Multirate Digital Signal Processing
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Lecture – 18 (Part-1)
DFT and High Resolution Spectral Analysis – Part1

Good morning. We begin lecture 18. Today's focus will be on using the DFT for High Resolution Spectral Analysis. Yesterday we said that DFT is the spectral analysis tool but inherently there are some limitations in the spectral resolution that you can have, and so we pick it up from there and study how we can modify the DFT to get high resolution spectral analysis.

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So this is a very, very important application and something that I am sure you will find very useful whenever you have to do spectral analysis. We also will touch upon a couple of multirate applications, multirate DSP applications. And these are useful tools again some of them are for filter design some of it are for various different settings, hopefully you add it to your toolbox. By the way the cascaded implementation of a sampling rate converter we have already discussed so I will not touch upon that again.

I am sure you would have probably heard of the term Transmultiplexer thing, transmultiplexer. So a transmultiplexer, the two forms of multiplexing that we are commonly encountering in communications one is Time Division Multiplexing the other is Frequency Division

Multiplexing, right Time division you interleave the signals in time, in frequency division you interleave them in frequency.

So something that converts from time division multiplexing to frequency division multiplexing and back that is a transmultiplexer. It is not used in terms of terminology too much because nowadays you do not have so much of focus on communications transmission; this primarily comes in the transmission side. We focus tend to focus more on the information theory the modulation, demodulation those aspects.

This comes in the transmission side so it is called Tmux again that is the term. Now Tmux seems to be in a completely different domain but it turns out that multirate DSP gives us a very nice interpretation of that. And it is actually that interpretation of the transmultiplexer that tells us that OFDM and and Multirate DSP are also very nicely linked. We are going to spend several lectures on the topic of Maximally Decimated Filter banks.

These are very widely used all the way from the latest hearing aids to the spectral analysis tools and so this would be something that, it is good for us to get a flavor for. The general case is an M Channel, M channel Maximally Decimated Filter Bank and the fact that you have maximally decimated basically tells us that there is a potential for aliasing because unless you have ideal filters you cannot decimate by; if you have unless you have a filter that cuts off at 2pi over M perfectly you cannot decimate by M.

But this is a possibility that aliasing will happen and you have to get rid of the aliasing. It turns out that this is a fairly challenging problem and so we will limit ourselves to M=2 and you will find that, that itself is a handful and will just indicate what are the ways in which you would extend it to be on M=2. But I think when we talk when we come to the topic of maximally decimated I will spend some more time on that.

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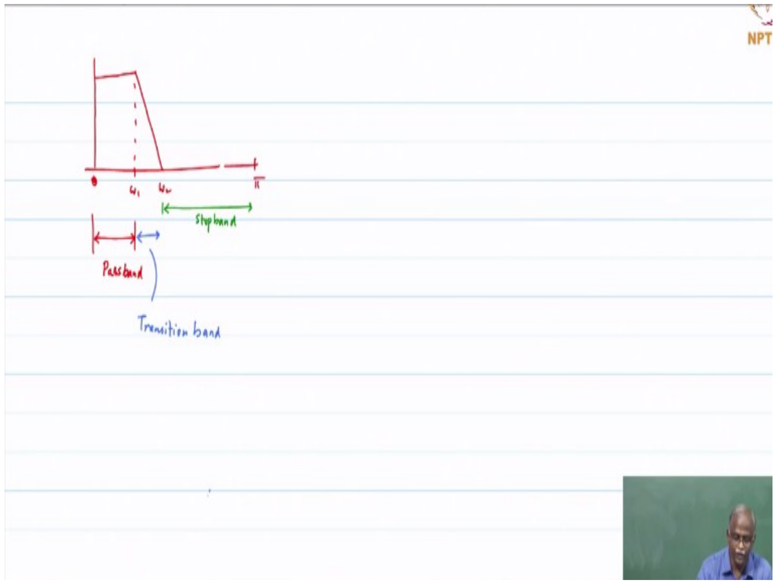

So a quick review of what we have discussed in the last class so that we can start to build on the topics that we have for today.

