Multirate Digital Signal Processing
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Lecture – 31 (Part-1)
Capacity of Wireless Channels - Formulation of Capacity Calculation - Part 2

Good morning. Very sorry for the late start. We will catch up. The plan for today's lecture is to pick-up where we left off yesterday. We were talking about the power allocation for achieving capacity. We will also build on today by adding a few aspects of OFDM history.

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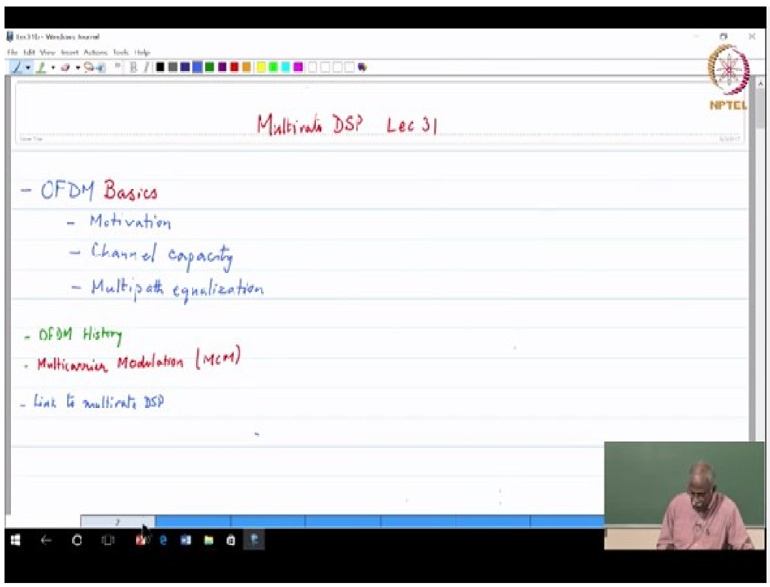

And OFDM history will indicate to us that it did not always start off by being called OFDM. It was called multicarrier modulation. So if you want to read up on the history of OFDM, you would have to search through, the code words would be multicarrier modulation and a very good overview is given in Andrea Goldsmith's book. So I would very much encourage you to take a look at that. So that is in chapter 4.

So the history of; but at the end of the day, what we are interested in is the link to multirate DSP. So those are the things that we hope to establish in today's lecture in addition to the basic, the advantages of OFDM. So multirate DSP is where we go back and rejoin where we are. So right now we are in the communications domain trying to understand and get a feel for the capacity of wireless channels.

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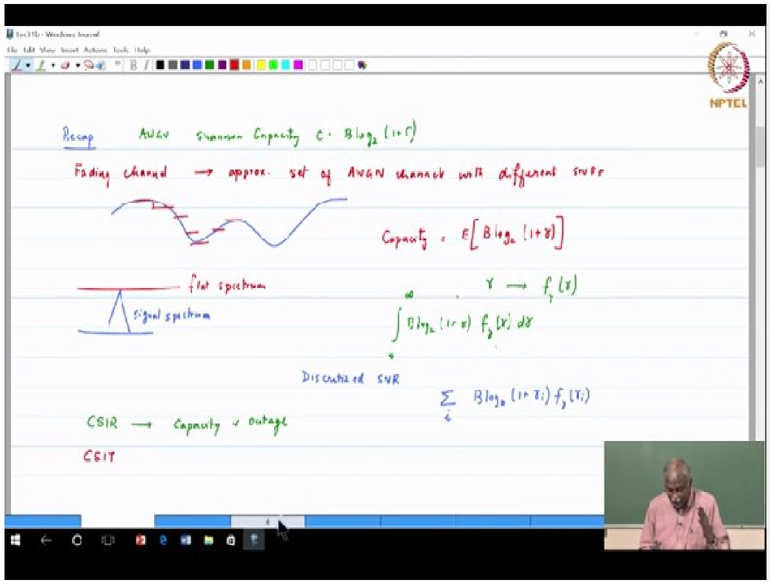

So a quick recap of the result so far. I will not use the slides. I will just write down all the key results so that gives you a chance to sort of refresh your memory. So for AWGN channels, we talk about a Shannon capacity. Shannon capacity, that is the underlying model that we have which says capacity=B*logarithm base 2 1+the SNR. So what we are doing is that trying to get a representation for the fading channel.

