Multirate Digital Signal Processing

Prof. David Koilpillai
Department of Electrical Engineering
Indian Institute of Technology – Madras

Lecture - 17 (Part-1)
Spectral Analysis of Filter Bank - Part1

Good morning, we will begin lecture 17. As I mentioned today is the day we discuss about the spectral analysis and this I believe will be a very useful tool and lot of interesting principles that we can build upon.

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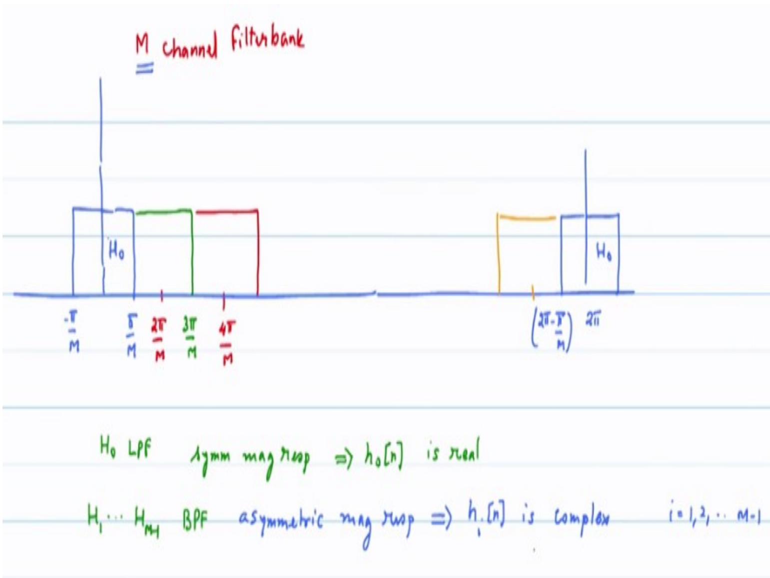

So a quick summary of what we have discussed in the last class and then we dive into today's lecture. We are looking at Filter banks where we use the alphabet M to denote the number of channels that is a very common term we call it as a M channel and of course there are special cases where M=2, M=3 and of course larger number like M=16 and 256 also. So M characterized the family of Filter banks the most general form.

And we are looking at uniform Filter banks that are obtained through the shift of the frequency of the magnitude response.

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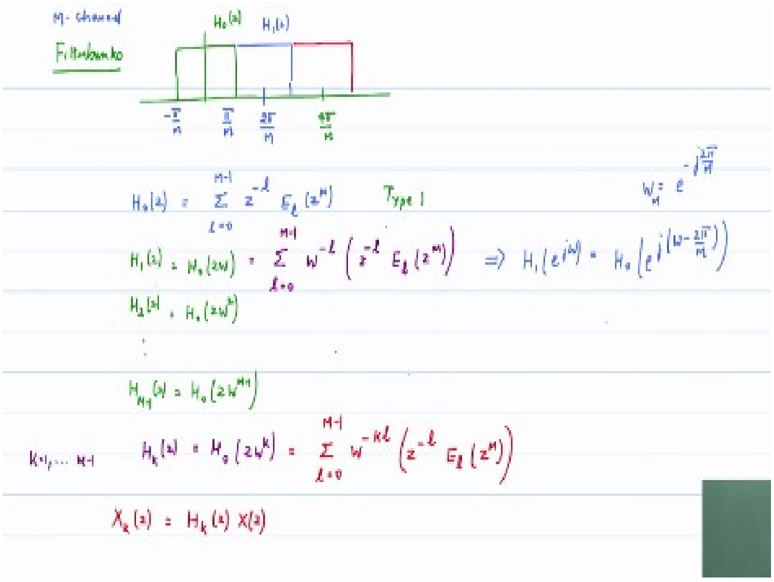

So in the last lecture we defined H0 of Z H1 of Z and H0 of ZW. So H0 of ZW if you look at in the on units circle it will be H1 e of j omega W is given by e power –j 2 pi over M this is W subscript M. In case I forget to write down the subscript the default is M. So if you write it in terms of the argument it will be e power j omega-2 Pi over M which basically says that the frequency response of H0 get shifted to a center 2 Pi over M.