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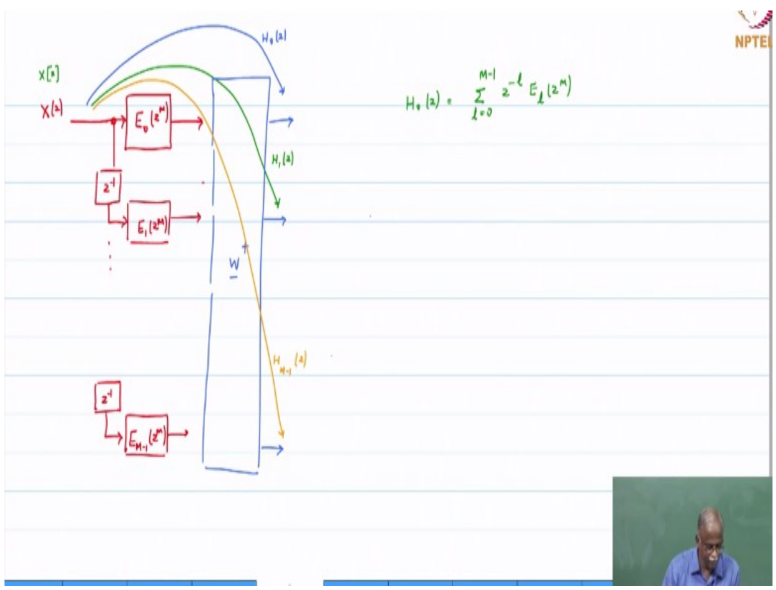

The fundamental principle that we are working with is the DFT Filter Bank. So basically we are looking at a bank of filters M filters each of which are shifted in frequency. We did a polyphase decomposition and the polyphase decomposition was H0 of z is equal to summation l=0 to m-1 z-1 1 e l z raised to the power m. And then we applied the shifted versions and we showed that what emerges is the following structure where you have the where you had the polyphase components of H0 and then the IDFT matrix. And again this structure was derived in the last class.

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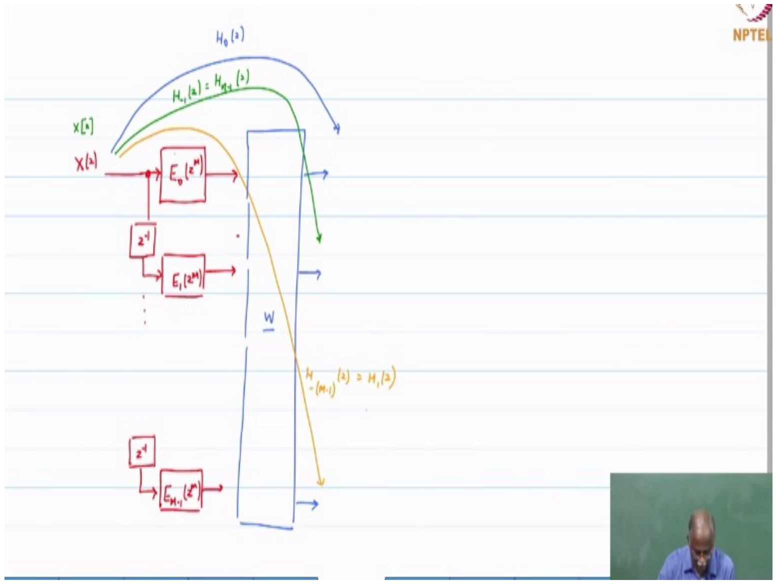

Now moving ahead, you could change it to the DFT matrix we showed that, but then the filters will be somewhat shifted it is not in the same positions but you get basically the same output even in that case.