The fading channel would be primarily a wireless channel. The fading occurs because multipath components, either add constructively or destructively causing you the changes, fluctuations in that. So if I have a fading channel, I am approximating it. You approximate it as a set of AWGN channels of different levels, okay. So that is the very key. It is a set of AWGN channels with different SNRs, okay.

This is very crucial in our interpretation of the capacity results with different SNRs. So what do we mean by this? The SNR in a fading channel will not remain constant. It will keep changing. What we are saying is this is going to be approximated by a set of SNR, AWGN channels which have constant SNR, okay. So we are going to approximate it by sets of; so when we say that this is how the channel is changing but at any given time.

I can approximate it as an AWGN channel with this and another important element, my signal spectrum, if this is my signal spectrum, my channel frequency response, I am assuming is flat. So though it is going to different levels, it is flat. It does not distort my spectrum. So this is my signal spectrum. This is my channel frequency response. So and the reason we have shown it as flat is we said that there is no dispersion. So flat spectrum, okay.

So is the whole picture clear? I have a flat spectrum but this flat spectrum is not constant as in an AWGN channel, it is sort of going up and down and depending upon what the current fading situation is, I will approximate it as an AWGN channel either with good SNR or medium SNR, low SNR, very poor SNR and so forth.

So given this framework of visualization of a channel, then we talk about a capacity of a fading channel not as a single number but as the expected value of B*logarithm base 2 of 1+gamma, where gamma is the instantaneous SNR. So associated with gamma, is a random variable. So we associate with it a probability distribution function, f gamma of gamma. Once you know that, we can always write this capacity as integral 0 to infinity.

Since it is SNR, it can only be, it is non-negative. So this would be B*logarithm base 2 1+gamma f gamma of gamma d gamma, okay. Now capacity can, this is assuming you had a continuous variation. But if you took up a discretized, for a discretized SNR case, let us assume that there are some number of discrete levels. It will no longer be an integral. It will be a summation.

It will be B*logarithm base 2 1+gamma i, where i is one of these SNR levels. And the Pdf, the probability that you get that and that would be my capacity expression. So I can view it as a discretized channel. I can view it also as a contribution channel but in both cases, we get a meaningful interpretation only when we relate it back to the AWGN channel for which we know that there is a well-defined capacity expression.

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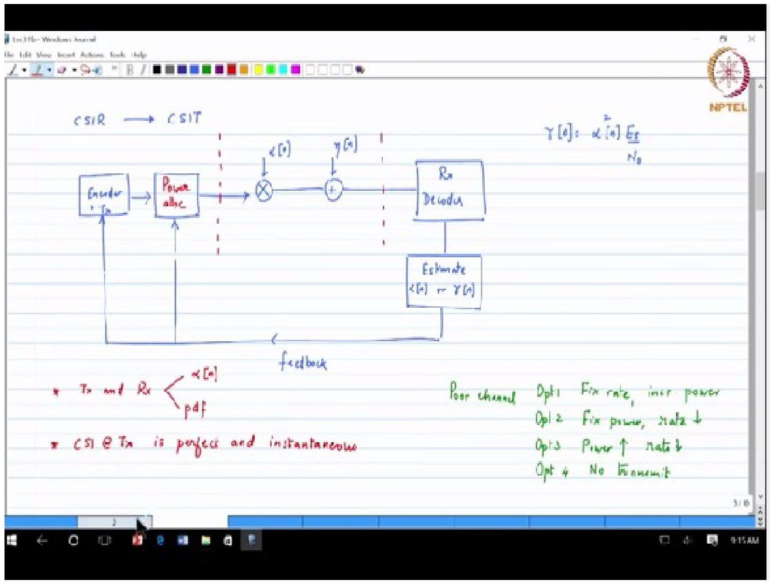

Okay, now we also said that the receiver always has knowledge of the channel. We refer to it as CSIR, channel state information at the receiver. Now if the receiver were to give, have a feedback channel to tell the transmitter that this is what the current instantaneous SNR that is being seen on the channel, then that could, transmitter could take advantage of that. Now what are the ways in which you could take advantage?

