Multirate Digital Signal Processing
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Lecture – 37 (Part-2)
Orthogonal Frequency Division Multiplexing - Part 2

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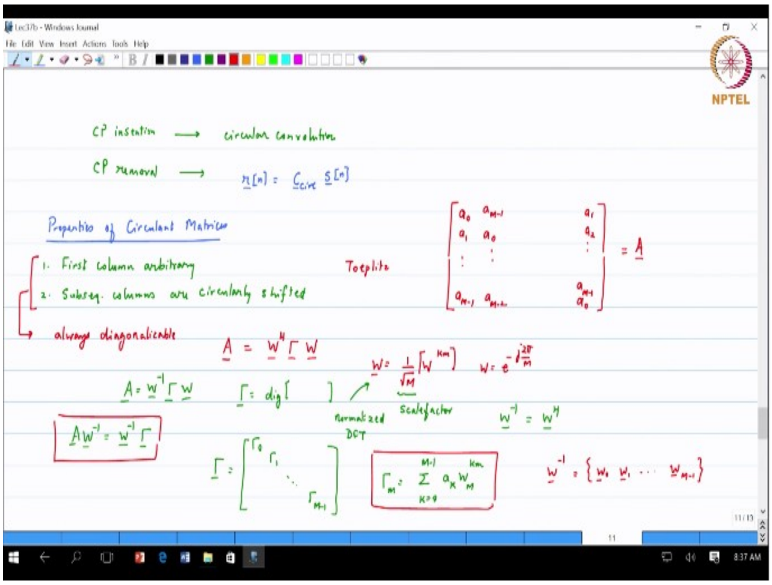

Okay so we will take a small detour because this is a very important result. We would look at one of the essential properties of a circulant matrix. Actually, for any circulant matrix of this form or circulant matrices and we are going to define circulant matrix or the structure that we are going to define is the following. The structure that we are going to define is the first column is arbitrary.

If you can choose the circulant matrix with this, first column is any values that you can think of okay and these subsequent columns are circularly shifted. So basically the lowest the last term in that column will show up at the top, circularly shifted okay. So clearly there should be one more family of circulant matrices where you do shifting of the rows. So we are looking at the shifting of the columns.

Because that is the matrix that we have to work with okay. The claim is for this family of circulant matrices that you are the matrix is of course there is this matrix will be Toeplitz okay. This type of circulant matrix is always diagonalizable okay. So always diagonalizable with actually a well-defined structure okay.

So if I take a matrix, so let me just take construct a matrix not necessarily our matrix. So any arbitrary one a0 a1 through aM-1, next column will be aM-1 a0 aM-2 and the last one will be a0 at the bottom a1 a2 aM-1 as the entry. So this is the general form of a circulant matrix. The claim is that this matrix if this is denoted as A.

A is always diagonalizable and the structure of the diagonalized form is W Hermitian lambda times W, where W is the DFT matrix that we are familiar with, with a slight modification that it is the normalized DFT matrix 1/root M W KM okay, normally the DFT matrix has got a 1/M factor in the IDFT, so you take 1/root M with that forward transform 1/root M with the inverse transform it exactly you get the same thing okay.

So where W is=e power –j 2pi/M exactly the DFT matrix but with this extra 1 over so this is a scale factor, normalizing scale factor and so this is referred to as a normalized DFT. Again, but for the scale factor it is exactly what we normalized DFT. The advantage of this is that W inverse is actually equal to W Hermitian. That is the reason why we look at this okay. So what we have said is this matrix A is=W inverse times lambda times W.

Post multiply both sides by W inverse, we get by the way and lambda is a diagonal matrix. We will just tell in a minute what the entries of the diagonal matrix are okay. So if you post multiply then what you find is that A times W inverse is=W inverse times lambda okay and the lambda is equal to a diagonal matrix which is lambda0 lambda1 to lambda M-1 where these coefficients lambda M are nothing but the DFT of the first row, first column of the circling matrix.

So this is K=0 to M-1 a subscript K WM KM, so basically so it is very tightly coupled to the whole DFT properties. Take the DFT of the first column and then you actually use the DFT matrix as part of the diagonal structure. So this is what this result is showing okay. So let just zoom in and look at this result. So if I can write W inverse, W inverse as in terms of its columns, let me call them as W0 W1.

These are the columns of W inverse matrix, WM-1. If you now visualize it in terms of the column vectors, this is nothing but the eigen equation. So basically the columns of W inverse are the eigenvectors and the eigenvalues are the DFT coefficient.

