Multirate Digital Signal Processing
Prof. David Koilpillai
Department of Electrical Engineering
Indian Institute of Technology - Madras

Lecture – 03 (Part-1)
Signal Reconstruction - Part 1

Good morning. Lecture 3, we will be covering, picking up from where we left of in the last lecture. We look at a quick summary of the key points that we have discussed in the last lecture. Look at some examples.

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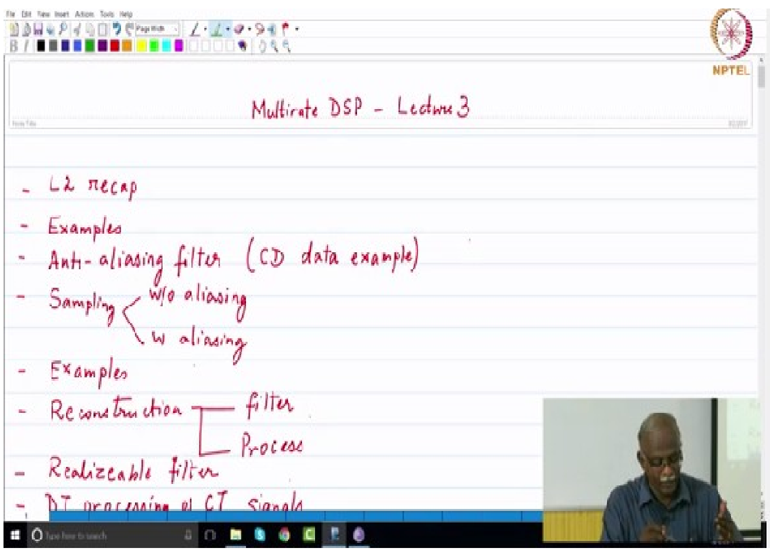

Build on what we have talked about in yesterday's lecture. Make another comment about the anti-aliasing filter used in the compact disc data example, and then today's focus will be on the reconstruction process and one of the applications of multirate signal processing or of DSP in general where you can move the continuous time processing into the discrete time. So the last part will be the discrete time processing of a continuous time signal.

So let us begin with the following task, a quick summary of the points that we discussed in the last lecture.

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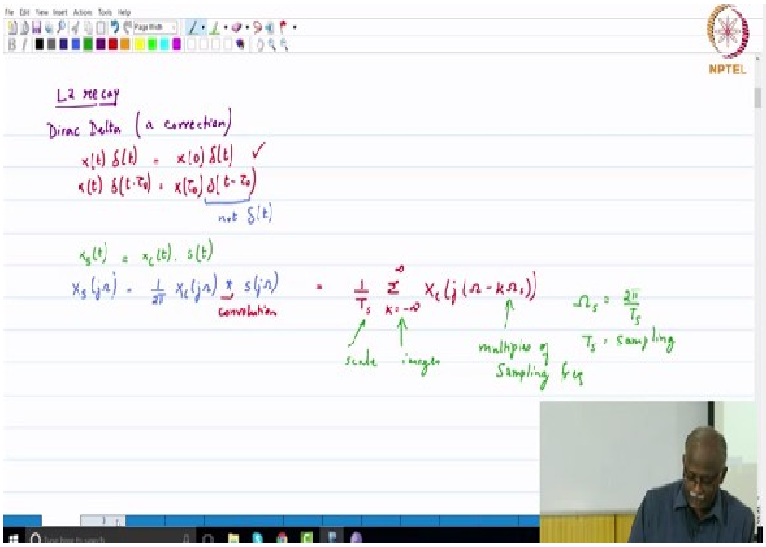

I will like to begin by pointing out that there was a mistake in the notes or what I had written in the class. I just wanted you to first rectify that. This was in the context of the Dirac delta. We talked about 2 properties of the Dirac delta function. The second one was, one was the area property. The other one was the sampling property and I just wanted to mention of a correction that you should make a note in your notes.

So yesterday we did write down the following result. X of t*delta of t, okay. So this gives me, this is the same as X of 0, because the function gets killed everywhere except where the delta function is non-zero, time the delta of t, okay. This was fine, the way we had written it down. The second one was where we had made an error, X of t*delta of t-tau 0. So this is a delta function, Dirac delta function, positioned at t=tau 0.