Four options. First option, you can increase the power. If you know that the SNR is bad, then what you can do is you keep transmitting at the same rate but increase power. The second one is says that okay I am not going to increase my power but I am going to reduce the transmission rate which means that I am going to send less information. Hopefully the signal will get through, okay.

Now you can do a combination of both. You can increase the power and decrease the rate. You can do that combination as well. And the fourth combination is that you do not transmit at all. That is the choice that we have to make. But all of this is possible only if you have knowledge at the transmitter because if you do not have information at the transmitter, then the best thing that we can do, so in the case of CSIR- that you have only CSIR.

The best thing that we can do to achieve capacity is capacity with outage. Do not design for the worst case. Sometimes the worst case will happen, you sort of say that okay under those conditions, the transmission will not take place. The other good conditions is when; so capacity with outage, okay and yesterday we saw an example of how you would take advantage of the fact that I do not know what is happening in the channel.

But I design and transmit for a reasonably good channel and therefore, when it is better than that, no problem. When it is worse than that, my transmission goes into outage. So get a capacity that is not the best but that is the best that we can do. Now the key question is what can you do with CSIT? With CSIT, I know that there is a variation in the SNR.

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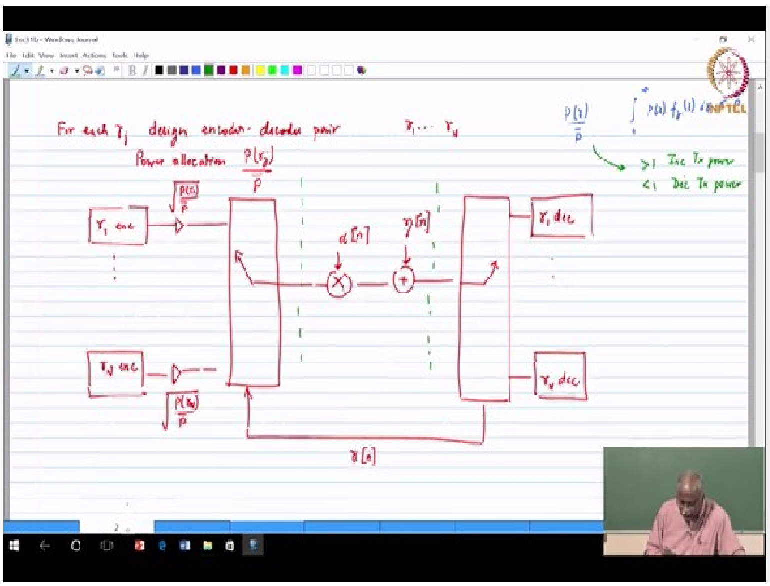

We could do this, where I have a set of optimized transmitter encoder decoders. This is one scenario. We also saw that there is an additional element that we have that we can vary that is the power allocation. Now power allocation is done by means, by the ratio P of gamma/P bar. P bar is your average value. The definition of P bar is integral 0 to infinity P of gamma f gamma of gamma d gamma.

This should be less than or equal to P bar. You can take it as an equal to sign, not a problem. But basically we do not want to exceed this average power under the transmission so that therefore, you can have a balanced reference point. So given this, P gamma/P bar is a dimensionless quantity. It is a ratio of 2 numbers. So this is a good metric for us to look at. When it is greater than 1, that means it will boost, whatever transmit power you are using, it will boost it further.

So basically says that you are increasing transmit power effectively. And less than 1, you are decreasing transmit power. Now the key question is, when would you increase and when would you decrease and that was what was the nature of the calculation that we were doing. So given that you have the option of choosing the best transmission scheme possible and the fact that you have the ability to choose the alphas, to choose the power allocation. So here is the statement that where we ended the last class.