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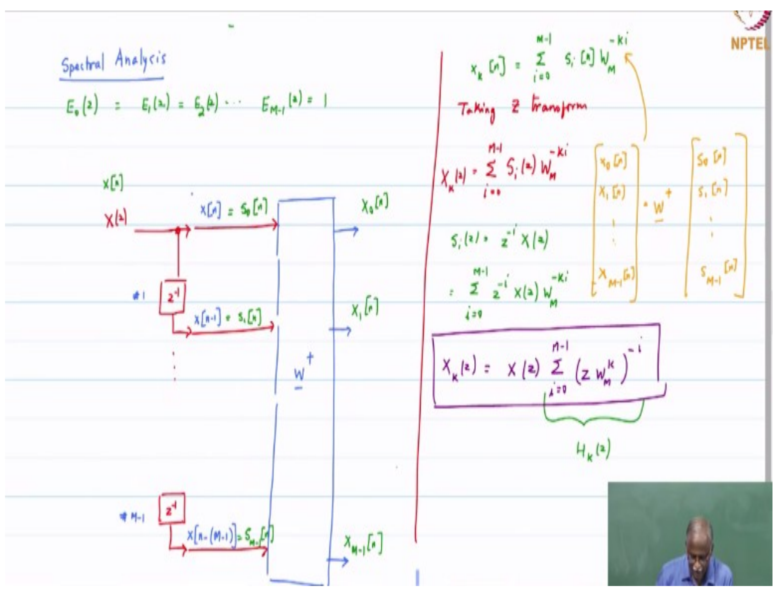

We then looked at a special case, so let me just insert a picture okay.

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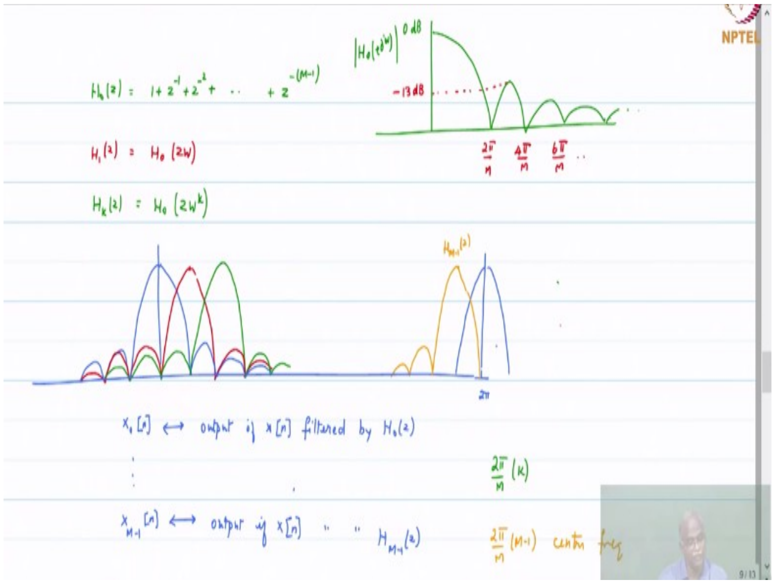

We had a special case. Now the special case was what linked us to the spectral analysis, special case was when all the polyphase components were equal to one, that is the same as saying H0 of z=1+z inverse …+ z power –m-1 So this is a special case and it turns out that this is nothing but what you do when you are taking the IDFT or DFT. And using the structure we showed that this corresponds to passing the input sequence through a bank of filters, right.

And the bank of filters and if it is the IDFT basically all the polyphase components have been set equal to one then you have the IDFT it is like passing through a bank of filters given by this graph. And the counter parts that we showed was that if you changed it to advance operators then and automatically the DFT appears in this case and then you get also the same it is as if passing the output through a sequence of staggered filters exactly the same as in the previous case.