So you will sample X of tau 0 and this should be delta of t-tau 0. Basically the impulse response. In the class, I had written it down, it is not delta of t, okay. The delta actually occurs, the position of the Dirac delta does not change. The only thing is that it samples the function there, okay. The second point or the other aspect that we had discussed at, in depth, was that we are trying to sample a continuous time signal.

So the sampling part comes by multiplying the continuous time signal, Xc of t with an impulse of, a train of Dirac impulses and this is the or the sampling function for which you have to specify the sampling period. And the analysis was that the time domain, we have the easy representation, a straight forward representation of uniform sampling. In the frequency domain, Xs of j omega, notice that I have not yet moved over to the discrete time, so it is still the continuous time Fourier transform.

This will be equal to a multiplication; the time produces convolution in frequency. So 1/2pi Xc of j omega convolved, this * is a convolution symbol, S of j omega just mention that. This is convolution, okay. And we wrote down that this finally results in the following expression, 1/Ts summation k=-infinity to infinity Xc of j omega-k omega S, where omega S is a sampling frequency or omega S can also be written as 2pi/Ts where Ts is the sampling period, okay.

And we did observe that there were 3 elements of our sampling process. The first one was a scale factor. The other one was multiple images and the shifts happen as multiples of the sampling frequency. And all of these are going to be exploited in our use of the sampling process, okay. So that was the sampling process.

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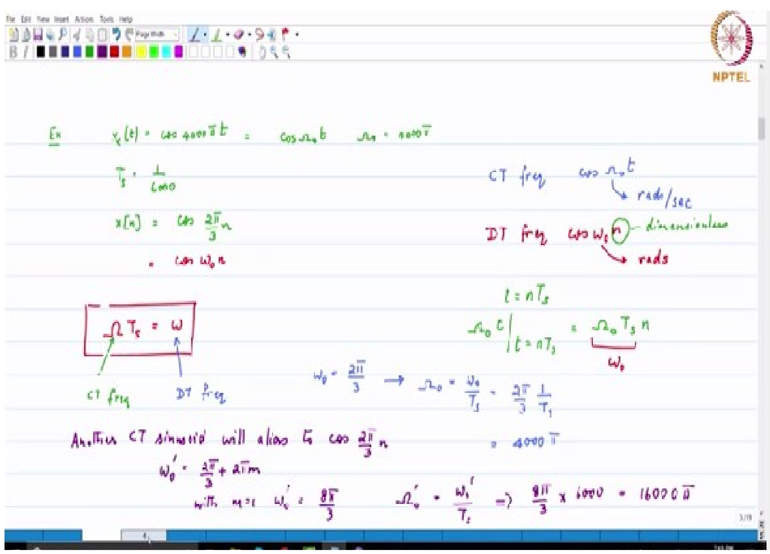

Let me revisit the example in order to introduce the new element that I want you to pay attention to, okay. Again, it is the same example but this time we will be revisiting it 3 times, to add something new each time. So look out for what the new element is and make a note of that. So we looked at a continuous time signal, Xc of t which was cosine 4000pi t. So I were to treat it like, using the continuous time notation, cosine omega 0 t, omega 0, if I write this as omega0=4000 pi, okay.

And we say that we would sample it at Ts=1/6000. We will sample it at 6000 Hz, sampling period will be 1/6000 and this gave us X of n=cosine 2pi/3*n. Now up to this is very straight forward. Just substitute the sampling frequency and you will get this result, okay. Now here is a point that I would like you to note. So the continuous time frequency is if you think of it as cosine omega 0 t, the units of omega 0 would be radians per second, okay.

Now on the other hand, if I wrote this as a, using a slightly different notation, cosine omega 0 n, that is how we would represent a sinusoid in the discrete time. Now the discrete time frequency is cosine omega 0 n, units of n? Is samples. Actually, basically it is a dimensionless quantity. So but if you want to take cosine of something, it has to have radians as the units. So omega 0 in this case has got radians as its unit, okay.

So there is a difference between what we call as continuous time frequency and the discrete time frequency. Again, this is something that I am sure you would have looked at in the context of DSP. But I just want to be sure that. So basically this is a dimensionless quantity, okay. It is just a number. So therefore, omega 0 n for it to have radians, omega should be. And so now if you were to compare, when you substitute t=nTs, okay.