And of course H2 will be shifted to 4 Pi over M and all the way to M-1 and we showed that if we were to apply the polyphase decomposition then we got a form of this type. I hope you are able to get into this and this part is comfortable.

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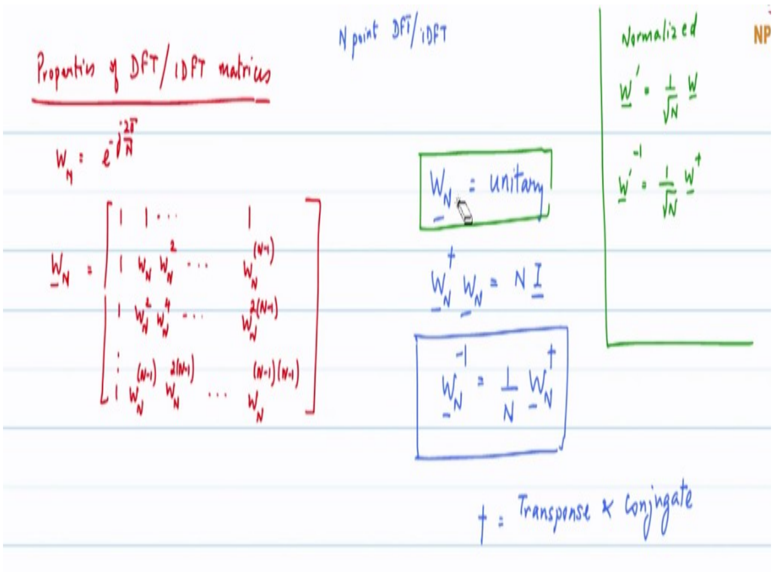

Then we look at the properties of the DFT matrix in general we talk about an N point DFT or IDFT and that is why we have chosen M there and N here. If you are talking about M channel

Filter banks then the DFT size will also be an equal to M. So N point DFT or IDFT that is what we have generally talked about and you find that the structure of the DFT matrix which you would have studied in DSP we made the following observation again probably a repetition for most people that the W matrix which represents the DFT is unitary.

And if you take W dagger W you get N times I that is a definition of unitary matrix and which also tells us that the inverse can be very easily obtained for a DFT matrix it is just a conjugate transpose with the scale factor. The dagger symbol is a matrix operation transpose followed by conjugation in the case of the DFT matrix it is symmetric. So therefore transposition this is the same matrix back so it is effectively just conjugation of the matrix.

So that sort of a very, very easy form that we have to have. There is what we refer to as a normalized form of the DFT matrix. So in the normalized form again this is more of an observation. So in the normalize form so that the forward and the inverse transform looks similar. We defined the forward transformation matrix as 1 over root N times W. So basically there is a scale factor of 1 over root N that is given to normalize the matrix.

So in that case W prime inverse will be 1 over root N W dagger. So basically you have a 1 over root N scale factor for both matrices and not just 1over N for the inverse and this is sometimes used but very often in signal processing. When you look at it primarily as a transformation and you want to have very similar looking transforms and you want to have a normalize transformation then you would take this form.

But in DSP our basic definition is W as given in the slide and W inverse will have the 1/N scale factor to get the appropriate identity matrix. The decomposition or the task that we were asked to do is to look at this type of structures and look at what happens when you replace this with a DFT matrix not the conjugate transpose of the DFT matrix. So I hope you had a chance to look at it if not we will just spend a few minutes.

So that you get a chance to sort of visualize what it is because that is an important element in the discussion that we are going to be having today.

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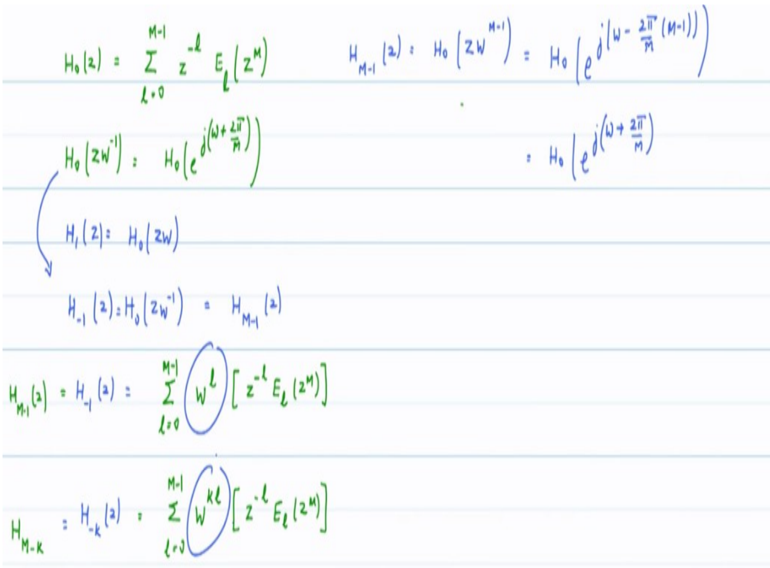