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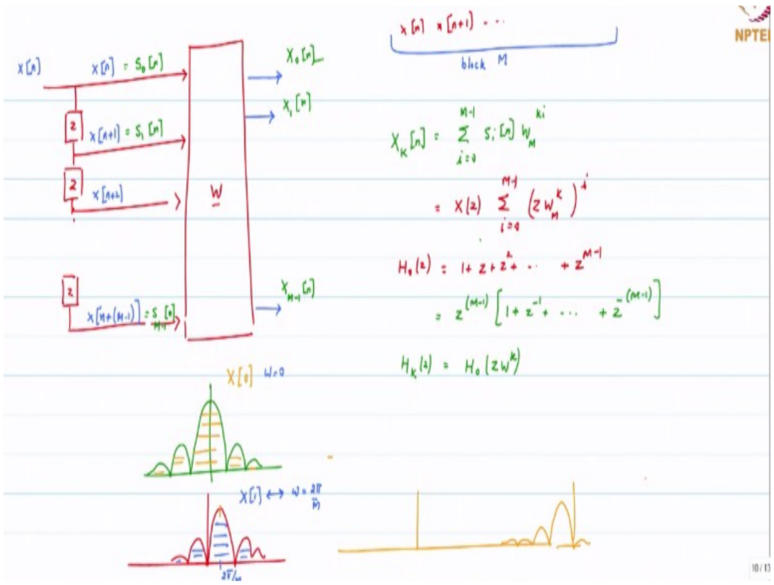

So this is more or less what we had developed in the last class and towards the end we said okay now if I wanted to use, so basically this is the way we would compute DFT. You take an input, create a block, pass it through DFT and then assume or call the output as the DFT coefficients and then interpret them as the spectral components the value of the spectral components at those frequencies. We said that okay these are not the values exactly at that frequency alone it is actually contributed to by several components even outside of that range.

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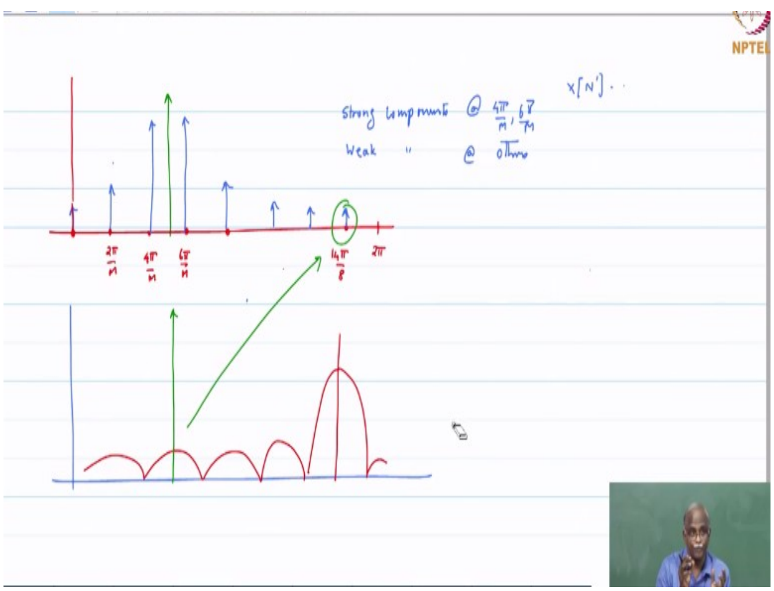

And a consequence of this is the observation that we saw here. So if I have my input signal has got a tone single tone which is lies in between two DFT frequencies, you will find that all of them have got non-zero values, what is puzzling is why would a frequency component which is far away from my tone still show something non-zero and the reason precisely is that if I if it is not a Delta function is actually a filter that is looking at the input spectrum.

So basically, then it does have some component comes out of it and that is what shows up as the DFT coefficient. So we see that the DFT in its inherent form is it valuable tool, no doubt no absolutely no doubt that it is a valuable tool. But it has its limitations. So the key point to remember is that when you interpret a DFT based in a spectrum you just need to be careful that you interpret it correctly.