So omega 0 t, if when you substitute t=nTs, what you get is omega 0 Ts*n, of which these 2 terms combine become omega 0, okay. So the general over arching relationship between frequency in the continuous time and frequency in the discrete time is actually embedded in this example. I just wanted to highlight that, that relationship is that Omega*Ts=omega. I have to use the word omega both times.

But this is the continuous time frequency, which is in radians per second. This one is the discrete time frequency in radians and of course, when you multiply Omega*Ts, you get the dimensions correct. So this is the discrete time frequency, okay. So this is the important element in our understanding and our interpretation of the results that we have. So for example, we said that we had a frequency 2pi, omega 0=2pi/3.

Now how do I convert it into the corresponding analog frequency? Omega 0=omega 0/Ts=2pi/3*1/Ts. Ts itself is 1/6000. So that basically gives me 4000pi, that was the result that we verified yesterday, okay. Now the second or one more addendum to the thing that we had discussed in the last class. So another frequency, continuous time frequency, another CT frequency, continuous time sinusoid, that will alias to.

The 4000pi is what gives you omega 2pi/3 or cosine 2pi/3n. That will alias to cosine 2pi/3n. I think we already wrote down the answer. I just thought I would highlight that. So this is a case where omega 0 prime, we will write it as 2pi/3+2pi m. We chose m=1. So with m=1, omega 0 prime will be 8pi/3. Now how do I convert it into the corresponding continuous time frequency? We know the relationship.

So this gives me, Omega 0 prime=omega 0 prime/Ts, same sampling frequency. This is the one that actually produced aliasing. So this basically says it will be 8pi/3*6000, that will be 16,000pi. So a continuous time frequency or cosine 16,000pi t radians per, will produce the same result as the case of cosine 4000pi t, okay.

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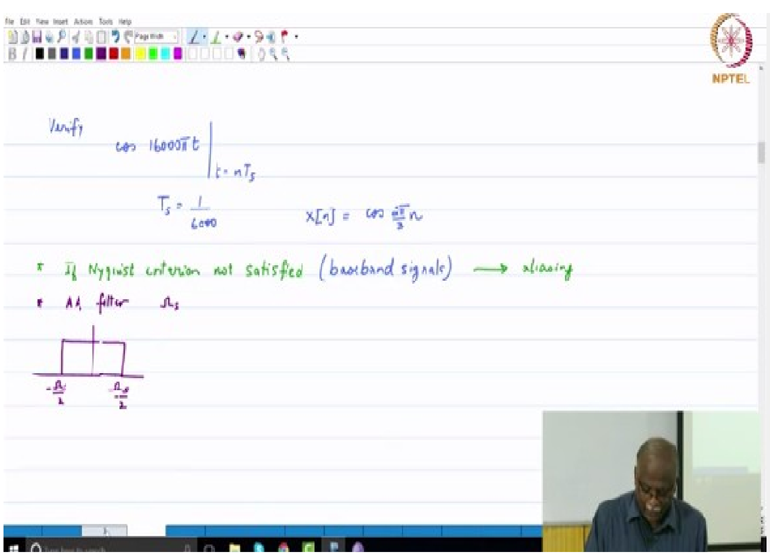

And may be a good idea for us to just confirm, or maybe you can do that as a simple exercise, verify that cosine 16,000pi t, when you sample it at t=nTs, Ts=1/6000, then what you get is X of n=cosine 2pi/3n, okay. So obviously we can see an example of aliasing and there are infinite number of such frequencies that are going to map to cosine 2pi/3n. So therefore, the statement that was made is that the unique representation is valid only between the continuous time and discrete time only if the Nyquist criterion is satisfied, okay.

So if Nyquist criterion is not satisfied, let me give you the reverse condition. If Nyquist criterion is not satisfied, then there is no claim on uniqueness, not satisfied, okay. And maybe there is even a further, we can make a further specification that we are talking about base band signals. Because yesterday, we said that there is notion of bandpass sampling which is different from the sampling of low pass signals, of base band signal.

So what we are talking about is in the context of base band signal. So if the Nyquist rate is not satisfied for base band signals, not bandpass, base band signals, then there will be aliasing. So the fact that the process of sampling is going to produce multiple copies is not the issue. The fact that somewhere there could be an overlap of the images. And if you want to ensure that aliasing is not present in a base band, when a base band signal is sampled, so we have to employ an anti-aliasing filter, AA filter.