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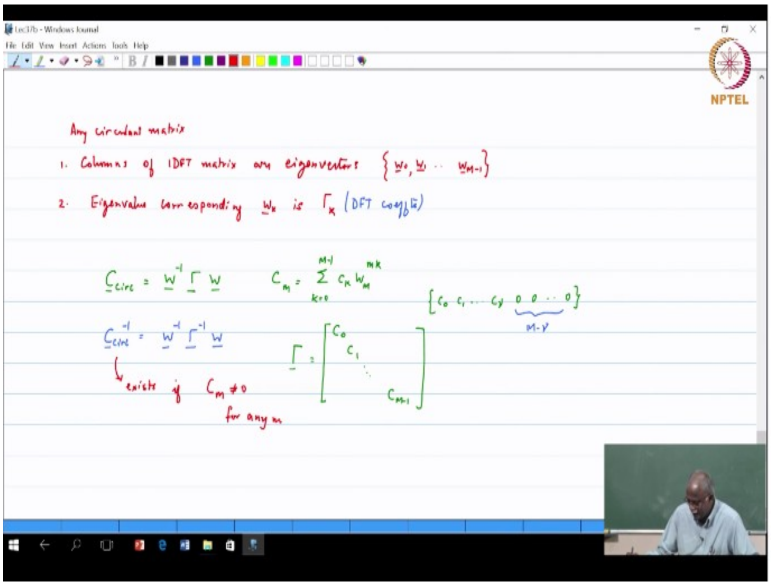

So let me just summarize that. For any circulant matrix of the form where you are doing certainly shifting of the columns, any circulant matrix the result is that the columns of the IDFT matrix are the eigenvectors so basically the eigenvectors are W0 W1 WM-1. These are the columns and the eigenvalue corresponding to any given eigenvector, eigenvalue corresponding to any WK is lambda K which is the Kth DFT coefficient okay.

So these are the DFT coefficients, the corresponding DFT coefficients okay. Very good, so the way it benefits us is that we now have C circulant, I can write it as W inverse lambda times W where lambda K will be=summation. So let me write it as lambda M, lambda M is=K=0 to M-1 CK WM MK okay and the vector that we are doing has got only up to nu-1, so nu so basically the corresponding vector that we are working with is C0 C1 up to C nu and then there are zeros effectively.

Because you are taking an M point DFT. So how many zeros are there? This corresponds to M-nu okay. So effectively the structure is very similar to a generic one but this is what we have as the expression okay. Now C circulant inverse, C circulant inverse is=W inverse lambda inverse W okay. So C circulant inverse exists if lambda inverses basically those diagonal elements are not 0.

So this one exists if lambda M not equal to 0 for, no wait, how did I define, defined in terms of okay. Actually, since we are talking about the C coefficients maybe we just let us instead of this we will call it as CM, let me call this as CM. So the diagonal matrix lambda is the DFT coefficients of the channel coefficients C0 C1 CM-1. These are upper cases, so please okay upper case C's DFT coefficients.

So the inverse will exist if none of the channel coefficients C sub M are not equal to 0 for any M okay. Now here is an important interpretation and I think this is where the coming at it from the DSP side actually adds a lot of value. Now what are the C coefficients? What are the C coefficients if you think about it? What are they? It is the frequency response of the channel, correct frequency response of the channel.

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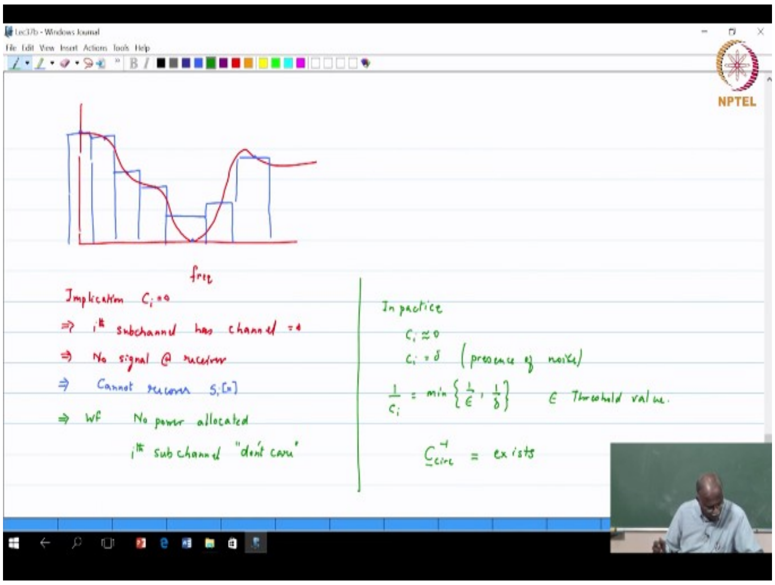

So if I were to think of the channel, this is frequency and think of some frequency response of the channel and deliberately I will make it touch 0 to make sure that we get this. So some channel variation where it touches 0 and something okay, some channel variation is there. Now what was the task of the DFT or the multicarrier modulation is actually to divide this into subchannels.

