sinc function, it has zero crossings at Ts, 2Ts, 3Ts, 4Ts and so on and in between, the ringing function, okay, so I have sort of shown you what it should look like this is what you know this is expanded view just to give you a feel for it.

But basically, the important thing is to look at the response of the impulse of the reconstruction filter.

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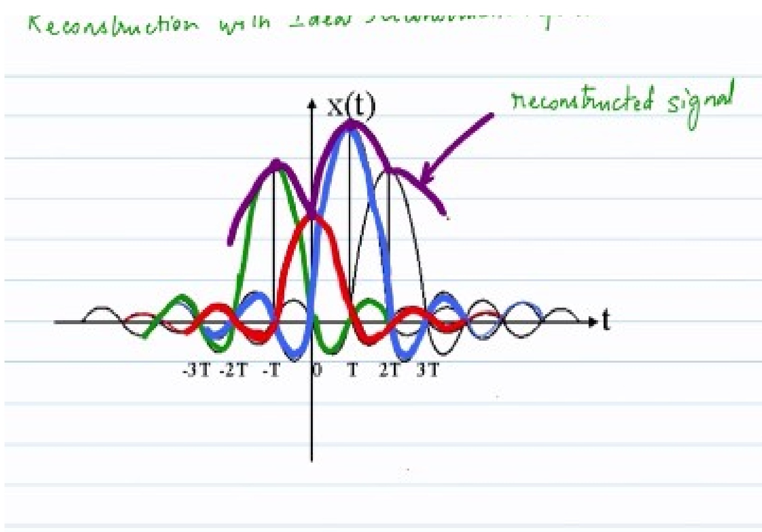

Now, comes the reconstruction process, so if I have which is the sample at time t = 0, I have to produce a copy of the reconstruction filter which is the red version, so that is the sample; that is the reconstruction filter centred at t = 0, then at t = 1, it is the blue copy of the reconstruction filter and then at t = -1 is a green copy, so you have the green copy sort of occurring the reconstruction filter the red version the blue version, notice at the sampling points all of them (()) (25:27) so that is the very, very important property.

Because what that tells us is when I (()) (25:36) the resultant signal will have same value as the sampling points, those sample values will not be altered everything in between gets filled it but the sample values will not be altered and how do I show that so basically, the purple one is the sum of all the signals, so basically it is the (()) (25:59) whatever of take all the signals that are following in a certain window and we will find that at the sampling points you are exactly coinciding with sample value x of n.

And then at all the other points (()) (26:12) signal, so the ability to visualise the reconstruction process in this manner is very, very important and what happens in terms of the process and why is the Nyquist property is so important, why is it that we have to worry so much about the zero crossings? Because if these zero crossings were not satisfied, then the original sample values would have gotten altered, so that is the reason why, so this is the one of the important criteria that we will use or we would like to have an insight into, that the impulse response goes through zero crossings at Ts, 2Ts, 3Ts.

And that is sometimes referred to as the Nyquist property, okay again we use the word Nyquist so many times on this in this particular discussion, this is a Nyquist filter or filter that has this Nyquist property in the time domain, okay. Another result may be we write it in the place where we have written all the equations, that we would need to, and this is the introduction of the frequency domain elements in our discussion.

So far this is the time domain version of it, the continuous time signal xe of j omega, reconstructed to form a continuous time signal, right and in the process we went from the discrete time omega to the continuous time omega and then passing through the reconstruction process, since we know that omega is related to omega times Ts, we can also say that yes, omega is just a variable whether is the discrete time omega or the continuous time omega.

So, I can very well write this xe of j omega as xe of j omega Ts, am I right because that is the relationship between the continuous time omega and the discrete time omega, now this is very important because this as a function with which is periodic in omega with period 2 pi, okay this is the discrete or the normalised frequency now, this happens to be a function that is periodic in omega, I have just mapped it.

So, with period; with period = 2pi/Ts, okay so the resultant; the reason for this particular expression is as follows; Xr of j omega, the reconstructed signals spectrum is the spectrum of the reconstruction filter, Hr of j multiplied by the continuous time representation of the spectrum of x of n, am I right, basically, the spectrum of the signal at this point and that I can if I; if there is no loss of information going from x of n into xs of t, then I can write this as x of e of j omega Ts, okay.

Now, we will come back to looking at this expression a little bit more because, this is a very, very key result, because what it is saying is there is a continuous time spectrum which has got lots of replicas. Where did the spectrum come from? The spectrum come from, it came from the discrete time signal, also got different spectra, so what did I do? I took the discrete time spectrum (()) (29:59) all spectrum representation, and then I chopped off everything with outside a certain window, outside of the window that is specified by the reconstruction filter.

So, this equation is a key question and we will come back to looking at this multiple time in our discussion, okay, so the reconstruction filter is a sinc function or it has got a Nyquist property, it does not have (()) (30:27) reconstruction filters as well but this property is very important that you do not change the values at the sampling points, okay and this is the reconstruction process it basically, it turns out to be a superposition of the several repetitions of the reconstruction filter impulse response with the appropriate scaling that is very important for us.