So if the sampling frequency was Omega S, the anti-aliasing filter would be from -Omega S/2 to Omega S/2. It will force the signal to be band limited, input signal going into the sampler and therefore, you will not have the effect of aliasing, okay. So that is a useful result for us, okay. So the key points to remember are the sampling process, the creation of images, the aspects of aliasing.

Again, this is something that we study very extensively as part of this course and we will come back to revisit it multiple times. Now let me go back to the last slide that we looked at in the previous class.

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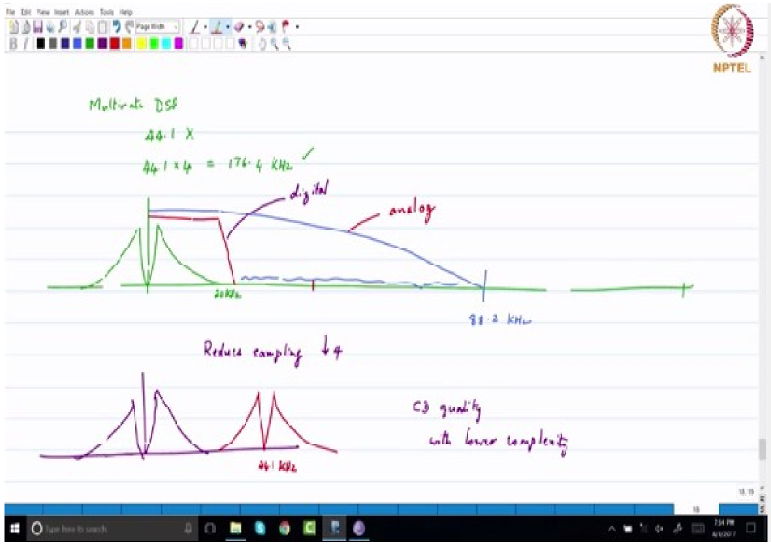

This was a case where we were trying to create CD, compact disc, compatible data. So you had analog continuous time signal with information possibly up to 20 KHz, that is what we have shown here, that it has got some range between 0 and 20 KHz, probably a lot of information in the lower frequencies but there is the frequency content. So you have to preserve that. Now if you wanted to sample it at 44.1, your anti-aliasing filter would have to be the one that is shown by the red line.

And for an analog filter to achieve such a sharp cut-off, would be difficult. So we said that multirate signal processing allows us the following option, treat it with sampling rate of 176.4 KHz. So that means the sampling point has moved sufficiently far and we now have the omega S/2 at 88.2 KHz. The analog filter can, it has to be flat up to 20 KHz and then can droop all the way to 88.2.

Everyone is comfortable with that? After we have done the sampling process, that means you will have a signal where the green is present and then there is some unwanted signal because your filter, anti-aliasing filter allowed those blue lines to be, this blue component to be present. You have to get rid of that. So this then is followed by a digital filter. The red line becomes the digital filter that will give you a spectrum with a sampling rate of 176.4 which you then have to down sample to bring it to this form.

This is the form in which it truly represents or is in the form of compact disc data, okay. Now a question that just want you to think about, so here is a case where aliasing occurred. But aliasing did not cause a problem for us because? Aliasing, is the question clear? Aliasing is occurring here? Yes, no? No aliasing? My question is, is there aliasing in this process? There is no aliasing. Why?

Because Omega S/2, right. Omega S/2, what we did was that was sampled at 176.4. So no aliasing there. I am going to down sample by a factor of 4, but before that I applied a digital filter. So digital filter was now, anti-aliasing filter is in the digital domain. So there are 2, your final sampling rate of 44.1 came through 2 stages. The first stage was sampling at 176.4 KHz for which the analog filter was anti-aliasing, no problem.

When we down sampled to 44.1, so which means that you must have another filter. If you did not have this red filter, what will happen? This blue portion will start overlapping with your spectrum. That is where the problem will come. That is where the aliasing will happen, okay. Now can I allow this analog filter to be a little sloppy? In other words, that it does not cut-off at 88.2 but goes a little bit beyond 88.2, okay.