So basically so you took this particular value that was the gain for this subchannel. The next one had a gain here and the next one had a gain like this. So these are the subchannels, in the multicarrier scheme these are the subchannels that we introduced and so somewhere here is the channel maybe just for let us take it at 0 okay. So there is one subchannel for which the tap is 0. Again, please draw it so that okay.

And then afterwards these are okay. So multicarrier scheme we know to do water filling all of those concepts are there. The key element that is okay I just want to okay, this is a little wider just to accommodate the fact that I want a 0 okay. So what is the interpretation that I cannot invert the C matrix? So which means that I cannot get back any information that was transmitted on that subchannel.

Just think about it, you transmitted information on some subchannel where the channel gain was actually 0. So which means there is no way you are able to recover it okay, so the implication of CI being=0. Implication is CI of CI being=0 implies that the ith subchannel has zero gain, zero channel gain okay, has channel gain=0. So in water filling what would you have done?

If the channel gain is 0, SNR is very bad, so you would have not transmitted any information. So actually we did not lose anything if you had actually done water filling. So ith has channel gain=zero which means in the general context this means that there is no signal at the receiver, so there is no way we can recover this if you had actually transmitted information on that, so there is this but that particular symbol cannot be recovered, cannot recover the if you had transmitted Si of n, it cannot be recovered.

Because the channel gain was 0, there is nothing from which you can detect this information, but in practice what would you have done; you would actually have not transmitted any information if you recall water filling. So the one that helps us is water filling, water filling says that no power would have been allocated to this channel. That means no symbol transmitted on Si of n, no power allocated okay.

Under this condition, the ith subchannel is a do not care okay if you had done it correctly okay. Now in practice what usually happens is that this one is not exactly 0, it is some small number right, it is some small number. So if you in practice when you take frequency response you will never get exactly 0. So effectively what you will find that in practice Ci is something which is very close to 0 but Ci is actually equal to some small value delta okay.

So because there are several reasons for it, one is the presence of noise. So when you estimate something in the presence of noise, you do not get exact values, you get something very small in the presence of noise okay. That is an important element, so we say that when you do 1/Ci okay 1/Ci we say that always take the minimum okay, do not let it go below a certain, you should take the minimum or maximum sorry, no minimum of correct, minimum of 1/epsilon, 1/delta okay.

Now if delta was very small, it will give you something very large but you always sort of put an upper limit on that. So this epsilon is like a threshold, is a threshold value. So you do not let the 1/C to go too large. So this is a way by which we manage. So this inversion of the circulant matrix is really not a problem. So effectively what we are saying is that C circulant inverse exists okay.

First of all, if it was something that there was no information there but you still have to invert it in the process, so which means that some small number will be there in that diagonal element, we will replace it with epsilon if it goes very small and therefore you get this information okay.

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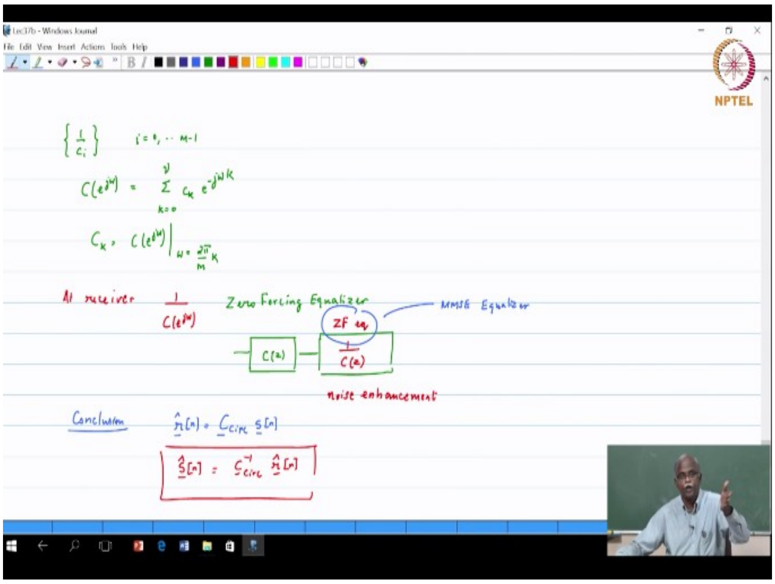

