Multirate Digital Signal Processing
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Lecture – 26
All-Pass Decomposition, The Study of Mth Band and Nyquist Filters

Okay let us begin, the we are continuing where we left off in the earlier lecture, lecture 25. So really do not need to do a full recap.

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What are we talked about in the last lecture was the all-pass lattice and it is properties. So basically that is what we will start from. We have also introduced some terminology, a bounded transfer function, a bounded real transfer function, lossless transfer function, lossless bounded real and power complimentary transfer function. So again these are all going to play an important role.

The focus of today's, this lecture is twofold, one is to bring us to the context of the all pass decomposition. So this is going to be a key part of today's lecture. In the process we are going to pick up some useful tools by way of the factorization of polynomials, the property of Mth band filters which are also such as also satisfied by Nyquist filters. So again revisit something with Nyquist something with M band when specifically, when M equal to 2, that becomes, instead of calling 2 band filter we call it a half band filter.

So M equal to 2 is a half band filter, so this is the subset of this. So from general M we go to half band and that is the last stage before we get to perfect reconstruction for a 2 channel filter bank. Perfect reconstruction as we have defined it, aliasing is canceled, aliasing canceled, no magnitude distortion or phase distortion. So basically you get exactly the same transfer function back. So that is what we have been trying to achieve and that is what we will be able to get by the end of today's lecture.

Again if time permits we go all the way down or we stop at some point which is reasonable. So let me start with a quick point about lecture 25.

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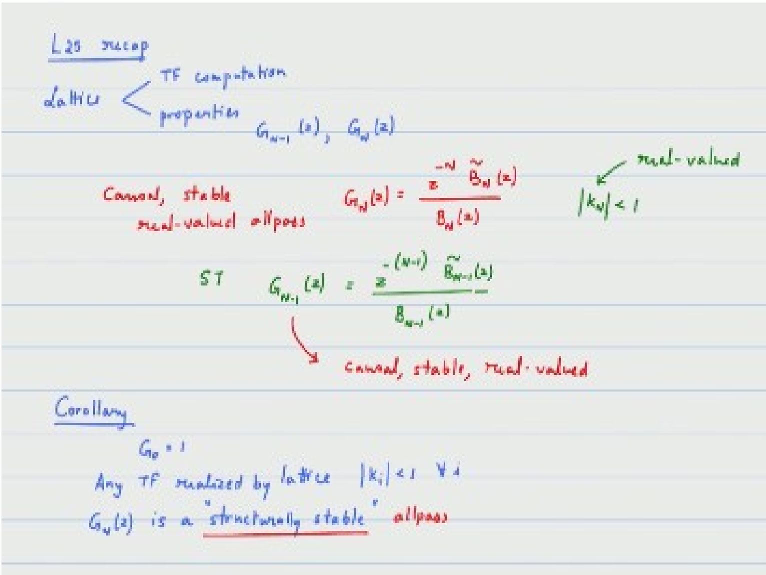

So lecture 25, this is what I would like you to carry away from lecture 25. So lecture 25 we saw a lattice. So with this being able to compute transfer function, transfer function computation. I think that is an important element. Then what are some of the properties and how do they affect the lower order transfer function G N minus 1 of Z and how do they carry forward to the higher order transfer function GN of Z.

So basically lattice preserve some properties and then how do we do that. So in our case the lattice structure did the following. So if I had a causal stable real valued all pass transfer functions, causal stable real valued, you do not have to specify real valued, but most of the time that is what we are interested in, all pass. Then we said that G N of Z can be written using the para conjugation property, a denominator polynomial BN of Z, Z power minus N, the para conjugate BN tilde of Z okay.

And in the lattice we enforce the condition that mod K N again this is with reference to the lattice, that mod K N is less than 1, that was because of the stability, to ensure stability. Now if we and of course we also said that K N will be real valued, will be real valued. We claimed and this is something that I would like you to validate, show that G N minus 1 of Z we have more or less done all the steps, but I would like you to go through and show that this is Z power N minus 1.

It is one order lower than the original polynomial. BN minus 1 of Z is the denominator, the numerator is BN minus 1 tilde of Z. So this is also an allpass okay, and the claim is that this all pass function is causal, stable and real valued; causal, stable and real valued; real valued mean by the real valued coefficients okay. Now there is a corollary that is written or that corollary that we can write down which you should be able to validate and prove that if you start with G0 equal to 1, that is basically G0 of Z or whatever you want to call the zeroth order all pass.

Then if you construct a lattice, any transfer function realized by lattice with the constraint that mod K less than 1 for all I; however, many stages you have. Then the overall transfer function GN of Z is a structurally stable all pass. That means the stability has been enforced by this structure, what you can show it is a stable all pass, but how did it achieve it is property, it is because of the structure and the way you construct from the lower order to the higher order.

And the property gets preserved. Structurally stable all pass, that is a useful result for us to verify okay. Now very quickly let us run through the other concepts and build on that.

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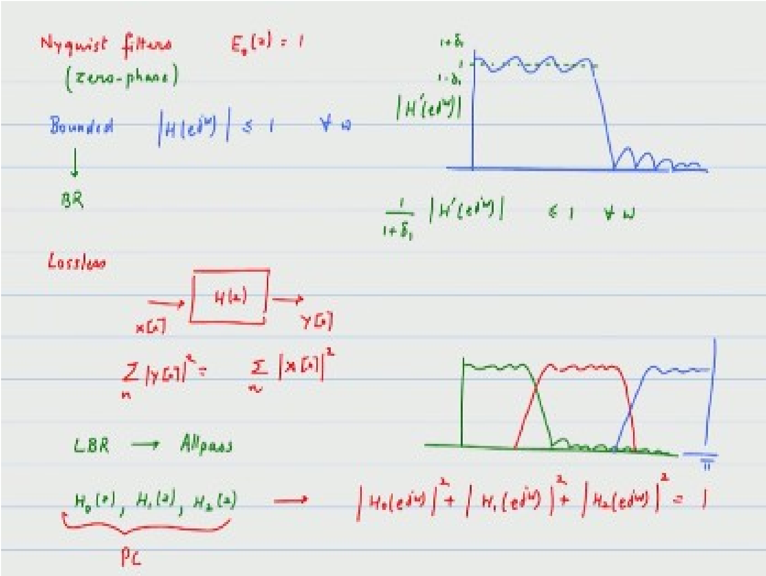

So we looked at the context of Nyquist filters, these are usually involved in interpolation or decimation. So we made the observation that when you have a Nyquist filter which is used for, for example interpolation by a factor of M then we saw that the zeroth polyphase component had only one nonzero coefficient. Basically if you view it as a zero phase filter okay. So if you view it in the zero phase form then that basically means that there is a middle point and then it is symmetric on positive and negative.

Of course if you shift it, this will become a delay, but basically one of the polyphase components has got only a single nonzero coefficient okay. That was an observation that was made, we will build on this in today's lecture. Now what are some of the other things that we talked about, one was boundedness. Bounded basically means that H E of J omega is less than or equal to 1 or some constant for all values of omega that is the property of boundedness.