So H0 of Z again going back to the polyphase decomposition. H0 of Z write it down in the polyphase form l=0 to M-1, Z power –l E0 of Z power M. So an important point that we want to make is previously we got the shifted versions by the following form. The previous case it was obtained by H0 of ZW. So if I want to get the transpose conjugate let me look at ZW inverse.

So I just want to look at what does H ZW inverse actually what does it represent. So if I were to look at H0 at ZW inverse you can see that this actually corresponds to I think it as H0 of e power j omega plus 2 Pi over M. If I were to substitute H ZW inverse this is what I would get. Now also look at what you get for H of M-1 as per the original definition all these are as per the original definition. This would be H0 of Z W M-1 that would be H0 of e power j omega -2 Pi over M times M-1 close bracket, close bracket.

Look at the argument of the exponent e power j omega-2pi over M into M-1. If you simplify you will verify you can verify this is nothing but e power j omega plus 2 Pi over M. So basically H of M-1 and this H0 of Z W inverse. So if you were to follow the same notation or the convention that we were using if H1=H0 of ZW. So basically the power of W becomes your subscript for the filter then this would correspond to H of -1. This would be H0 of ZW inverse this is nothing, but H of M-1 of Z as per the original representation.

So basically if you were to write down in the following manner H -1 of Z which is actually equal to H of M-1 of Z we have to write down the expression for this. This would be equal to summation l= 0 to M-1 again substitute and verify that what you get is W raise to the power l and within bracket Z power -l El Z power M. One more general case if you were to write down for H of –K.

So H of -1 is H of M-1 this will correspond to H of M-K this is equal to summation l=0 to M-1 W Kl within bracket the same expression Z power -l El Z power M. So if you go back and look at the figure that we had previously obtained. Now basically what we have is that this matrix is no longer the W dagger it is actually the W matrix if you go back and look at our expressions and notice that these are all the coefficients of the DFT matrix.

Did I get that correct yes? If you had negative powers, then it would be part of the IDFT matrix yes this is correct. These are all coefficient that you will find in the DFT matrix.

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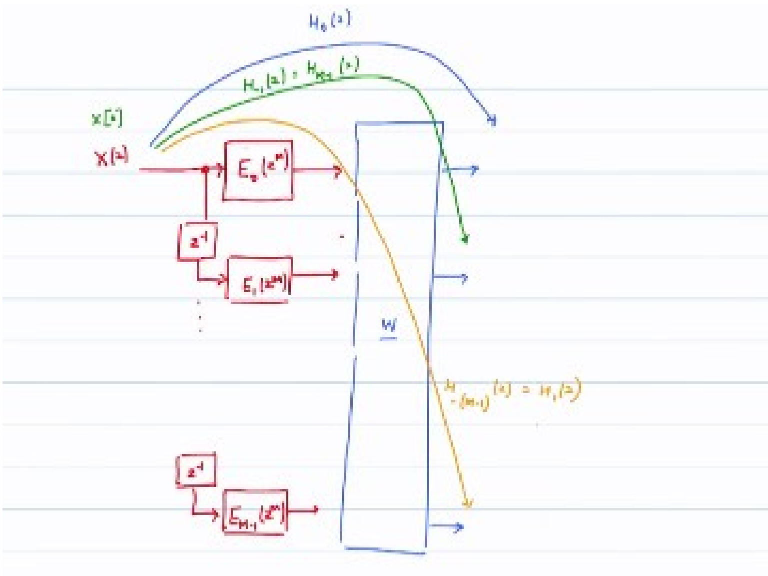

So that is why we now have the DFT matrix here and very quickly let us write down the transfer function. So between this point and this point the transfer function is H0 of Z if you go back and look at the expression there. Now between this and this point the second branch that will come out to be H of -1 of Z because of the DFT matrix. The polyphase components are the same the delay chain remains the same.