So now here comes the insights that are coming in. So effectively what are we computing? We are computing 1/Ci for i=0 to M-1 okay and what are these Ci's if you think about it, C e of j omega is the frequency response of the channel. This is given by K=0 through nu CK e power -j omega K okay and C subscript K is nothing but C e of j omega sampled at omega=2pi/M times K.

Basically, it is a sampling since the DFT coefficients are samples of the discrete Fourier transform okay. So effectively what are we doing? In the process, we are at the transact receiver multiplying by 1/CK is the same as doing 1/C e of j omega in some sense right. At those frequencies, you are doing 1/C of j omega. This in communications you would have learnt it as a zero forcing equalizer okay.

Zero forcing equalizer basically says if the channel is C of z, the zero forcing equalizer, this is the zero forcing equalizer is 1/C of z. The problem with zero forcing is noise enhancement because if C of z becomes very small, 1/C of z will become large. What happens to the noise? Noise gets boosted up. So the problem with this is noise enhancement.

So if you do not want noise enhancement, there is an alternative to zero forcing equalizer which is called no it is in the same form of channel inversion. Decision feedback is a completely different family in this itself, it is called MMSE equalizer. Basically, MMSE will set a threshold and then do the inversion okay. So MMSE equalizer, it does not allow enhancement of noise.

So basically it controls the enhancement and what we have done here is exactly the same as we are trying to do zero forcing, if it turns out that zero forcing is going to cause a problem then therefore we can achieve it through. So in other words, the inversion is equivalent to the MMSE type of solution okay. So the conclusion, the conclusion is that we can solve this equation where r hat of n is=C circulant times S of n.

And we obtain the answer, the estimate at the receiver S hat of n to be=C times C circulant inverse. Basically, that is the MMSE approach, times r hat of n okay.

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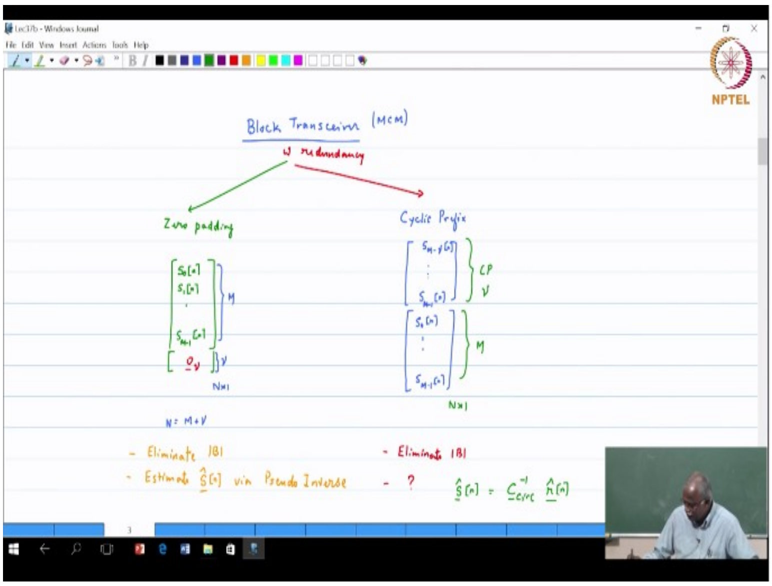

So we have managed to completely solve the second part of this portion also we have been able to solve. Eliminate IBI and the solution for the result, the estimate is that S hat of n is=C circulant inverse times S of n you know r hat of n r hat which is the received vector with the CP portion removed okay. Now you look at this and say well did you really solve any? Did it make any benefit at all? Both cases you are inverting a matrix okay.

And well so is there really any advantage to the cyclic prefix? And the answer turns out to be yes and that is a very key element as to why we even worked with the method. So here is the part that we would like to really focus on but before that maybe just a summary of all the results that we have had so far okay. The summary of results that we have is the following.