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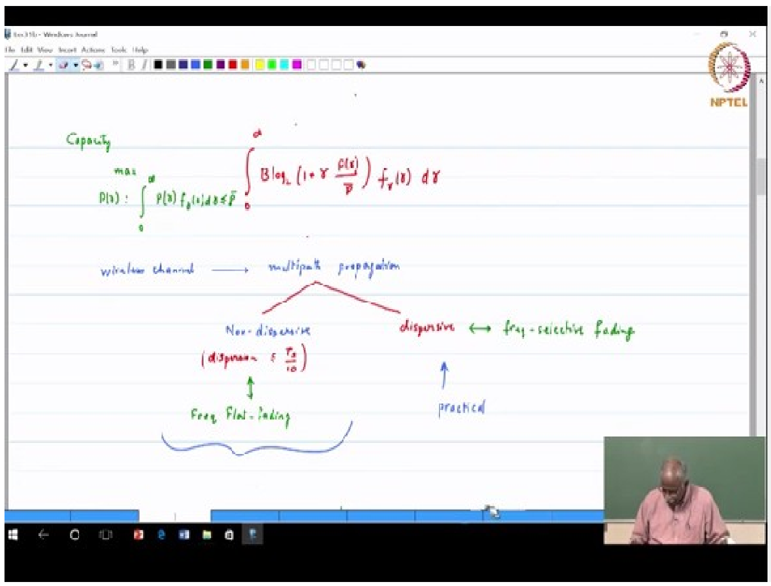

So the capacity that we are trying to maximize, capacity is to, we want to maximize under the following condition that the P of gamma satisfies the following, integral 0 to infinity P of gamma f gamma of gamma d gamma is less than or equal to P bar. Under this condition, this is the maximum power constraint. We were trying to optimize B*logarithm base 2 1+gamma*P of gamma/P bar*the PDF*d gamma integral, 0 to infinity, okay.

So this is the condition that we are trying to optimize and we wrote down the Lagrangian, let me not waste time by writing it again.

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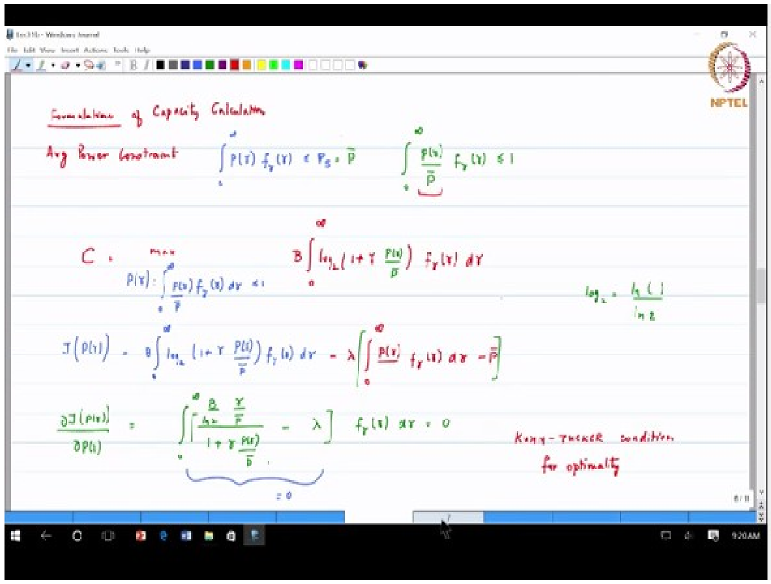

Basically, the Lagrangian had an objective function, Lagrange multiplier. Again, this is a maximization problem. We have an inequality constraint that the total power allocation should be less than or equal to P bar and so when we did the first differentiation with respect to P, we got an equation and we said that there is a sufficient condition which satisfies this. There was a very valid question after the lecture how do you know that it is a sufficient condition.

But how do you know it is optimum. Let me just write down the, we have a set of conditions called the Kuhn-Tucker conditions. Kuhn-Tucker conditions for optimality. What we will need to show is that whatever assumptions that we are making actually satisfies the Kuhn condition, it is the sufficient condition for solving for what we are doing now. But it has to be backed up with the verification that the Kuhn-Tucker conditions for optimality are satisfied.