And typically in the context of filters this is very easy for us to visualize. So for example if I had a low-pass filter which had a certain ripple and a stop band. It had a ripple of this type. I am just showing it with equiripple for illustrative purposes okay. If this line is 1 then the upper line is 1 plus delta 1, the lower part that reaches is 1-delta 1 okay. So it is basically bounded between 1 plus delta 1 and 1 - delta 1.

Now if this is H dash E of J Omega okay, magnitude. Now if I did 1/1 plus delta1. Again we are going to do this type of manipulations in today's lecture. If this is the waveform that is given then 1 over 1 plus delta 1 scaled H E of J Omega will always be less than or equal to 1 for all values of omega. So this scaling is a very important element in boundedness, but because the filter response you have basically have drawn, you can always make sure that there is an upper bound for it to be satisfied.

So when you have filters that have got a well-defined frequency response these are bounded transfer functions. So again this is a useful concept okay. Now of course a subset of this will be bounded real. I will just write it as R real coefficient. Then there is another set which is lossless. Lossless basically says if I input X of N to this filter H of Z and I get an output Y of N, the energy is preserved, summation over N mod Y N squared is equal to summation over N mod XN squared.

So that means you do not lose energy in the signal, again this is a very useful class of filters, class of transfer functions and you would call this as lossless. Now you can combine both of them and say there is a class of transfer functions that is lossless that is real by coefficient and is also bounded everywhere on the unit circle and this we said basically points to all pass functions okay, all pass functions this is the class that we are interested in.

However, there is another interesting class of filters, transfer function that we are interested in. So this is if you have for example H0 of Z. I am just going to write a different example from what we considered in the last class H1 of Z, H2 of Z and these satisfied the following condition, magnitude H0 E of J omega magnitude squared plus H 1 E of J omega magnitude squared plus H2 E of J omega magnitude squared is equal to one or a constant.

Then this set is said to be power complementary that means wherever one of them is large, the other ones will have to be small, so that they have to add up to the constant value and again this is a very interesting observation that we can make. So power complementary filters that means if I have this as one of the filters okay. Now the H1, H2 probably need to be something of this type okay.

You can visualize this as one combination of power complementary filters okay. This is all the way to pi okay. So you can visualize them as filters that look like low-pass bandpass and high-pass and when you square and add them they add up to a constant. This is a very special class, power complementary class and of course we are very interested in the case where you have 2 power complementary functions.

Because we are looking at the 2 channel case because we want to preserve the signal everywhere between 0 and pi, one of the filters being low-pass, the other filter being a high-pass okay. So this is a quick run-through of the key elements that we have discussed in the last lecture. Now we will spend the next 10 minutes or so looking at the all pass decomposition.

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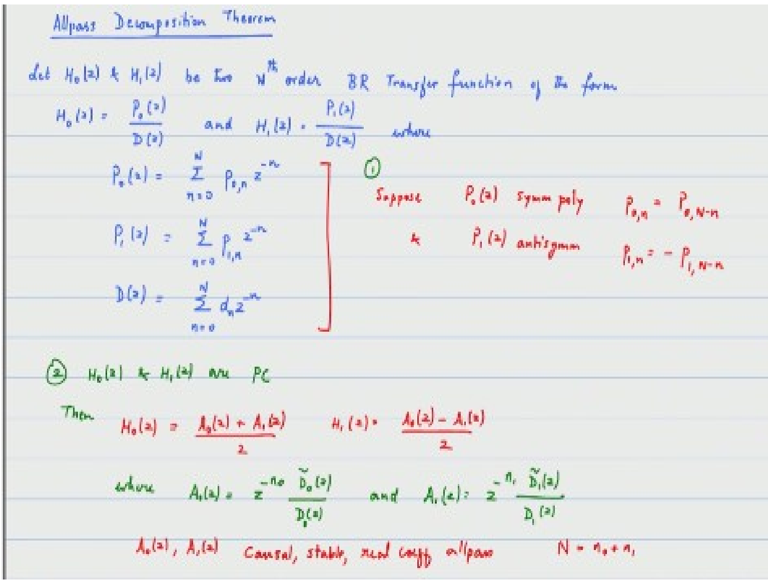

I would definitely encourage you to look up this portion in Vaidhyanathan chapter 5. It is presented as a theorem. Theorem meaning it is a result that can be proved, the proof is assumed to be have been read okay. So we will only summarize the result because at the end of the day we will give you enough steps in the proof, but the important thing is how do we take it forward to construct the two-channel case okay.

So here is the statement of the theorem and I will make it verbatim. So you can verify it in Vaidhyanathan's book. It is basically exactly stated so that there is no confusion as far as the theorem. Let H0 of Z and H1 of Z be 2 Nth order. Okay we are talking about IR transfer functions. So which means that they have numerator and denominator. Once you have taken off all the common terms, both zero cancellations what is left is the transfer function for whom we can define the order.

Are Nth order bounded real transfer functions, BR transfer functions. They are individually bounded real transfer functions okay. That means they are well behaved. There is no pole or something that is something that will shoot up. On the unit circle they are bounded okay. Of the form now some of it may seem unintuitive or counterintuitive at the beginning, but you know just take it down, we will justify the details subsequently.

H0 of Z is a polynomial P0 of Z in the numerator and a denominator D of Z and H1 of Z remember both of them are of the same order. So this one also has got the same denominator D of Z, P1 of Z, where the polynomials themselves P0 and P1. So P0 of Z is if FIR polynomial of order N. So N equal to 0 to N, that means the highest power of Z will be Z power minus upper case N. P0, N subscript zeros representing P0 polynomial N is the coefficient index.

Z power minus N, P1 of Z same structure N equal to 0 to upper case N, lower case P1, N, Z power minus N, D of Z is also a polynomial of order N. N equal to 0 through N. D subscript N, Z power minus N and under this framework we are going to make the following additional constraints. So if we further constrain it to B suppose that P0 of Z represents a symmetric polynomial. Symmetric polynomial which basically means that P0, n is the same as P0, N minus n.

So there is symmetry, it is a finite length polynomial. There is a symmetry that is there and P1 of Z is antisymmetric. So which means P1, n will be equal to minus of P of N minus n, N minus n, okay, so what have we said so far. There are 2 bounded real transfer functions which are numerator denominator, same denominator. The numerator polynomials one of them has got symmetry, the other one has got anti symmetric okay.

Now if in addition to that you add one more condition. So this was condition number one, a second condition that if H0 of Z and H1 of Z are power complimentary. Then the all pass decomposition theorem states that we can write down H0 of Z as A0 of Z plus A1 of Z divided by 2. Notice that this is slightly different from the earlier decomposition where we took the polyphase components to be this thing.

These are not polyphase components basically the filter itself is the sum of 2 all pass functions. H1 of Z is A0 of Z minus A1 of Z divided by 2 where the A0 and A1 are all pass functions, where A0 of Z is a all pass transfer function where you have Z power minus N0 D0 of Z D0 tilde of Z and A1 of Z is Z power minus N1 denominator D1 of Z, D1 tilde of Z okay, where these are causal stable transfer functions.