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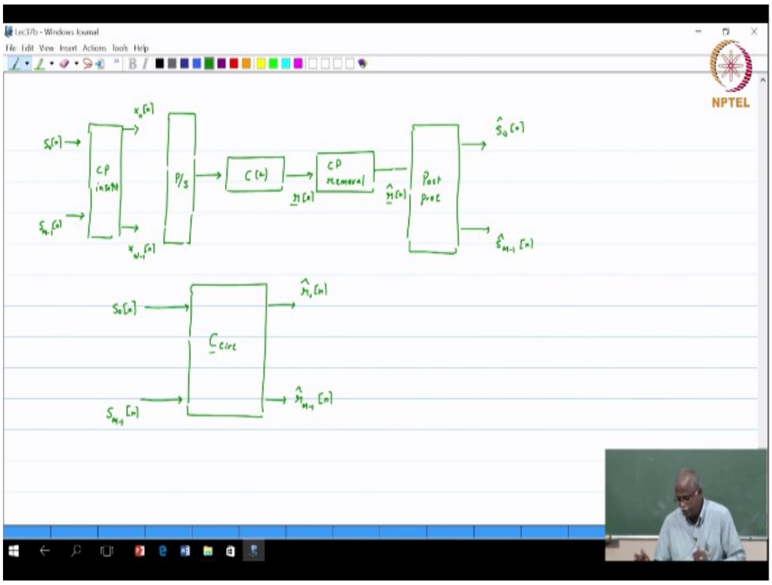

We take a vector of data S0 of n SM-1 of n okay and then we do the cyclic prefix insertion, CP insertion which means basically this will become a larger vector okay. So this becomes the X vector, X0 of n Xn-1 of n okay, that becomes my then I combine it to pass it through a channel. So this is where the unblocking occurs, the parallel to serial conversion okay. So take this information basically we have inserted the cyclic prefix.

Then, you pass it through the channel, pass it through the channel C of z. Then, at the other side we remove the cyclic prefix or throw away those portions CP removal okay. CP removal and then the post processing that needs to be done. In our case, there will be that inversion of the pseudo-circulant matrix. This is the post processing and then you come out with the vector S0 hat of n SM-1 hat of n.

So at the receiver, we have r of n and after the cyclic prefix removal it is r hat of n and then the post processing and that is what we have okay. So the input-output relationship is basically given to us in the form input-output is given to us in this form that there is a circulant matrix. We call it as C circulant, which takes in M inputs S0 of n to SM-1 of n, output r0 hat of n rM-1 hat of n and of course the estimation process basically will do the channel inversion okay.

So I hope the overall picture is clear. We have done something very similar to the zero padding but in this case we have obtained a circulant matrix and shown that this matrix can always be inverted. So here are some comparisons of the two, we had mentioned that we will compare.

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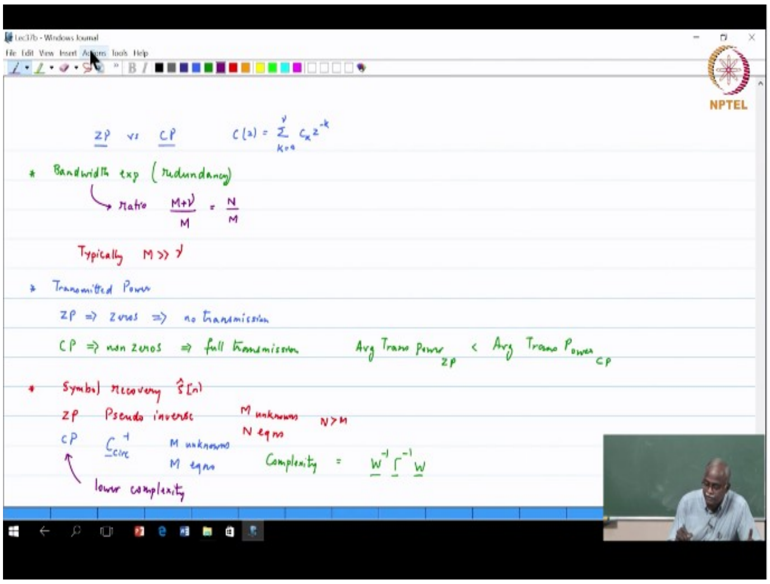

So zero padding versus CP. Are there any advantages disadvantages of each of those? So in both cases, the channel length is the same C of z. C of z is=summation K=0 through nu C subscript K z power -K that is the same. The bandwidth expansion in both cases that is the amount of redundancy that we have introduced, bandwidth expansion that is another name for redundancy, both cases we have used the same amount of redundancy.