So basically it is a constraint optimization using the Lagrangian, we get a solution and then validated by verifying the Kuhn-Tucker conditions for optimality, okay. So the Kuhn-Tucker conditions we will do it as an exercise in the assignment but for now, let us just look at what the result is going to give us. So this is where we stopped and one more result which I want to mention before we move on.

We also mentioned in the last class that the wireless channel has a fundamental property what we refer to as multipath propagation. There are multiple ways by which the signal will end up at the receiver, that is referred to multipath propagation, okay. Now this is what leads us to 2 classification of the wireless channels. The first one of course is dispersive. That means these multipath components arrive at different instances of time and that will result in frequency selective fading, right.

Your channel frequency response now is not going to be constant across frequencies. It is frequency selective means different frequencies will see different gains, frequency selective fading, that is one side of it. The other side is the class of channels where we say it is non-dispersive. Now you may ask a very valid question, as long as they travel by different paths, how will you guarantee that they are arriving at the same level.

So yes we do not, I cannot necessarily guarantee that they are arriving because these are electromagnetic waves travelling at the speed of light. So even a small perturbation in the length could cause a variation. Now the key thing is, as far as the signal that we have transmitted, it has got a finite bandwidth and a finite bandwidth means it has got a finite baud rate. A baud rate means it has got a symbol period.

So non-dispersive effectively for us means that the dispersion that is happening, there will be some amount of dispersion between the path. If that dispersion is less than or equal to one-tenth of a symbol period. I really cannot detect it. Basically my time resolution of the signal that I am working with and therefore, I declare the channel to be non-dispersive.

Non-dispersive effectively means that we now have a flat fading channel which make me, can allow us to assume that whatever bandwidth that I have transmitted, is going to experience the same gain transmission gain across all the frequencies. So flat fading. So it is an interconnected set of concepts but I hope these pieces are helpful in getting a broad picture. So frequency flat fading, okay. So wireless channel, multipath components, dispersion will lead to frequency selective.

If there is no dispersion or dispersion, non-dispersive by our definition, means that you will have a frequency flat. So all that we have been talking about is this category of function, right.

Everything that we have been talking about so far is in this category. We are talking about frequency flat channels but where the gain of all the frequencies is going up and down, this is what we have been talking about, but what is practical in most scenarios.

This is what is practical. Any wireless channel that you take, is going to see some level of dispersion and frequency selectivity. So the task that we are going to do is, how do we link these 2 and where does OFDM come in. That is the goal for today's lecture. Hopefully, we will be able to put all the pieces together in today's class, okay. Let me just backup. We are starting from this equation where the objective function said to obtain the derivative, set it equal to 0. Basically we said that from this, we derive a sufficient condition which is given by this expression.

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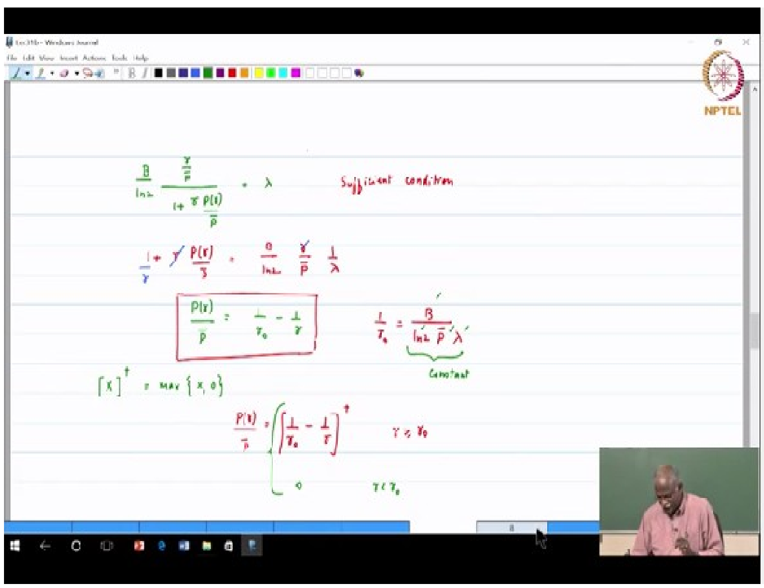