And we know that H of -1 is nothing but H of M-1 of Z and go on all the way to the last one. The last transfer function between the input and the last transfer function that comes out to be H of - M-1 of Z that will come out to be H1 of Z. So the summary of the observation is that you could have very well done it with the DFT matrix as well if you wanted to do it. There is nothing different or unique that you will get.

It is only that the filters now have been shifted around. In fact, basically H1 has gone to the bottom and good point to sort of leave this discussion and say that the shifted Filter banks we can obtain the straight forward version that you will get when you substitute these shifted version is one which has got the IDFT form or the transpose conjugate of the DFT matrix.

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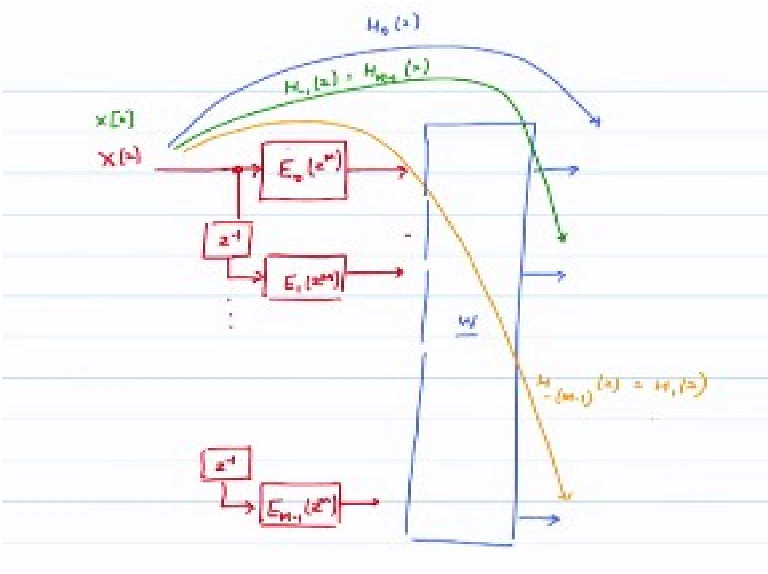

And so if you were to draw in this case this is H0 of Z second one is H1 of Z is shifted to the right by 2 Pi/M. And in this case this one is H M-1 of Z. So this is more of the intuitive order basically you have H0 shifted to the right by this H1, H2 all the way to H M-1. So this gives you the intuitive orders we can very well work with a DFT matrix as well and we get a slightly flipped order.

Now we move into probably one of the most important elements of this particular unit which is the following.

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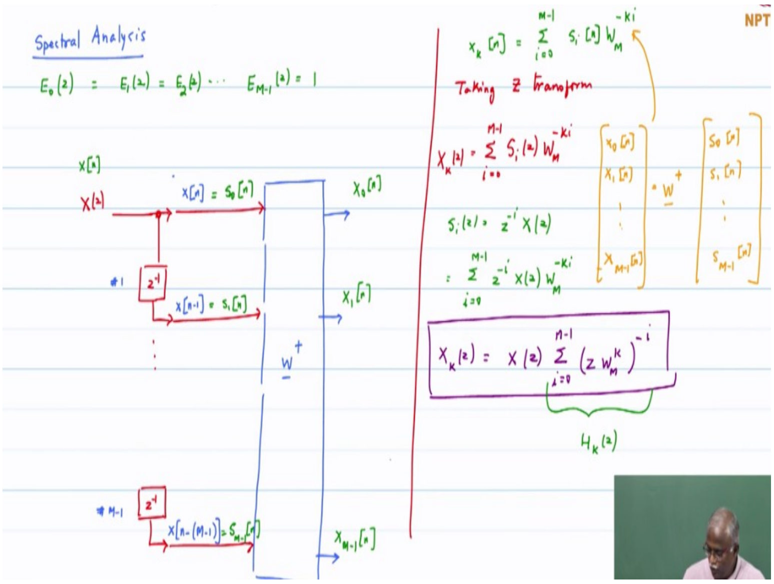

Spectral analysis very, very important please if you have any questions in the material that is being presented feel free to stop and ask we will spend some time discussing that. We are going to take a special case and it is a very important special case. The special case is that you have the polyphase components E0 of Z you have E1 of Z polyphase components instead of writing it as Z power M the polynomial will be E0 of Z.