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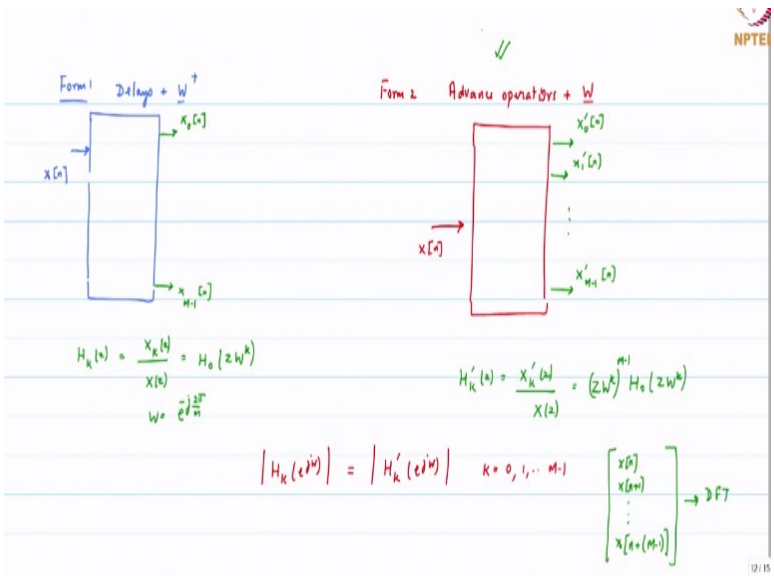

So let me just ask you to do a couple of things and we will build on that for today's class. So Form 1 or Structure 1. Structure 1 is a combination of Delays plus the transpose conjugate of the DFT matrix, right. The input is x of n; input is x of n, okay. I am going to draw this as a complete black box, what is inside is a delay chain with the IDFT matrix okay. And the outputs are M outputs; the labels on them is x sub 0 of n; x sub M-1 of n.

And the transfer function Hk of z between the input x of z and the output x k of z, this is nothing but H0 of z w k where w is actually w subscript M e power –j 2pi/M, okay. So this is Form 1, which we have studied and so this is the interpretation of that. Now using this as a reference can you tell me what this will be? Form 2. This is Advanced Operators. A chain of advanced operators plus the DFT matrix, okay.

So I have x of n going into my box that box has got the chain of advanced operators and the outputs x0 dash of n x1 dash of n … x M-1 dash of n, okay. Now as I did in the previous case I am interested in the transfer function H k prime of Z = X k prime of z divided by X of z. Can you tell me what this transfer function is? It is H0 of XW k, Z W k raised to the power of M-1, okay Z W k is raised to the power. Okay you can look at the expressions and then and derive that.

But basically what this says is that modulus Hk e of j omega the magnitude response of what I get from the component that is there in Form 1 is the same as what you will get from Form 2, Hk prime e of j omega magnitude the phase terms will go; this is for all the values k=0 to M-1, okay. So these are equivalent forms both of which are useful for spectral analysis; both of them are using the rectangular window as the underlying filter.

It is not a just a delta function that is sampling your signal spectrum; it is your signal passing through a filter which has got a well understood, well defined frequency response. So this is useful for us to keep, so again I assume since most of us are from the communications background our understanding of taking the DFT is taking a sequence of x of n, x of n+1, so basically you would form a vector x of n; x of n+1… x of n+M-1.

Typically, that is how you would block the vector, create the block and then you would take it through the DFT. So this is the form that we have and basically this also gives us the exact same interpretation.

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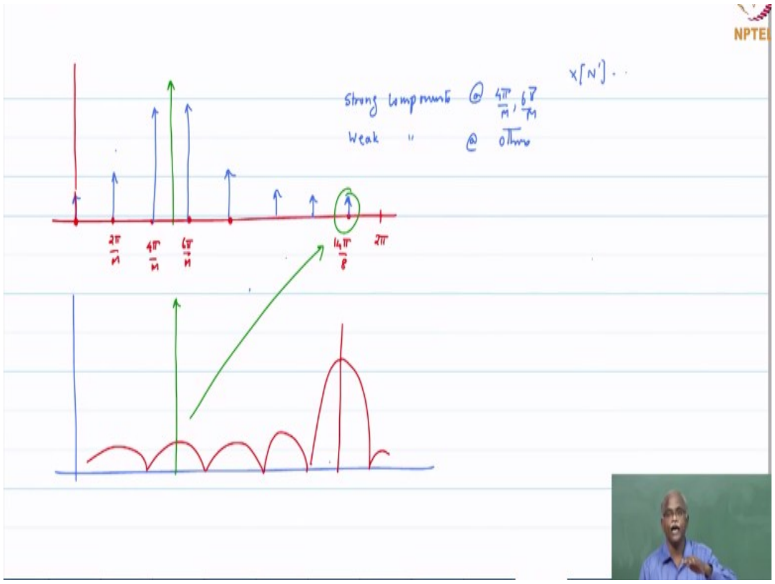