A0 of Z, A1 of Z are causal stable real coefficient all pass functions; causal, stable, real coefficient all pass and the condition is that the overall order of the filter now comes out to be N0 plus N1. So basically what you have done is you have split the Nth order transfer function into 2 transfer functions one of order N0 and order N1, they are two all pass functions of those orders and by combining them you are getting the final structure.

Okay now quick checks you can do to confirm that the denominator order is N because if you take these and you take the common denominator it will be D0 into D1 so order will come out correctly and then of course you can make arguments about the numerator as well. Okay now this is the theorem okay. Now it is actually not as complex as it sounds. All we are saying is there are 2 transfer functions which are power complimentary.

And if I choose the splitting correctly then I will get the two factors A0 and A1 to be causal stable all pass functions and this is exactly what will help us to address this issue, but we are going to basically take away from this whatever we can and learn from this. So here is the interesting parts of it, again I am going to give you just the highlights, but please do fill in all of the details okay.

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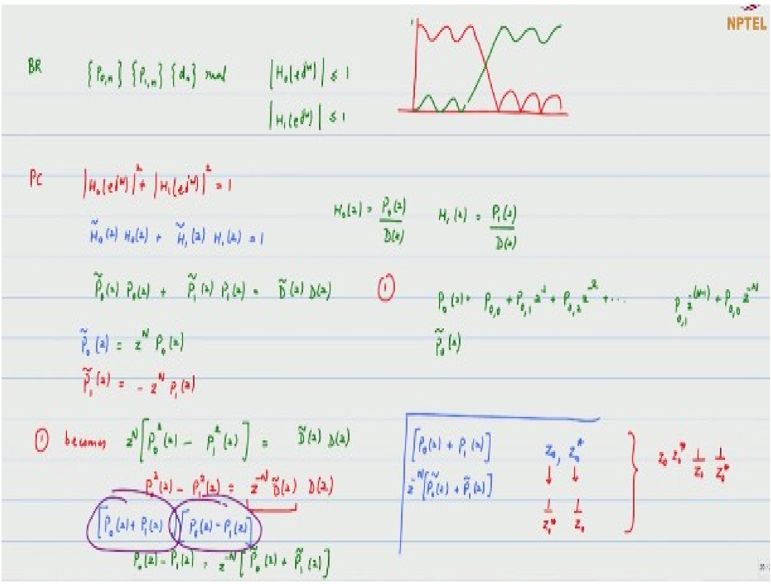

The first condition that was imposed that they are both bounded real okay. So and that means the polynomial's coefficients P0, N that set P1, N and D subscript of N, these are all real okay. That is the real part of it and the bounded part comes from H0 E of J omega magnitude less than or equal to 1, H1 E of J omega magnitude less than or equal to 1 okay that is the bounded real part.

It can be any constant, we have chosen it to be equal to 1. Okay second condition that they are power complimentary. So which basically says: magnitude H0 E of J omega magnitude squared plus H1 E of J omega magnitude squared must be equal to constant. Okay and now, this is not counter intuitive because they are power complimentary and they are bounded, which means that wherever H0 touches the value equal to 1, the other one has to go to 0 only then they can satisfy.

So actually, it will give you a nice picture if you want to visualize it. If you have a transfer function that is equiripple on both sides okay. Now the complementary transfer function will be exactly 0 where the other one is equal to 1, this is equal to 1, right that is the peak value. Wherever that is equal to one and basically wherever that reaches a minimum this will reach a maximum again it will touch 0 like this.

And then it crosses over because in the interim region also they have to add up to equal to one. Wherever this one hits zero, this one will reach a maximum and then subsequently wherever there is okay, so the power complementary property actually gives you a very nice relationship between the two and one of them being a low pass will force the other one to be a high pass so this is a very useful property for us to have.

By analytic continuation we say that this is equal to H0 tilde of Z H0 of Z plus H1 tilde of Z H1 of Z equal to 1 okay, that is the first step. Now if I can substitute for the expressions for H0 and H1, both of them have got expressions. H0 of Z is P0 of Z divided by D of Z, H1 of Z is P1 of Z divided by D of Z; substitute into this what we get a very useful form which is P0 tilde of Z times P0 of Z plus P1 tilde of Z P1 of Z okay.

And this is equal to D tilde of Z D of Z. Basically cross multiply okay. Now comes some interesting observations as to why some of those results or conditions were imposed P0 tilde of Z okay. So now I want you to sort of visualize, let me just write it on the side. If P0 of Z can be written in the following form P0, 0 P0, 1 Z inverse, P0, 2 Z minus 2 dot dot dot because of the symmetric conditions the last term will be P0, 0 Z power minus N.

The one before that will be P0, 1 Z power N minus 1. So that would be the relationship. So now if I do P0 tilde of Z what I have to do, I have to do the conjugation does not make any change, these are all real coefficients and I have to replace Z with Z power minus N okay, now easy to verify that if I do that then what will happen? I will get this polynomial will all powers of Z. if I pull out Z power N, I will get back P0 of Z because of the symmetry.

So P0 tilde of Z is nothing but Z power N times P0 of Z. That is a very useful result based on the symmetry that we have enforced. Similarly, P1 tilde of Z is equal to same type of relationship but now there is a minus sign because of the anti-symmetry. So this can be written as minus Z power N, P1 of Z okay. So we now say that equation 1 becomes with these 2 substitutions, it becomes P0 squared of Z.

There is a Z power N sitting outside, P0 squared minus P1 squared of Z equal to D tilde of Z, D of Z. Nice form - is P0 squared of Z minus P 1 squared of Z taking the Z power N to the other side. Z power minus N D tilde of Z that is a nice form because that makes it causal D of Z okay and left-hand-side of course you can factorize it into P0 of Z plus P1 of Z into P0 of Z minus P1 of Z okay. So some interesting observations which we will utilize and then quickly move forward into the key the crux of the, the reason for this.

Keep in mind that what we are doing is giving us some tools for us to develop the results. So in addition to the mathematics the tools that are being used are very important. Now the zeroes that are captured by P0 plus P1 and P0 minus P1 are the zeros of D of Z and D tilde of Z. So basically whatever are the zeros and it is reciprocal conjugates are going to be captured okay. Now there is yet another interesting flavor to this particular discussion, let me just add that also here.

P0 of Z minus P1 of Z if you use the results that we obtained just a few lines earlier, remember these results. If you use that then this is equal to Z power minus N P0 tilde of Z plus P1 tilde of Z, even better okay. So basically, P0 plus P1 will capture some set of zeros. The reciprocal conjugate zeros will get captured by P0 tilde of Z plus P1 tilde of Z. Now again this is a very important element.

Another observation which is useful for us okay, here it is P0 of Z plus P1 of Z. This is a real coefficient polynomial right, that is what we have assumed. These are P0 and P1 are real coefficient. So that means if there is a 0 at Z0 there must be a 0 at Z0 star, Z0 conjugate. So these zeros cannot occur individually except if they are on the real axis. Anything that is not on the real axis must occur as a conjugate pair okay.

