Multirate Digital Signal Processing
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Lecture – 30 (Part-3)
Capacity of Wireless Channels - Formulation of Capacity Calculation - Part 1

So we now want to sort of state the problem statement and say, is there something that we can do more than this? And now you may wonder why are we asking such a question?

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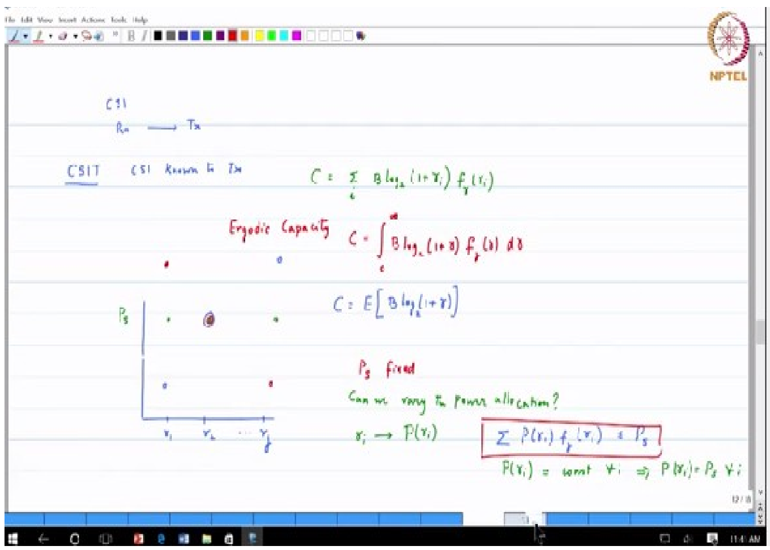

The question comes from the following observation. Here are the SNR conditions. This is gamma 1, gamma 2…, gamma j, okay. And each of them has got their own probabilities. Let us take the discrete channel. What are we saying is a transmitted power for each of these, dot dot dot. All of them are transmitting the same power. Thus we did not change, we did not touch the transmitted power, right because all you said was the transmitted power, you transmitted at some power.

The signal fluctuated because of the channel conditions, noise power remained constant, so therefore, this is the transmitted power, Ps. Now does it have to be constant? Now is there any intuition that says well you know what maybe I should do something a little bit differently. So for example, if the channel condition is not so good, maybe I boost the power, right. So maybe I can look at a scenario where I do red here and something here.

Maybe I try to equalize the channel. Where SNR is not so good, I try to do that. Or I might do something really counterintuitive. When the channel is bad, reduce the power, why waste time, why waste power on a bad channel and transmit it with more power. So depends on which way you want to do it. So one thing that we have not yet looked at is the power allocation. So one option is to keep Ps fixed.

The other option is can we vary Ps? And what should be the criteria or something should be, it should be somehow linked to the SNR, can we vary, but the question is then what is the optimum mapping? That is the very important question. Can we vary the power allocation, okay? This is an important question because it leads us to, basically we are building our way towards OFDM. So this is an important question that arises.

So now what do we say; for each gamma i, I am going to apply a transmit power which is proportional to P of gamma. I have not defined the function here. It is P of gamma i, okay. But I have to assign these powers such that, very important condition; summation P of gamma i f gamma of gamma i is less than or equal to Ps. Am I right? On average, I have to keep the Ps constant. I cannot do some allocation which suddenly looks like I have transmitted more than Ps.

On average, I have to do Ps. Now quick sanity check. If I transmit the same power level for all gamma i's, P of gamma i=some constant for all i. Then notice that will come out and you will find that P of gamma i=Ps for all i. That is the sanity check. So what you are saying is instead of assigning constant power, can I do some differential power allocation. So this is a very important observation and a question. Why do we have to do constant power allocation? Maybe it is better for us to do something different.

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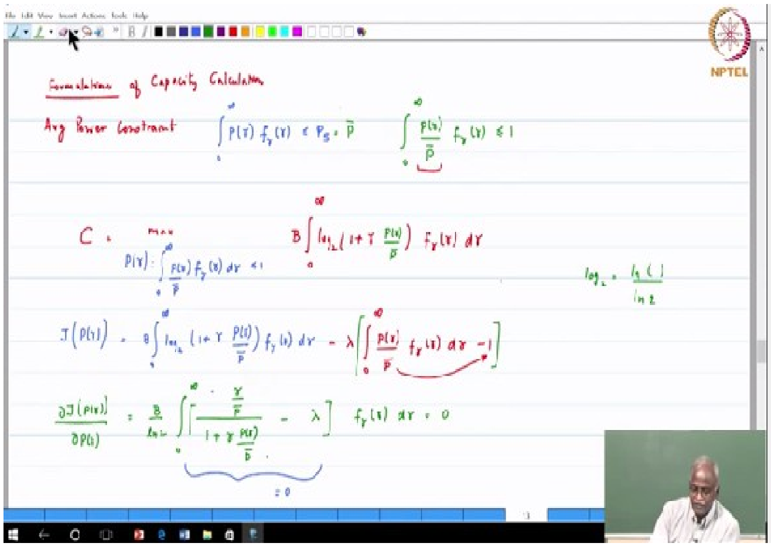

So here is now the problem formulation. The formulation of the capacity. How do we now define formulation of the capacity calculation? So we have a power constraint, okay. It is usually called the average power constraint and the average power constraint says that we have to allocate power in such a way that the integral 0 to infinity P of gamma f gamma of gamma should be less than or equal to Ps, okay.