Please make a note that this is a sufficient condition which will have to be validated using the Kuhn-Tucker conditions for optimality, okay. The sufficient condition says that if this is satisfied, the derivative will go to 0. Rewriting this equation we now see that P gamma/P bar is given by 1/gamma 0-1/gamma, where 1/gamma 0 is defined in terms of the quantities given, all of which are constants.

Just as a quick check, B is the bandwidth, laun 2 is a constant, P bar is your average power which is a constant, lambda is your Lagrangian multiplier which is also a constant. So basically this is, 1/gamma 0 can be treated as some constant value. Now we of course cannot allocate negative power. So this quantity on the right hand side must be greater than or equal to 0. So we actually introduce a following notation which is X+ which indicates that it is always a maximum of X,0.

So if it is less than 0, you will truncate it and make it equal to 0. So P gamma of P bar, the optimal solution, we are constraining it to be equal to 1/gamma 0-1/gamma with a + sign which means that it will not allocate negative power. And this is true for, so basically you will start having allocations of power for gamma greater than or equal to gamma 0, that is when you will have nonzero values here.

And of course, because of the + sign, what will happen is this will become equal to 0 if gamma is less than gamma 0, okay. So we are going to do it in a couple of ways. We are going to first of all get a feel for what this equation is and what exactly we can derive from here.

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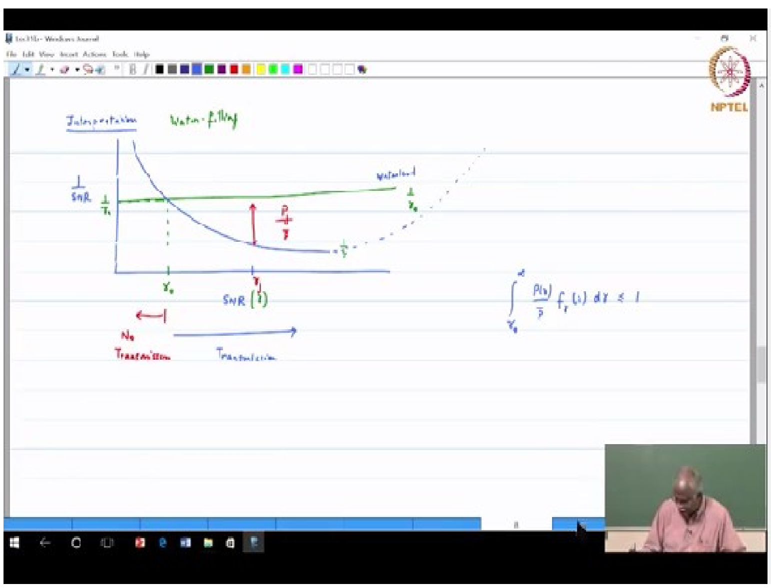

So here is the interpretation. Interpretation of this result is extremely important because that is what gives us the insight into understanding capacity of wireless channels. So the first thing that we are going to plot is 1/gamma, okay. If this is the, if the x axis is SNR and y axis is 1/SNR, basically I have got some kind of inverse function, okay. SNR, 1/SNR. Now this graph is 1/gamma, SNR is gamma, okay.

Now what is gamma 0. Gamma 0 is some constant. So let us say somewhere you find on your axis, gamma 0 is present. So corresponding to this is 1/gamma 0, okay. So this is 1/gamma 0 and we draw a line with respect to 1/gamma 0. So this is 1/gamma 0, okay. That is a constant line, 1/gamma 0. So What does the power allocation algorithm say? That if your SNR is less than gamma 0, no power allocation.