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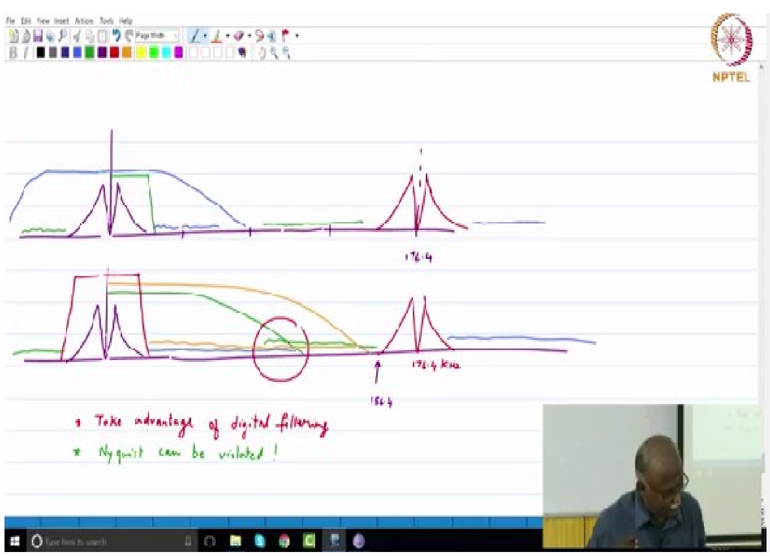

Let me explain what I mean by that, okay. So here is the drawing. Let us quickly reproduce the part of interest to us, okay. I have the portion of the desired signal, okay. And so this is 44.1, sorry this is at 22.05, this is 44.1, 88.2, 132.3 and 176.4, okay. So this is 176.4, okay and I have a copy of the spectrum because I am going to sample it at 176.4, okay. Now first option that we have done is an anti-aliasing filter, which is flat in this portion and then drops off up to here.

So no aliasing will happen in the process of sampling at 176.4. There is some unwanted signal that is present here, okay. There is some unwanted signal that is present here because this is a filter that will go on this side as well, okay. And when I do the sampling process, the images will be produced. So there is some unwanted portion here, there is some unwanted portion on this side.

All of that is present but aliasing is not present. So then when I applied the digital filter, it will remove all the unwanted parts and then I get the signal that is of interest to me. So now the question that is being asked is that I am going to have the same scenario. And here is the replica of the signal, that is when I sampled it at 176.4 KHz, this is where it is going to come. Now if I allow my analog anti-aliasing filter to be a little sloppier than before, what will happen?

It will allow more of the blue to leak, correct, and there will be, green will be present here and when the copy comes, these 2 will overlap. See there is an overlap which previously was not present but now is present, okay. Is everyone clear with this one? Okay? This is aliasing. But has it affected your desired signal? No. So in fact, you do not need to restrict yourself to this point. You can actually allow the analog filter to be even more gradual, okay.

Now we get a little bit ambitious, how far can I go? How far can I go? How easy can I make it for my analog filter? Can I make it all the way to 132.3? How far can I go? It turns out that I can go all the way up to? This is 176.4 KHz. This edge will be 20 KHz away. 156.4, it turns out that I can go all the way up to 156.4 or before 156.4. So I can go there because some unwanted stuff will come, so basically let me show it with; so I go all the way to the maximum level possible.

So that means this orange filter is going to allow some unwanted stuff. And when I do the sampling process, that will be sitting here, but my digital filter will take care of that, okay. So the digital filter will take care of that. Notice the orange stuff is removed in the process. So one of the things that I wanted to introduce through this example of course, one is that you know multirate signal processing is a very practical tool which you are using everyday whether you may not have realized it.

But it is all about understanding the underlying framework of sampling, aliasing, what is my desired signal, how far can I allow the aliasing to come, all of that is a very important element. So the comment that I would leave here for you to think about is that we always take advantage of digital filtering, okay. So that gives you a huge advantage.

When you do, particularly when you are working in the area of multirate DSP, we always keep in mind that we have a very powerful tool in digital filtering because you can design more or less any type of filter, low pass, high pass, bandpass, differentiator, integrator, anything you can build in as a digital filter. Take advantage of digital filtering, okay. So this also means that, when you say take advantage of digital filtering, reduce your requirements on analog filtering.

But you had to be very careful that you do not compromise on the signal fidelity, the signal in the range of interest. So you can deliberately violate. So there are examples like in your music players where you may be violating Nyquist criterion deliberately. But because you know the portion of signal that is eventually going to be retained, so you take advantage of that, okay. So Nyquist criterion can be violated but you have to be careful with the way you do that so that you do not lose some of the essential parts of what we are trying to do.