Okay. Now I want to solve this problem, right. I do not want this; I do not want this kind of faulty or misleading interpretations. What are some of the ways; yesterday after class several questions were there and we discussed it, but more importantly I just would like to get your feel for what are the some of the ways in which we could address this issue. Why is the spectral lobe so high? If you go back this is the frequency response.

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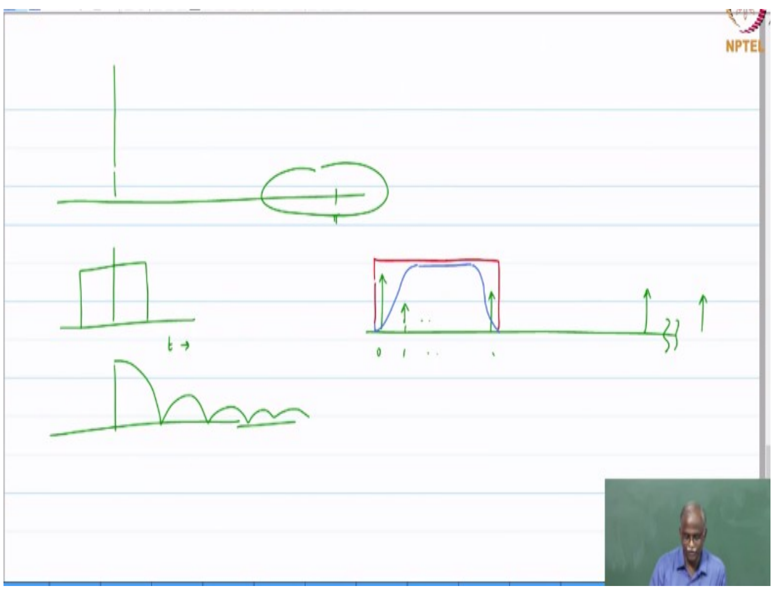

Now you will get high frequency response or so if I were to plot the frequency response of an impulse response or any signal, this is pipe. If I tend to see a lot of energy around the high frequencies right which basically means that my frequency response has got components all the way from DC to, that indicates that there have been some sharp transitions. Usually sharp transitions are the ones that cause that.

So for example, if I were to have a brick wall in the time domain; this one this is in the time, okay this is in the time, you know that this is infinite in frequency in the continuous time. So basically what you will get is something that will keep going, okay. And so did; was there a sharp transition in the time domain, yes because you took a rectangular window. So what did you do you took a segment.

So basically you had; okay I am just going to quickly sketch these things okay. So 0 1 so something some large number and the signal is continuing; what we said was okay I am going to take the DFT of this portion. So I set it is effectively like setting all the other terms to 0 these terms are taken without any modification. So basically a rectangular window was applied to this data. So the first option would be is other than a rectangular window; can I apply some other window basically.

So and there are several windows; you can have, let us say a window that has a smooth response okay. That sort of removes the sharp edges. Can you guess the frequency response of this?