So which means that in this region up to here, anything in this region beyond this point, there is no transmission. No transmission means no power allocated, no transmission, okay. Now beyond this point, yes there is transmission. So this is the region where transmission and power allocation happens and very interesting, how much power do you allocate, it depends on 1/gamma 0-1/gamma.

So for any value gamma j, let me say I want to take a value gamma j, the power allocated will be 1/gamma 0-1/gamma j. So this will be actually the power allocated. So this will be Pj, sorry, P of j/P bar, okay. So this is the allocation that will happen. So this is going to be labelled as P of j/P bar, okay. Now as I go towards the right, that means gamma is increasing. Those are good channels with good SNR.

What is the optimum power allocation algorithm telling me? Keep adding more and more power to it. So the power allocation algorithm basically is giving us probably a counter intuitive recommendation, thinking would have been when the channel is not so good, why do not you boost the power and get the data through. But what the power allocation algorithm is saying is, below gamma 0, do not even bother, do not even allocate.

Above gamma 0, start allocating but allocate more power when the channel conditions are good. Now what does this mean when we go back and look at this particular, this diagram. It says you can, if you allocate more power, let us say this is the good channel, if you now going to boost its power by adding more transmit power to that, then what does it tell you about the encoding scheme and the decoding scheme that you can use?

You can go to a higher level modulation. You can reduce the amount of overhead which means your throughput will increase. So basically what the capacity optimization technique is telling you is that focus more on the good SNR channels. When you get a good SNR channel, increase the power, pump more data through. This is the best one, okay.

Now you may ask the question, why not carry it to the limit and say forget all the bad channels and wait for the good channel to occur, what is the problem with that? That means you say forget it, forget all the bad channels. I am just going to wait for the good channel to occur. **"Professor - student conversation starts"** Average power time. Well you can transmit for whatever duration. You just blast the power.

You will have to reduce the power. No, assume that you have the ability to, let us say there are 10 channels. 1/10th of the time the best channel occurs and I am going to say okay, I will transmit to 10 times the average power. (23:01) you may not be able to transmit anything, say for example. No 9 times out of 10, you are not transmitting anything. So whether 1/10 times when I get the good channel, I am going to blast 10 times the power. (23:12) It is probalilistic.

Okay. It is a probabilistic. Now the question is, is that the best in terms of data rate? It turns out that you cannot, excluding all the bad channels is actually not a good idea. You have to use even the reasonably okay channels in order to get the maximum capacity. The other side of it is from a practical standpoint, most of your equipment will also have a peak transmission power transmission capability.

So you cannot blast at 10 times the power. Because you know, that will mean that you have, your system design becomes very complex. **"Professor - student conversation ends."** So again, given all of these constraints, you do want to throw away the very bad channels, that is intuitive. You want to keep a reasonable number of good channels and among the good channels, you give more power to the channels.

Now if you just extend your imagination, just a little bit, think of this as a complete, sort of draw mirror image, okay. You just think of it as a mirror image. Again, this is not part of the graph, okay. This is just the visualization part. So what does it look like? Looks like a vessel and this green line is like water level. You have poured water into a vessel. So this is actually referred to as the water level, okay.

So the green line becomes your water level and so the power that you allocate is actually referred to as water filling. Because you know how far are you away from the bottom, that much. So basically this algorithm has the name water filling algorithm, okay. Water filling algorithm is a short form for saying I want to get the maximum out of, maximum capacity out of my channels where there are a whole range of SNRs.

And the immediate thing that comes to mind is okay, below a certain SNR, I will not transmit anything. Above a certain SNR, I will transmit but I will transmit with appropriate power allocation. How much power will you allocate? The better the channel, the more power I will allocate. However, we cannot violate the average power constraint which says integral, now remember it is no longer 0. It is gamma 0 to infinity, that is where we do the power allocation.

P of gamma/P bar f gamma of gamma d gamma=1, okay. So basically this is what we will try to achieve. You can even say less than or equal to 1 but the best we can achieve will be to get it equal to 1. So this is how the power allocation technique works. This is where the name water filling comes from if you think of it as a water level and then this is the gap as the water filling, amount of water filling that you are doing. This is how the whole thing works together.