So the bandwidth expansion ratio, how much did you expand by? It is M+nu/M, the same as N/M. So in both cases, we added new additional samples, in one case it was all zeros, in other case it was. So typically we find that the block length, the data that we are processing is much larger than the channel length. So therefore this bandwidth expansion is not a very significant element.

So again there is a bandwidth expansion but it is nominal compared to that. Now is there a difference in the transmitted power? So no change in terms of bandwidth expansion, transmitted power. Now transmitting a zero effectively means you are not transmitting anything. So transmitted power there is actually an advantage for the zero padding because zero padding means zeros which is not a zero information symbol.

That means there is nothing to transmit, so this is not a bit zero, so there is actually no transmission. So actually you are saving power whenever there is a zero to be transmitted okay, zero value to be transmitted. On the other hand, the CP's are all non-zero values okay, so which means that you will transmit all the time okay. So in some sense, this is full transmission okay like normal transmission.

So the CP actually requires slightly more power okay, very slightly because you know the overheads are there but so the average transmitted power for the zero padding will be marginally less than the average transmitted power for the cyclic prefix, power for the CP case okay. Now this is an important one, actually the symbol recovery, what happens at the receiver? Obtaining the so symbol recovery part, symbol recovery step where you estimate S hat of n.

In the case of the zero padding, it is a pseudo-inverse because it is an over determined set of equations okay. Now what is it in the case of CP? Basically, in the previous case there are M unknowns and N equations okay, so where N is greater than M. So you do the least squares there. In the pseudo-inverse case, we do C circulant inverse, this corresponds to a case where we have M unknowns and M equations okay.

So in some sense the pseudoinverse is probably a little more robust in terms of noise okay. Now what about complexity? What is the complexity of the inverse of the circulant matrix? It is very less because it is can you just tell me the final expression. What is it? It is W inverse that is basically you implemented in a fast algorithm. These are diagonal matrix, so I just take the DFT coefficients and take the reciprocal.

This will be W again okay, so definitely CP has an advantage, so this one has got lower complexity because of the structure of the matrix. In the case of the Moore-Penrose pseudoinverse, there is no structure that can be exploited. You will also have, now in the case of the pseudoinverse the zero padding.

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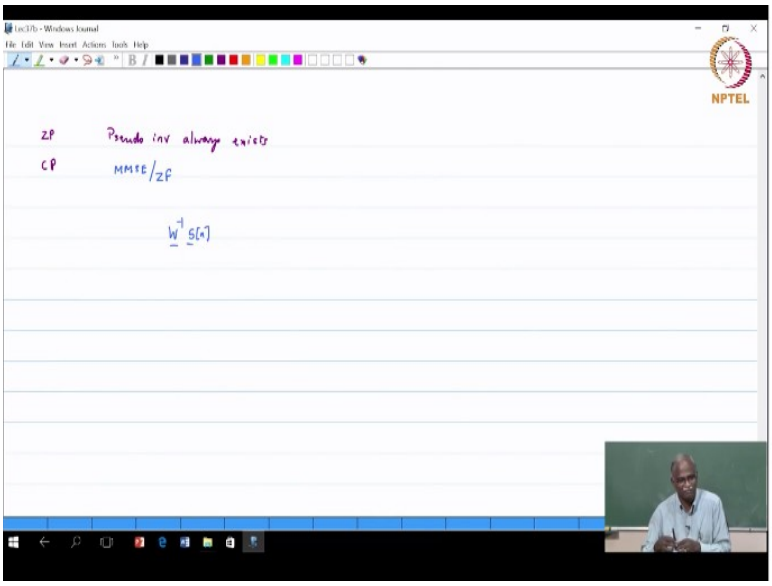

In the case of zero padding, the pseudoinverse always exists. In the case of the cyclic prefix, we may have to replace the zero forcing with the MMSE. So MMSE/zero forcing will help us do the inversion, so there is no problem okay. Now at the end of the day, you may say well you know did we do all this work for the sake of you know just a little bit of gain in this aspect, actually not.

So here is where the crux of all that we have discussed so far oh sorry I have already crossed my time okay. So at the climax you have to stop the thing but we will pick it up from here. Let me just sort of give you the flavor of what is going to happen okay. The transmitted signal that actually I want to transmit is not I would not transmit S of n, what I feed to the system is not going to be S of n but it is going to be W inverse times S of M okay.

If you now plug in to the equation that we now have, you will find what advantages you are going to get and this is really the climax. I was hoping that it will end with that but it will have to wait for the next time. Thank you.