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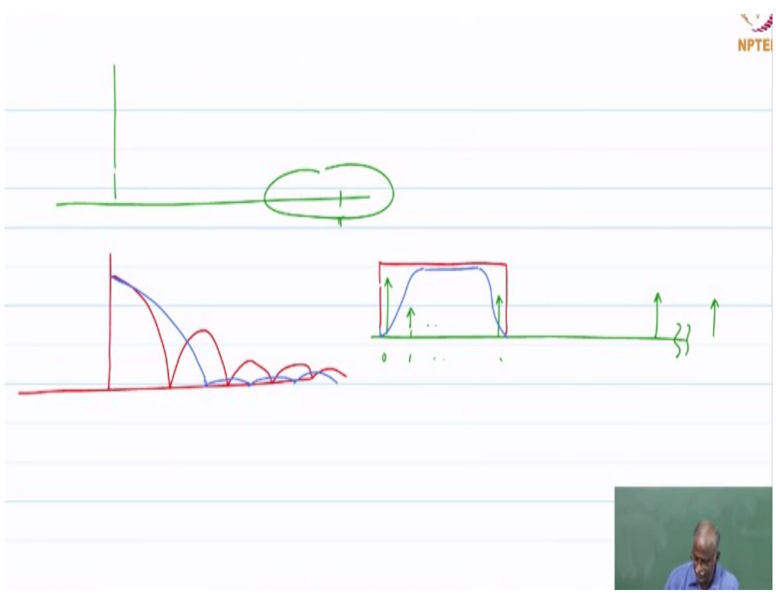

So let me just draw the corresponding so the red version is a rectangular window so let me draw the spectrum of the rectangular window, okay for the blue. It will have lower side lobes; what is the price that I pay? It will have a wider transition. But it will have a, okay what is the price that you will pay? Do you get the spectral analysis the resolution that you are looking for? What will happen if you go back to this case?

You will get a strong result for 4pi/M you will get a strong result for 6pi/M. Actually 2pi/M also maybe slightly higher because you have got a wider transition band, right that is the price that we pay. But the good news is that these high frequency components would not be there because those would have died down because of the window that you chose. Now question, how will my structure get modified for this?

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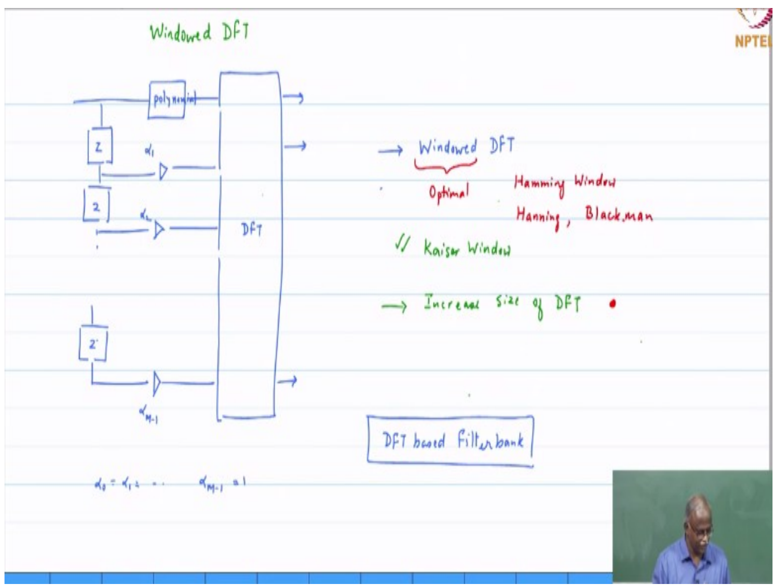

So the structure will be as follows. The original structure we said would have a set of delays advanced operators said z z … z. These branches, previously we were taking it directly to a DFT. Am I right? And effectively that was saying that all my polyphase components are constants and we set them equal to 1. Okay. So let us since they are constants let us call them as Alpha0, Alpha 1, Alpha 2, Alpha M-1.

And the special case is when Alpha0 = Alpha 1 = Alpha M-1 = 1 that is your conventional DFT with the rectangular window. Now if they are not equal to 1 and they are shaped in some response then you get the windowed DFT. So this is what you would refer to as the windowed DFT. Okay. So you solved one problem that is frequency components far away from where the tone lies those will be substantially reduced.