We have E1 of Z all the way to E M-1 of Z. The special case that we are going to consider that this is equal to, this is equal to E2 of Z …all of them are equal to 1 that means they are all scalars they are not polynomial they just equal to constant. Now very important for us to analyze what is going on. So I would like to once again go back to the filter that we derive and see if I can just use this to simplify my drawing maybe it would have been easier for me to redraw the whole thing.

Let me just see if we can reuse this one. So I am going to erase those unwanted lines keep only the ones that are necessary and basically we have said all of these are constants so these are gone. It is only a straight line that is connecting them. So just erase this we will redraw this lines. This is a special case that we have chosen to study and let us just so that we are clear this is delay number one and this is delay number M-1.

It is a M channel case so because of the structure if this is x of n this will be x of n-1 this will be x of n-M-1 M units of delay. We are going to give them some new labels just for convenience of working with these signals. Let us call this as S0 of n that is a top branch. Second branch is S1 of n it is a delayed version but just for convenience we are writing them in this fashion.

S of M-1 of n and then you have the IDFT matrix and you have the outputs. Now if you notice that what we are computing is like the inverse DFT, but I do not want to call it the inverse DFT because then we start getting confuse between time and frequencies domains. It is a transformation of this S vector into another vector orthogonal transformation and let us call these outputs as X0, X1, X of M-1.

And we are also going to give it a time index that is why I do not want to call it as think of it as frequency and time at time n what is your input S0 of n at the top most branch S1 of n at the next branch S M-1. I am going to apply the IDFT or the transpose conjugate of DFT matrix and this is X1 of n this index is just time index n is my notation clear. I have got these vectors these signals S0 S1 S M-1 if I take a particular snapshot in time I get a column vector.

To that I apply the transpose conjugate matrix and I get another column vector I call that as X0, X1 X M-1. I am clear with this notation because I am going to work with this notation you will find in a moment that actually very, very helpful so I do not forget the arrows. The arrows are signals going in M signal is going in M signal is coming out. Now this is a linear transformation S0 of n going towards X0 of n.

So I am going to take the Z transform and represent this equation in the Z transform. So this particular structure taking the Z transform then I get Xk of Z any one of these branches if I were to write down the Z transform of that that would be summation please pay attention I mean follow this if there is any doubt please let me know. So before that maybe I will write down the time domain equation Xk of n can be written as summation I= 0 to M-1 S I of n times WM -K1.

I will give you a moment just sort of make sure that we have written the equation correctly. The expression for these outputs there in terms of the inputs time that corresponding row of the W dagger matrix basically the input vector if you were to visualize it maybe I will just write it and then erase the input vector is S0 of n, S1 of n …S M-1 of n that is the input vector.

You are processing it through the W dagger matrix so and this produces X0 of n, X1 of n, X

M-1 of n. And what we have captured here is this equation one special case you have taken a particular row of the W dagger matrix and that is what you get. Now taking the Z transform of this. This W are constant so they do not affect. So XK of Z just rewrite it neatly. XK of Z is equal to can you help me i=0 to M-1 it will be Si of Z times WM -Ki is what I have done correct basically XK is a linear combination of some signals if you want to take the Z transform of XK you have to take the Z transforms of all the component signal and it is a correct equation.

Now what is Si of Z Si of Z is nothing, but Z power -i times X of Z. So if you now substitute back into this equation then it becomes i=0 to M-1 Z power -i X of Z times WM -Ki. So the input output relationship can be written in this form XK of Z= X of Z is common it does not depend on i it comes outside the summation you get summation i=0 to M-1 Z WM K raise to the power-i is a very key equation.

So if this is the input output relationship then this becomes the expression for Hk of Z the transfer function and now this is the point at which all the insights start to flow.