Ps or since we are talking about calling it average power, let us also call it P bar, okay. So this can also be rewritten in a very useful way which is; 0 to infinity P of gamma/P bar f gamma of gamma less than or equal to 1. Now let me tell you why this is a preferred model. What are the units of this? It is dimensionless, right. So it is some number. If you did not have any power allocation, what will that be?

Equal to 1. Because P gamma will be the same as P bar but now what will it be? If you want to boost it up, it will be 1.1. If you want to reduce the power, it will be 0.8. So basically it is some number in the range 0 to 1, maybe slightly above 1 for some cases but it is a real number which is a positive real number which is in that range. So this is a very useful form for us to visualise and to keep.

So the capacity calculation or the capacity formulation is given in this fashion. So I would like to compute the Ergodic capacity under the condition of the capacity calculation being the maximum. So I want to maximize the Ergodic capacity. So which means, it is a maximization problem; under the power constraint. Basically it will be the power allocation P of gamma for the different SNRs such that the power constraint is satisfied, average power constraint.

Let me just write it down, 0 to infinity P of gamma f gamma of gamma d gamma less than or equal to P or I am going to write it in the normalized form, this divided by P bar less than or equal to 1. You can take it equal to 1 basically if you want to maximize it. Okay, now this is the constraint. What am I optimizing? I am optimizing B*integral 0 to infinity logarithm base 2 1+, notice I have to write down SNR.

This is the SNR without modification. This is what the channel gave you. But you are going to modify it by the following term which is P gamma/P bar. It is a dimensionless quantity. It either boosts the SNR or reduces it by tampering with the transmitted power signal. So this is very important step. This times, basically f gamma of gamma d gamma, okay. So I just want to pause for a moment, just to make sure that you are comfortable with the statement of the problem.

I am introducing the flexibility to allocate power based on the SNR. I cannot do it arbitrarily. I must do it subject to an average power constraint. I want to interpret it as a dimensionless quantity which either improves or increases the transmitted power or reduces. So basically it is a dimensionless quantity, P of gamma /P bar. How does it affect the capacity? It now becomes logarithm of 1+gamma P gamma/P bar.

This term can either be greater than 1, less than 1 and that will accordingly affect. So under this condition, I want to know what is the best that I can achieve. So what am I optimizing over? I am optimizing over all possible power allocations, okay. And this is a detailed derivation that is given to us in Goldsmith, but I would like to just give you the highlights of it. So it is actually framed as a Lagrangian problem where we say that we would now like to define an objective function.

The objective function is J, the variables that I am optimizing over is the power allocation, P of gamma. So it is actually a function. This is given by the capacity expression, B*0 to infinity logarithm base 2 1+gamma P gamma/P bar f gamma of gamma d gamma, subject to the constraint, the average power constraint -lambda*integral o to infinity P of gamma/P bar f gamma of gamma d gamma, that is it, okay. So this is the, -1.

So I need to put a bracket around this, okay. So this is my objective function. So if you just quickly, in the afternoon or whenever you get a chance to look at this, basically we will now differentiate, typical Lagrangian type optimization, differentiate the objective function with respect to the power allocation function, okay. So which means that you will have to differentiate the objective function, wherever P of gamma is present.

What I will suggest is take a look at it but let me just give you the final expression. Logarithm base 2 is difficult for us to differentiate. So basically we will write log base 2 as natural logarithm of something/on base 2, okay. So basically we will use that. So it becomes B/ln 2, that is just the, I am taking out the, making it a natural logarithm. So then what we are defined with is you will have to differentiate the integrand.

So it will be integral 0 to infinity. The derivative of ln of 1+gamma, which will come out to be 1/gamma P gamma/P bar, differentiate the argument of the logarithmic function. So you are differentiating with respect to P gamma. So basically it will be gamma/P bar, okay and the other term comes out to be, you differentiate with respect to P of gamma. It comes out to be lambda. Then what you do is basically, this times f gamma of gamma, I am basically combining the 2 terms, f gamma of gamma d gamma=0, okay.

So basically write this, take this P bar to the other side, okay. So basically it will be integral of P gamma f gamma of gamma d gamma-P bar. So when I differentiate, I will be left with only this term, okay. So this integrand is a strictly positive quantity. So 1+something, right, logarithm of 1+something is, it is a positive quantity. So therefore, the condition is that if this has to go to 0, then this integrand has to be 0.

“Professor - student conversation starts” Where is lambda/P bar. I am sorry. That is why I said I took the P bar to the right hand side, right. So it will be, so when I differentiate, P bar is stated as a constant. **"Professor - student conversation ends."** Let me just rewrite that in case this is causing confusion, -P bar, right.