Now if you now say that Z power minus N P0 of Z tilde plus P1 tilde of Z, if I look at this that basically says that wherever there is a 0, the reciprocal conjugate also has to be present so this will be 0 at 1 over 0, Z0 conjugate. This will give me a 0 at 1 over Z0 because okay. So effectively between the two of them you will get Z0 Z0 star 1 over Z0 1 over Z0 star or in other words you could make the following statement okay.

If you treat this as one entity, if you treat this as one entity and this as the other entity, you can make a statement which says that the zeroes of these 2 will be disjoint, one will have a set of zeros and the other will have the reciprocal zeroes. Now the question will be is how do you know that they are disjoint. What happens if they are on the unit circle? Then the reciprocal will also be on the unit circle.

Now what did the assumption on H of Z? Stable. So which means that you cannot have 0 on the unit circle. So zeros of D of Z must be strictly inside the unit circle. Zeroes of D tilde of Z will be strictly outside the unit circle okay. So and also keep in mind that so between D of Z and D tilde of Z you have 2N zeroes that will also have to be captured by these 2 subsets in our discussion okay.

Is what we have said so far is it reasonably clear? Okay, now deliberately I am going to leave this and ask you to think about another topic that you have already studied in DSP, but it is very useful for us to revisit at this point okay.

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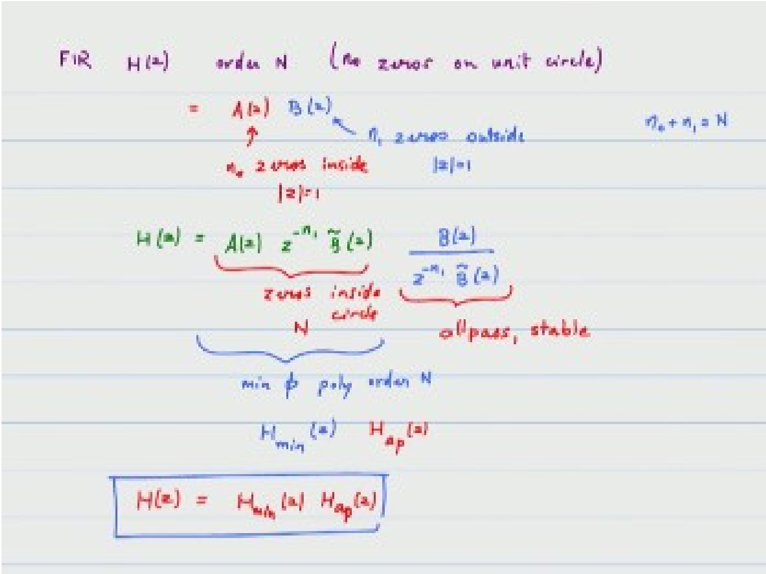

Now supposing I have a FIR filter H of Z, H of Z. Now it has got a finite number of zeros, supposing it is of order N, it has got N zeros. Now some of the zeros are inside the unit circle, some of the zeros are outside the unit circle and no zeros on the unit circle. This is the assumption that we are making. No zeros on the unit circle. Now this is an aside maybe you wanted to write this on a new sheet of paper so that you can continue the earlier discussion subsequently.

But for now this is a useful observation, no zeros on the unit circle okay. So now if I group the zeros that are inside the unit circle which I will call as A, A of Z. So and there are let us say N0 zeros inside the unit circle, inside mod Z equal to 1, multiplied by B of Z which captures the zeros that are outside the unit circle and this says there are N1 zeros outside mod Z equal to 1 okay and of course N0 plus N1 must be equal to uppercase N, total number of zeros that are present okay.

So here is the key statement which I hope will remind you of a result that you have seen previously. I can always write H of Z in the following manner, A of Z instead of B of Z I am going to write the para conjugate. So para conjugate means I must write Z power minus N1 B tilde of Z. Now if B of Z has zeros outside the unit circle the para conjugate will have zeros inside the unit circle.

So which means that this whole setup has got zeros inside the unit circle, zeroes inside unit circle. How many zeros? N, we have got N zeros inside the unit circle. So this actually is a minimum phase polynomial of order N okay. So this combination is actually a minimum phase polynomial of order N. In fact, we could write it as H min of Z. Why can I write it as H min of Z?

It has got the same magnitude response as H of Z, but it has got all the zeros inside the unit circle, am I correct in making that statement? because the para conjugation gives you on the unit circle, it gives you the conjugate response yeah. **"Professor - student conversation starts"** this would not be equal to H of Z. Yes. I have not finished that yeah. **"Professor - student conversation ends"**

So I would have to now write the remaining part which would be B of Z divided by Z power -N1 B tilde of Z correct. Now this is an all-pass, is it stable? The zeroes of B of Z are outside the unit circle, the para conjugate will have zeroes inside the unit circle, so therefore this is the stable all-pass and all-pass means magnitude equal to 1. So this is HAP of Z. So this is where the statement comes from where we say that any FIR transfer function can be written as H min of Z times HAP of Z where HAP is a stable all-pass transfer function.

This factorization I am sure you would have encountered and this is a very useful form because if suddenly someone says I want to have a minimum phase filter you can actually very easily do it by reflecting the zeros inside the unit circle, the magnitude response will not change, but you would have affected the phase response because the all-pass function has got a phase response, but no magnitude response.

So again it is a useful result, but it plays an important role in what we have to do in the discussion now okay. Now what are the things that we are, what has this sort of corollary result has shown is when you think of polynomials, think of zeros inside, zeros outside and always keep in mind that the para conjugation flips the location of the zeros, keep that picture in mind very useful for us.

And we will now in next few lines we, so now we go back to picking up from where we left off okay. So where did we leave off.

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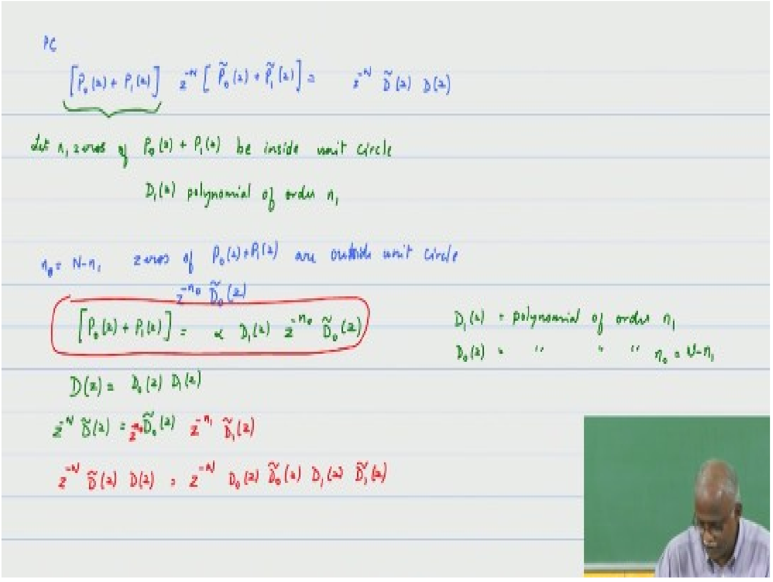

We said that the power complementary constraint can be written as P0 of Z plus P1 of Z okay and Z power minus N P0 tilde of Z plus P1 tilde of Z this is equal to the expression that we have Z power minus ND tilde of Z, D of Z okay. Did we get that correctly, I think so, okay, if I did make a mistake just catch me on that okay. So now let us look at the polynomial P0 of Z plus P1 of Z okay. Now let we do not know where the zeros like. It is a combination of 2 polynomials. So let N1 zeroes of P0 of Z plus P1 of Z be inside the unit circle, inside the unit circle okay.