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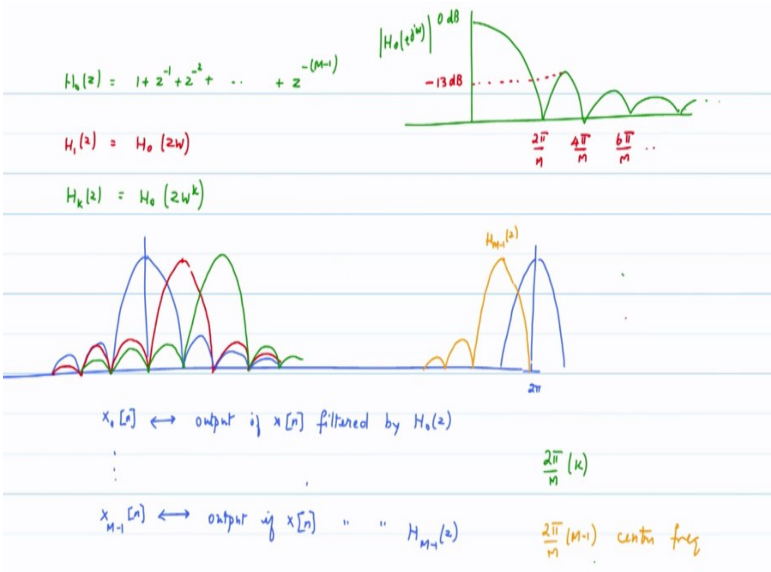

So what is H0 of Z? We said all the polyphase components were set to 1 so which means this is equal to 1+Z inverse+Z-2 …+ Z power –M-1. First let us look at what does this filter look like this is a rectangular window we have studied this many times ...So if this is 0 dB magnitude of H0 e of j omega the first 0 will happen at 2 Pi over M. Second zero will happen at 4 Pi over M, 6 Pi over M basically multiples of 2 Pi over M …. Another important element is that the first side lobe is at -13 degrees.

I am sure you would verified this in the course of your filter design. Basically if you think of this as a low pass filter it has the behavior of loss pass filter because it allows the low frequency components and suppresses the high frequency component, but it is not a very good low pass filter because the attenuation of the unwanted components with respect to the signals that are pass through is not very significant.

You would typically you would like to have 40 dB or 60 dB difference between the signals that are passed and the signals that are stopped in this case it is only 13 dB, but does not matter. This is the rectangular filter so this is what H0 of Z is. Now if you go back and look at our expression what is H of 1 which will be H of ZWK. So H1 of Z will be H0 of ZW if we go back and substitute you can verify that the expressions are consistent and in the general case of HK of Z is H0 of ZWK.

And so if we have to now sketch these filters unlike the ideal filters that we had let us see if we can quickly sketch them. This is H0 the first H1 let me sketch it in red the shifted version of it and likewise H2 get another shifted version. The books give nice figure so I am sure you can augment what we have drawn here with some good figures. So basically you see that you get a filter.

These are the DFT shifted Filter banks, but it turns out that basic filter was of this form rectangular window. So the subsequent shifted versions also came up in this fashion everyone clear with what we have done so far. So H of M-1 the last filter I request you to sketch that so this will be 2 Pi at 2 Pi you will get back the H0 so that is present there. H of M-1 will be a filter that is just shifted with respect to so this is H of M-1 of Z these are all shifted versions.

So basically we span from 0 to 2 Pi with this version. So we can write the following statement X0 of n if you were asked to give a description of this you would say that this is the output that you would get. If X of n is filtered through a filter H0 of Z output if X of n filtered by H0 of Z that is a filter and you pass it through and that is the output that you will get. Similarly, you will all the way to X M-1 of n corresponds to the output if X of n filtered by H M-1 of Z which is the rectangular window.

Shifted all the way to the last one the center frequency for this will be 2 Pi over M into M-1 this is the center frequency for the last filter and in the general case the center frequency will be 2 Pi over M into k that will be the center frequency and you span all the way from here to here. Now here is where we quickly start to apply the concepts that we have developed and let us see if we can everyone comfortable with this figure what we have done.

The structure, the input coming through a delay chain passing through the transpose conjugate of the DFT matrix gives me a systematic set of outputs where the first output corresponds to filtered by a rectangular window its rectangular in the time domain in the frequency domain it will be a sink function and then these are all shifted versions of the sink function.