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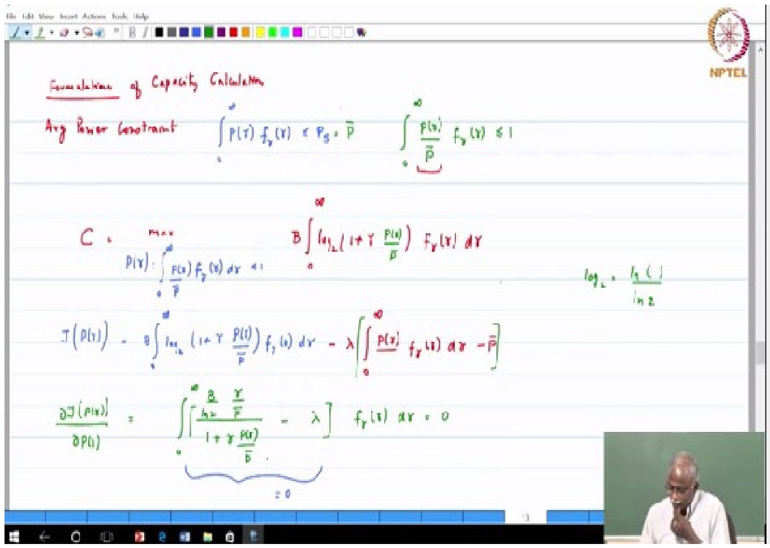

So when I differentiate with respect to P of gamma, that term goes off, right. That does not contribute anything, okay. **"Professor - student conversation starts"** So, I am sorry? (()) (12:55). Common band width, yes. (()) (12:59). Let me put it inside. Let me put that inside. Write as lambda B. Okay, I need to be a little bit careful. So I need to, let me keep this, let me put the B/lambda 2 inside. That is the easiest, that is correct. So let me put the B/lambda 2 inside. (()) (13:32) Right, is that okay. **"Professor - student conversation ends."**

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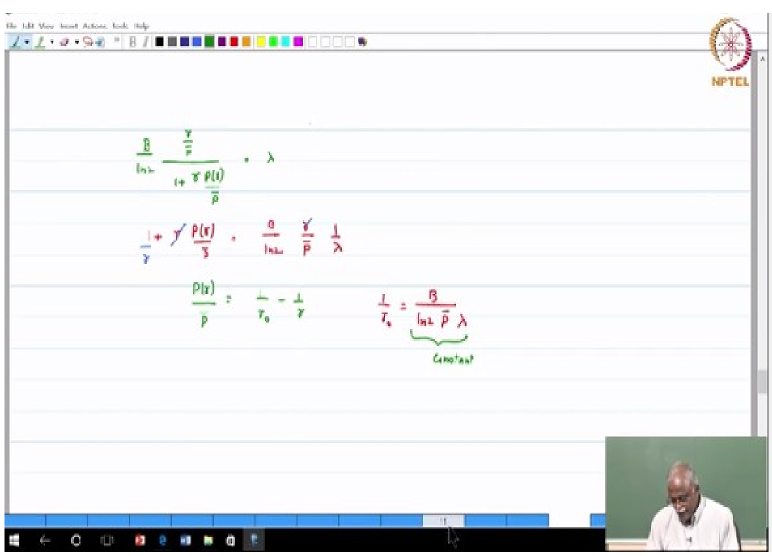

So let me just leave you with the thought that, this is the solution of the Lagrangian, we are just basically 1 step away. Let me just write it down and then pick it up from here in the next class. So B*ln 2 gamma/P bar/1+gamma P gamma/P bar, should be equal to lambda, okay. So rewrite this equation. Did I make a mistake? No, this is correct, okay. So 1+gamma P gamma/P bar, cross multiplying.

This comes out to be; B/ln 2 gamma/P bar*1/lambda, divide throughout by gamma. So then what we get is, this is a very important result. P of gamma/P bar, that is my power allocation, is actually comes out to be 1/gamma0 /1/gamma, where 1/gamma0 has been defined as the following. 1/gamma0 has been defined as B/ln 2 P bar lambda, okay. Now at the end of the day, B is a constant, bandwidth.

Ln 2 is a constant. P bar is your average power constraint, that is a constant. Lambda is a Lagrangian. So this is actually a constant. That is why we have written it as this. So the power allocation is actually given by 1/gamma0 - 1/gamma, okay. Now how do we interpret this is the crux of why OFDM was even born. So basically what we done so far is we have gotten a channel where there is variations.

We now need to find out what is the best I can achieve in terms of the Ergodic capacity. One way was to allocate equal power. The other way was to allocate differential power with an average power constraint. We went through the formulation as a Lagrangian. We showed that the power allocation, the optimum power allocation should be of the form 1/gamma0 - 1/gamma. Now what is the interpretation of this and how will it affect us, how will it lead to OFDM, that is the important question which we will pick it up in the next lecture. Thank you.