And the others are outside the unit circle. How can I make that statement? Why not on the unit circle? Because what is there on the left-hand side must be on the right-hand side and the right-hand side does not have any zeros on the unit circle. That has be the inside or outside okay. So we will call this factor as D1. So D1 of Z is a polynomial of order N1, polynomial of order N1 okay.

So where will these zeros come from D1 that will be a subset of D of Z because D of Z has got zeros inside the unit circle. Some subset of it got captured by P0 plus P1. Now P0 plus P1 has got some zeroes outside the unit circle. How many zeroes outside the unit circle basically it has got N0 equal to N minus n1 zeroes outside the unit circle, zeroes of P0 of Z plus P1 of Z are outside the unit circle okay.

So basically these will have to be factors of D tilde of Z okay. So let us call this as D0 tilde of Z. Basically these are factors from there. So in other words because it is a order N0, if I want to make this causal, this must be Z power minus N0. So then I can write an expression for P0 of Z plus P1 of Z okay. Now this is equal to some proportionality constant alpha times D1 of Z. Z power minus N0 D0 tilde of Z okay.

Zeroes outside the unit circle they have to be part of D tilde. So I am using the notation D0 tilde to be some factor of D tilde of Z and so this is a useful form for us okay. So D1 of Z is a polynomial of order N1, D0 is a polynomial of order N0, which is equal to N minus N1 okay. So now what we do is basically what we have done is we have said D of Z got split into 2 parts, one of which we call it as D1.

The another one which called it as D0. So if I now visualize it as D of Z as D0 of Z times D1 of Z, Z power minus N D tilde of Z is D0 tilde of Z which will be Z power minus N0 then Z power minus N1 times D1 tilde of Z. If I multiply the two together Z power minus N D tilde of Z times D of Z then what I will get is Z power minus N0, Z power minus N1 will add up to Z power minus N, D0 of Z, D0 tilde of Z, D1 of Z, D1 tilde of Z okay.

I have just done the factorizations okay. Hopefully there is nothing confusing everything is straightforward just go through. So keep this expression in a visible place.

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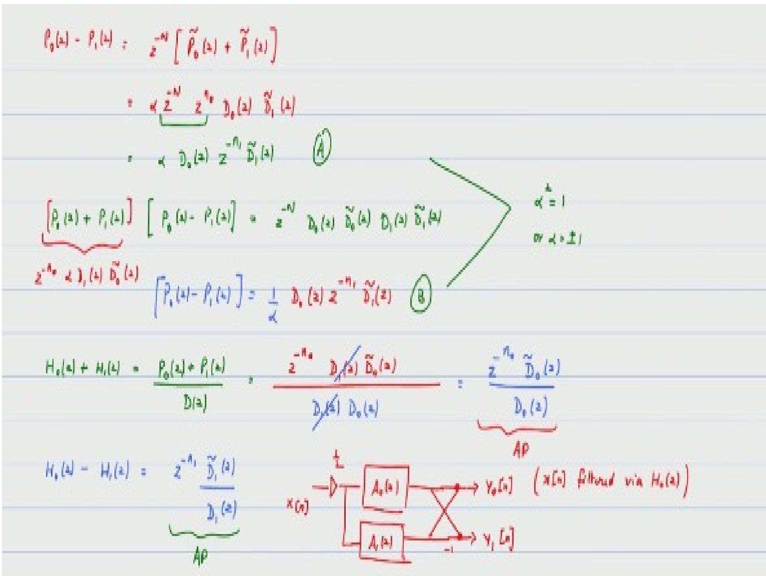

Now what about P0 of Z minus P1 of Z, it must contain all of the zeros that are reciprocals of P0 of Z plus P1 of Z right. Because this is equal to we have showed the expressions that Z power minus N P0 tilde of Z plus P1 tilde of Z okay. So P0 plus P1 you can take a tilde of the entire expression. So this will be equal to alpha times Z power minus N whatever was there in the expression for P0 of Z plus P1 of Z you have to take the conjugate of that.

So this will become Z power N0 because you are now what we are doing is replacing Z with minus Z and this one becomes previously we had D0 tilde. So now what we get is D0 of Z. Previously you had D1 of Z, now it will become D1 tilde of Z. So basically this would be the. So combining these 2, combining these 2 you find that what we get is alpha D0 of Z, Z power minus N1 D1 tilde of Z very nice okay, that is a nice expression that we have been able to get okay.

So this is an expression that we have been able to get. Now if you also look at this following expression P0 of Z plus P1 of Z we said that this is equal to this is a factor this is equal to alpha times D1 of Z, D0 tilde of Z, then the second term here is P0 of Z minus P1 of Z, this is equal to Z power minus N D of Z, D tilde of Z. So which basically can be written in the following way. D0 of Z, D0 tilde of Z, D1 of Z, D1 tilde of Z right.

So from this you get another expression for P0 P0 of Z minus P1 of Z okay. So this will have taken this to the other side you will have 1 over alpha basically take off the, did I make any mistake? if I have made any mistake just let me know so that I do not get. Z power minus N0 is also there okay. So there is a Z power minus N0. So take it to the other side, what you will be left with is D0 of Z, Z power minus N1 Z power minus N1 D1 tilde of Z okay.

So basically we got 2 expressions let me call that as A and B, both are valid because what did we say P0 of Z minus P1 of Z, we use the direct expression. We also looked at the overall expression and obtained it. If you look at these two A and B will tell us that alpha squared must be equal to 1 or alpha equal to plus or minus 1. But for this plus or minus1 scale factor the expressions are correct okay. So here is the crucial step, combine the results that we have so far H0 of Z plus H1 of Z.

This is given by P0 of Z plus P1 of Z divided by D of Z. So P0 of Z plus P1 of Z we already have an expression for that. It is Z power minus N0. I am going to take alpha equal to plus1 D1 of Z, D0 tilde of Z denominator Z power it is actually D of Z. So D of Z can be written as D1 of Z into D of Z okay. So this can be written as D1 of Z into D0 of Z that is the denominator. I am to cancel D1 present in the numerator and denominator. This is Z power minus N0, D0 tilde of Z divided by D0 of Z okay.