But I am little bit ambitious, I want better resolution close to the signal as well; increase the window lengths. So which means that take a larger sample why because if I double my sample space the length of the sequence that I am working with then what will happen to the window the window will automatically be reduced in half, to half, it is half its width and basically will give you a better resolution around the spectrum that you are looking for.

So the technique that I suppose we would we would tell for improving the spectral resolution one is do a windowed DFT. Am I right? Windowed DFT. And there is actually a complete area of research in the 80s which looked at optimal windows. And it turns out that the most commonly used windows are the windows you call as a Hamming window and occasionally you may use a Hanning window or a Black man window.

These are all the common used windows because they want to increase the spectral resolution of the DFT; they want to minimize the spectral leakage. This solves the spectral leakage issue because you do not want a DFT coefficients far away from where the tone lies to actually give you value. So that is what the windowing window you will do. But these are not the optimal windows.

Actually it was shown that the optimal window is something called the Kaiser window. This is your optimal window; it tells you how wide the spectrum or how wide the main lobe be and how much attenuation you want. So you can get even 90 dB attenuation 90 dB attenuation on this on the side lobes and Kaiser Window basically gives you a way to design these windows and a tool to design this windows and this would be the right way to do it if ever you have to do a spectral.

And we also said that that one is looking at the window; second if it is possible you would want to work with a higher size DFT; increase the size of the DFT; increase the size of the DFT; there are times when you cannot increase the size of the DFT; when can you cannot increase the size of the DFT, well that is one such one such limitation. The other limitation is that it is a non-stationary signal, because the signal has got time varying characteristics.

Let us say a burst of tone appears for a period of time. If you take a very large window you will you lose that information. So you may not be able to have the luxury of increasing the DFT side. So sometimes this may be a problem, right. Of course windowing you can do; this one you may not be able to do. In which case let us say that the maximum DFT size that you have is M, okay.

Now I cannot go; some number, but then I would ask you and I want better resolution I want better resolution. I want you know, I want to have a spectral analysis that is both well resolved in terms of frequency and spectral leakage. And then we go back and ask look at this structure if; from a DFT point of view you will say, nothing I can do. I applied the best possible window but you are constraining the size of my data set to be this so therefore I actually I am limited.

And this is where if you go back and look at this structure and say okay. Now why is this just a multiplier? We choose it to be that way. What is the most general case? It can be a polynomial. So remove this and go back to the original form where you now have E0 z M or basically; bring back polynomials. You have to be careful when you write this thing. So bring back a polynomial, okay.

So basically let me just erase this portion and indicate that this could be; it does not have to be a constant it could be a polynomial. And of course you can change all of them to be polynomials they do not have to be the same they can be different and that takes us back to the DFT based Filter Bank. So the DFT based Filter Bank tells us that we can achieve a bank of filters with very low complexity of this form.

As a special case we made the polyphase components to be equal to one constants in which are equal to one and showed that, that is the DFT. And we showed that that has got limitations in terms of spectral resolution when you want to do spectral analysis; when you go back a full cycle and say okay if you want to get very good spectral analysis then take design a very good filter; of course use the DFT or IDFT whichever one you want does not really matter both of them are equivalent.

And once you have that as your underlying framework then your ability to work with spectral analysis significantly improves and so; I hope that helps you get a perspective again the more you work with these tools in Multirate DSP you will get more insights into the techniques and the tools that that we have developed and hopefully these are things that are useful. Any questions on spectral analysis, how to get high-resolution spectrum analysis, yeah.

Definitely, because if take for some cases; if you are at one; if this becomes one of your DFT points frequencies then you will get a tone; all the others will have 0 crossings so therefore you will actually get precisely the spectrum that you are looking for, yeah. So basically if you take a filter and you take its polyphase components so each of these will be a polynomial, right. Those will be the polyphase components, yeah.

So in the general case those are polynomials and but it is not a random polynomial it is a polynomial that is derived from and filter impulse response. Okay, very good I am very glad from the questions that you ask. I am very happy that there is a good understanding of the concepts that I had wanted to emphasize.