Likewise, please verify that H0 of Z this is an all-pass function okay. Now H0 of Z minus H1 of Z, H0 of Z minus H1 of Z can also be verified and you can show the following and you can show that this is equal to Z power minus N1 D1 tilde of Z by D1 of Z which is also an all-pass function okay. So H0 plus H1 can actually be implemented in the following scale factor of one half all pass function A0 of Z, all pass function A1 of Z, a crisscross.

The upper branch goes with the plus1, lower branch comes to the minus 1, the overall transfer function will be the 2 filters that we are interested in H0 and H1 okay. So if this is X of N, this is Y0 of N which is X of N filtered by H0 of Z, filtered via H0 of Z and the other one will be the filtered by H1 of N okay. So what is it that we carry forward from here that if I enforce the poly phase components.

So let us go back to the theorem. If I enforce the symmetry, enforce the power complementary property and with the appropriate order then we can get the transfer functions H0 and H1 to be a sum and difference of all pass functions okay. So this is a useful result in the context that we have been able to you develop a couple of interesting concepts where we talk about separation of zeros inside the unit circle outside the unit circle and then the factorization and then how to get the results from there okay.

Any questions on the all pass decomposition that we have developed? All pass decomposition part that okay. **"Professor - student conversation starts"** (()) (51:22) **"Professor - student conversation ends"** The symmetry basically comes in this part, the symmetry and the anti-symmetric that is where the, that is what helps us do the factorization otherwise you would have gotten H0 squared plus H1 squared.

When you get them as differences then you can then get the sum and difference of the okay. Now are these stable all passes, are they stable all pass functions? Yes, because both of them have got factors of D and D is stable all the zeros of the polynomial are inside the unit circle. So therefore everything is stable, everything hangs together. This is an interesting result, a very nice result okay.

But more important what did we get out of it all the tools keep that in your mind. Okay so now we move on to the next topic.

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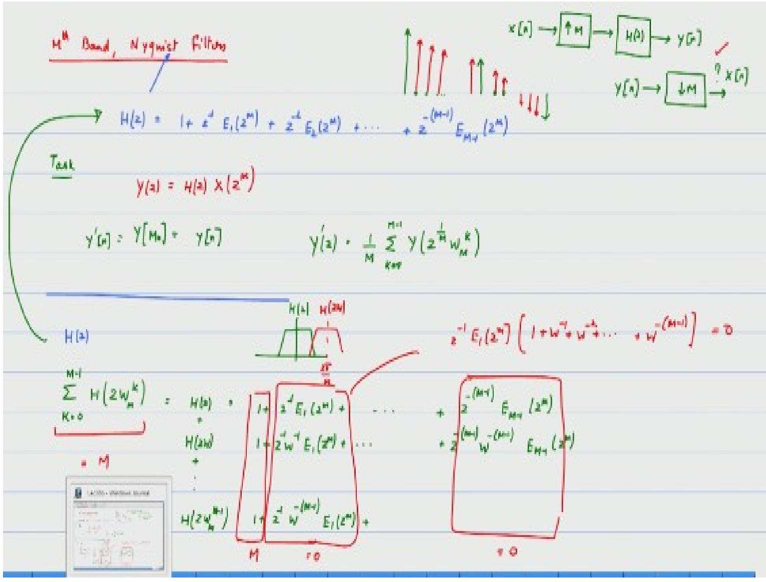

That is the Mth band filter and the Nyquist filters okay. Now we said the following that if I were to do a Nyquist filter okay I am looking at the Nyquist portion of it, the Nyquist filter, we said that we could write the filter in the following form H of Z as some constant one, because the one of the poly phase components has got only one coefficient plus Z inverse times E1 Z power M plus Z minus 2 E 2 Z power M dot dot dot plus Z power minus M minus 1 E M minus 1 Z power M okay.

Now here is a task for you to do. What is this structure of the interpolation filter tell you that there are 0 crossings of the impulse response correct. So when I convolve, when I convolve, when I do the interpolation initially I have zero valued samples some number of them, these are my input samples. When I do the convolution with this filter what happens is these zero values get filled in.

But the original coefficients do not get modified, notice the green ones remain in the same value, they do not get modified. So for example this is how this interpolation will fill in these values, but those zero valued and that is a very important result. So if the green was the input signal X of N went in, got up sampled by a factor of M and then you did the interpolation filter then you get the complete signal okay.

Now let me call this as the interpolation filter H of Z after which you get the composite signal which is Y of N with the zero valued samples. Now if I take Y of N and down sample it by M will I get back X of N? If the interpolation filter had been a Nyquist filter then the input has to because you did not modify those coefficients remember it is a zero phase filter, so there is no shifting of the signals happening okay.

So this is actually true, there is no question mark, you will actually get this okay. Now can you show what you know from intuition and through the verification graphical verification you have shown this. So can you show that Y of Z equal to H of Z times X of Z power M right that is the equation that is left for us. Can you show that Y of M N equal to Y of N. It is slightly non-trivial exercise, but once you see the trick you will see that it is fairly straightforward.

I do not want you to use the graphical argument, I want you to use a mathematical framework to because now Y of MN basically means that Y of Z. So basically lets me call this as Y dash of N. So Y dash of Z is 1 over M summation Kequal to0 through M-1, Y of Z power 1 by M WMK right you have to do that. Now go in and substitute and show that this hangs together. So assuming that you have done that exercise I want you to pay attention to this result H of Z which is the filter that we have shown here, the filter of that form.

Can you tell me what happens if I add shifted versions of this filter K equal to 0 to M minus 1 H of Z WMK okay, what is H of ZWMK? If H0 of Z was a low-pass filter this will be shifted by 2 pi over M, this is H of ZW if this was H of Z okay and now if you take all the shifted versions remember H of Z will have to be a low-pass filter with cutoff pi by M right plus minus pi by M. So if I shift all of them they will cover the entire range 0 to 2pi.

But I want you to see what happens. So basically the first will be H of Z plus H of Z W plus I am going to just write it in this form just for ease of WM M minus 1 okay so basically this is the summation that we are trying to do, H of Z is what 1 plus Z inverse E1 of Z power M, sorry E1 Z power M plus dot dot dot plus Z power m minus minus M minus 1 EM minus 1 Z power M, H of Z W that means wherever there is Z replaced with Z W this one will become 1 plus Z inverse W inverse E1 Z power M remains as E1 Z power M because W raised to the power M.

So this remains as Z power M, likewise this becomes Z power M minus 1 W minus M minus 1 E M minus 1 Z power M okay, now please go through and write down the expression for the last one 1 plus Z inverse W minus M minus 1 E1 Z power M plus dot dot dot I will just take the first equation and so I have to add all of these. If I add this vector, this vector I get M because all of them are plus 1s. Now if I add the next column, notice I have E1 Z power M Z inverse as common.

And then what I will have inside so this can be written as Z inverse E1 Z power M multiplied by 1 plus W inverse W minus 2 plus W minus M minus 1 roots of unity. So therefore this is equal to 0. So similarly you can you can, so this adds up to 0. Similarly, each one of these columns you can verify adds up to 0. So basically what we said was this expression for a Nyquist filter will add up to M.

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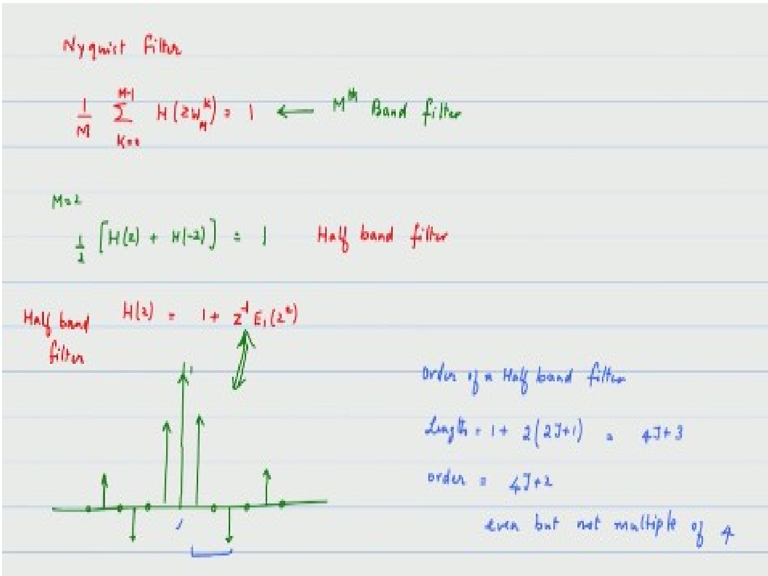

Or in other words a Nyquist filter also satisfies the following condition that 1 over M summation K equal to 0 to M minus 1 H of ZWK must be equal to 1 okay. This is what we refer to as the Mth band property. Any filter that satisfies this condition is called an Mth band filter. Now does an Mth band filter always have to be a Nyquist filter? Need not be, it just so happens that the Nyquist filter satisfies this property.

So from the Nyquist filter we derived a certain property which is a very interesting property which we then say okay this is something that we would definitely be want to keep in mind in our toolkit. Now the case when Mequal to 2 can you write down the expression for me? This is H of Z plus H of – Z add up to one half scale factor of one half must be equal to 1. So this is the condition of a half band filter okay.

Now that means the a half band filter when you factorize it, for a half band filter you will get the following form, it will be equal to 1plus Z inverse E1 of Z squared which is of course you know take the Mth band case your polynomials you are dividing this. Now is this, I want to ask you if this is intuitive? If this is intuitive that this is what will happen and answer is well it is actually fairly straightforward.

Because the Nyquist filter when I want to do up sampling by a factor of 2 will give me a nonzero value which is equal to 1 then the next sample will be 0. Then I will get a non-zero well, wait a minute, every second sample will be 0, so there will be a non-zero value next and then it crosses 0. So this is how a Nyquist filter with pi/2 as the cutoff will look like because every second coefficient will turn out to be 0.

So now if I do the polyphase decomposition on both sides this is a symmetric filter notice that you will get one coefficient of one and then all zeros elsewhere. So this would be the repeating structure. So this actually validates this particular form okay. So the first zeroth order poly phase component is equal to zero. So now may ask you a question may seem a little bit trivial. What can be the order of a half band filter?

Again so it all hangs together so there is a 0 valued coefficient. If I want to look at the non-zero value, supposing either I want to take up to the second nonzero term what would be the, it is some odd number because I can always leave off the last one. So the length of half band filter is going to be 1 plus some odd number of terms. So I will call it as 2J plus 1 and there will be a symmetric one on the other side.

So the total length will be 1 plus 2 times 2 J plus 1, why 2J plus 1 because it has to be an odd order, odd number of samples because I can either get 3 samples or 5 or 7 or 9 likewise in a half-band filter, I am only talking about a half band filter. So which means that the length is given by 4J plus 3 that is the length of the filter okay. If this is the length of a half band filter, the order is always 1 less than when assuming that I have shifted them all to make them causal, the order will be 4J plus 2 okay.

Which means it is even but not multiple of 4, an interesting observation, interesting filter, interesting results, but still not quite sure where this is going to take us and how we are going to work with the whole thing. Let me just end with one last result for today again this is a very straightforward result. Once we have this in place we will be able to finish the discussion in the next class okay.

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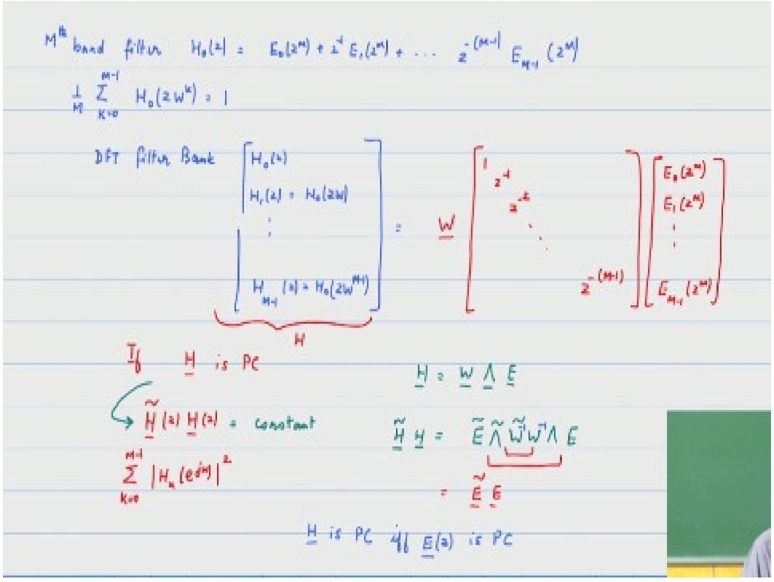

So let us take Mth band filter, Mth band filter basically says that we have the following condition 1 over M Kequal to 0 to M minus 1 H0 of ZWK equal to 1, equal to a constant okay. So H0 of Z is the Mth band filter, the property that it satisfies is that the sum of the shifted versions scale by 1/M is the constant. Now I would like to write down the DFT filter Bank, okay which is basically H0 of Z, H1 of Z which is equal to H0 of Z W and last one is H M minus 1 of Z equal to H0 of Z W M minus 1 okay.

So this is the DFT filter bank, in terms of the poly phase components basically I would now have the poly phase components as E0 of Z power M plus Z inverse E1 Z power M, E0 of Z power M will be a constant, but for now I am just writing it in its full form, Z power M minus 1 EM minus 1 of Z power M okay. Now please verify I think we may have already written this, but in case we have not this can be written as the DFT matrix times a diagonal matrix which is 1 Z inverse Z minus 2 dot dot dot Z power M minus 1 times E0 of ZM, E1 of ZM EM minus 1 of ZM okay.

Now here is a very important result. If you want the vector H, if this is the vector H, it has got M minus 1 M filters. If this vector should be power complimentary what does that mean H? H H of Z that tilde para conjugate times H of Z, because this is the same as saying this is summation K equal to 0 to M minus 1 mod H, HK E of J omega magnitude squared when you equate it on the unit circle right that is power complimentary.

But if I want to write it in a general form the power complimentary property should be enforced in this following fashion. This should be equal to a constant okay. Now what condition does that impose on the polyphase components? If you just substitute the fact that H can be written as W times delta which is a diagonal of times the vector E. Now H tilde times H, I am just dropping the Zs for now.

This can be written as E tilde, okay now this is delta tilde W tilde but W is only the matrix it is a constant matrix so basically there is no powers of Z. So it will become the conjugate times W delta E. Notice that these 2 will cancel each other, this will be powers of Z, this will be powers of Z inverse, they will cancel each other leaving you with E tilde E okay. So there is an important result I will do it in a minute.

H is power complementary if and only if E of Z is power complementary. Now this result actually comes into play in a intricate way in understanding the M channel case not the two-channel case, but what is very important for us is the last result that I will write down before we end today. There was a question **"Professor - student conversation starts"** (()) (01:12:45) **"Professor - student conversation ends"**

Okay so maybe this one should be W inverse W inverse tilde okay yeah I will have to verify because I think you are right because what you will get is the negative powers of W which is the IDFT matrix correct yeah it is absolutely correct yeah okay. So the relationship between H and the polyphase components is an important one okay. Now I just want you to look at one more element before we close today's discussion because this is also useful in important result for us.

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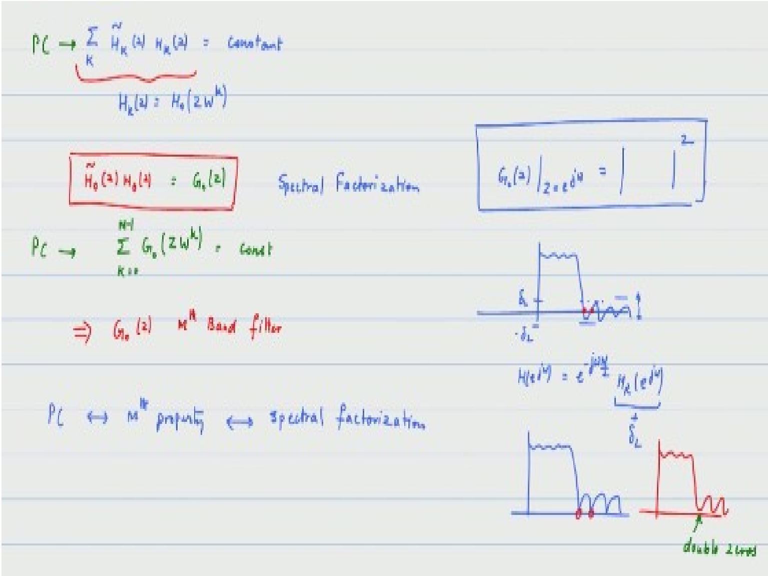

So what did we write down summation over K HK tilde of Z HK of Z must be equal to a constant okay and we have the relationship that H K of Z is nothing but H0 of Z W raised to the power K. These are all shifted versions of the filter. Now a very important observation. So the first term in this summation, the first term in the summation will be H0 tilde of Z H1 tilde of H0 of Z.

Let me call this as G0 of Z okay. So basically H0 tilde H0 I have called it as G0. So this is the power complementary constraint. Now rewrite the power complementary constraint with G0, it will come out to be summation K G0 Z WK K equal to 0 through M minus 1, am I correct okay. So what does that make G0, this has to be equal to a constant. So this means G0 of Z must be a Mth band filter okay.

Now how do you get an Mth band, how do you get a power complimentary set of functions? M functions that are power complementary, trivial. You design an Mth band filter, you have to do this. So G0 equal to H0 of tilde H0, now what can you tell me about the zeroes of H0 and H0 of tilde? They are reciprocals of each other, they are no overlapping unless it is zeros on the unit circle okay.

So what do you have to do? You have to take G0 take its zeros and map them into either H0 or H0 tilde - that process is called spectral factorization okay and you may have come across this in control system, we do that when we want to have a certain design constraint on our filter, this is called spectral factorization. So design an Mth band filter, take a spectral factor, I will give you a set of power complementary functions.

Because all the DFT shifted versions are power complimentary. Now Mth band filter has got a very unique property that it's polyphase component one of them is a constant and other things okay, but a very interesting interplay is seems to be present between the following conditions, power complementarity Mth, band property and spectral factorization okay. So there is a very interesting interplay of these 3 elements.

So now I want to ask you, I want to get only 2 power complementary filters. I want you to design for me 2 power complementary filters, what will you tell me? Take a half band filter take its spectral factor and that will give you power complimentary property okay. I hope it is pointing you in the direction of how do we get perfect reconstruction, we will answer that in the next class.

The question is if I have Mth band filter satisfies the Mth band property and in this particular case what we have shown is that now maybe, if I rephrase your question the question is can I always factorize an Mth band filter as a polynomial and it is para conjugate, that is a very important question because this is possible only if the, if I look at G0 of Z on the unit circle it must be of the form something magnitude squared.

As long as I it satisfies that then magnitude squared can always be factorized as polynomial and para conjugate, no issues with that. So I would come about to answer your question I would answer the question I mean I would say that when can you always guarantee that G0 of Z can be factorized in this form, when this condition is satisfied right. Now again a very important follow-on question is take the half band filter, half band filter we showed we sketched it here.

H E of J omega need not be, if I plot this response it could be something that does this right. It can have positive or negative values. How do I ensure that this is spectral factorizable? If this is delta 2 right, it oscillates between plus delta 2 and minus delta 2 plus delta 2 and minus delta 2. If the filter H E of J omega, if it is a linear phase filter E power minus J omega N by 2 HR E of J omega this is real valued.

If I add delta 2 to this what will happen? It will shift the response to be strictly positive greater than or equal to 0 okay. It does a very important thing. If I take the magnitude of this what will it look like? the magnitude response? It will look like this, am I right. Notice where are the zeros, they are here and here. If I, that means they are here and here. Now when I keep shifting this response higher what will happen these 2 zeros will come and merge.

So what happens in this case if I shift this by delta 2, I will get a response that does not have the sharp edges, but has smooth values and these turn out to be double zeros because those 2 zeros actually merged into one. So these become double zeros, okay. So you must have double zeros only then you can factor them. If it was a single zero, then there is no way I can factor them.

And remember the zeros of H0 and zeros of H0 tilde must be reciprocal of each other right so that also has to be satisfied. So spectral factorization, so the next lecture I will begin by saying when I have something of the form of magnitude squared okay what is the structure of the zeros? You will find that the zeros have a very specific structure and that lends itself to spectral factorization.

So basically G0 which is a half band filter if you design it shift it up so that all zeros become double zeros then you are permitted to do a spectral factorization of this form where you can actually satisfy the half band property and the power complimentary property okay. We will pick it up from here at the next class. Thank you.