Multirate Digital Signal Processing
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Lecture- 16 (Part-1)
Applications of Multirate - Part1

Good morning let us begin. We pick up from Lecture 15 and since there has been a little bit of a gap. Let me just quickly summarize what, where we are and the topics that we will be covering in today's lecture.

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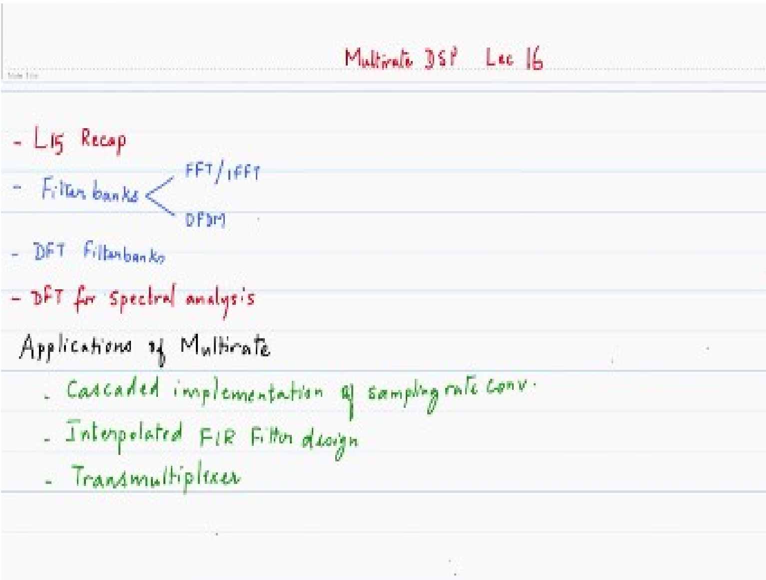

Today we plan to introduce probably one of the most important concepts of Multirate signal processing that is in the context of the use of the DFT for spectral analysis. So that is a important concept, it is an important application so something that definitely I would like you to pay close attention to. Of course there are several flavors of applications to Multirate and what we will be doing is just touching upon few of them and in today's lecture and tomorrow lecture time permitting.

We will look at some elements of the Multirate and the techniques that we have developed, but basically we are going to introduce filter banks and one of the manifestations or application of filter bank is in spectral analysis and that is what we have been looking at in today's lecture. So let me start by a quick question so that you can think about and give me an answer.

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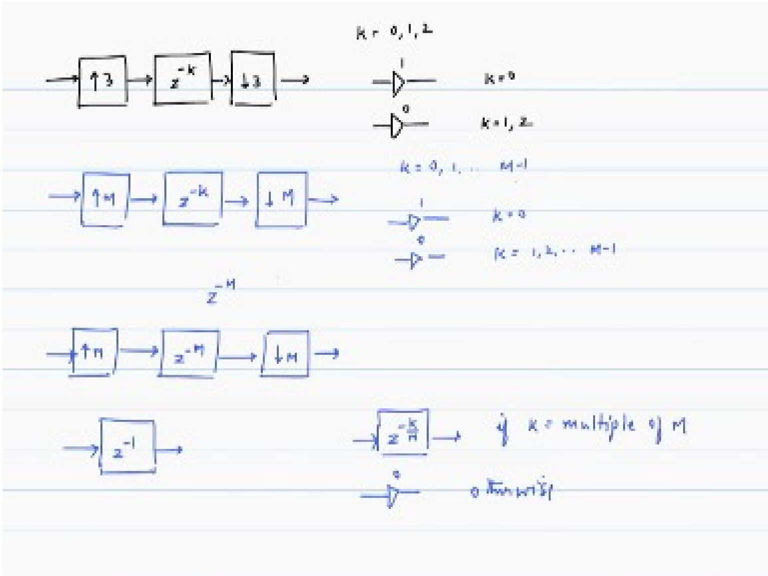

Supposing I have the following set up and I have up sampling by a factor of 3 a delay Z power –K and down sampling by a factor of 3 and where K can be 0, 1 or 2. The output is if K=0 you know up sampling, down sampling cancel each other. So this will be equal to the identity system with gain 1 for K=0 so that is equal to gain of 0 for K=1 and you can also verify that for K=2 is also going to be the same case.

So the general extension of this result is actually a very important result in Multirate. If this is up sampling and this is Z power -K and this is down sampling by a factor of 3 then I am not specifying that K has to be equal to 0 to M-1. In the general case what is this equal this is equal unity gain system not strictly unity gain. It will either by unity gain or it will be Z power –K itself. Let me make it as two separate cases so that we do not have confusion.

So take the case where K=0, 1 all the way to M-1 so then this becomes a unity gain term for K=0 it becomes 0 for all others. So it is actually this sort of a structure actually eliminates a lot of the combinations that are possible. Now what happens if K=M if Z=-M it will be the same as how would you prove that you can move the decimator to the left then it will be. So this should not be 3 this should be M.

So if this is M so if you had Z power –M up sampling by a factor of M down sampling by a factor of M in that case then what we get is Z power -1 as the delayed element. So any multiple of M is also okay for K, but it will come out to be Z power –K over M if K= multiple of M, equal to 0 otherwise. So I hope the reasoning is clear and you are able to follow. So where does this actually get applied.

Usually these questions are not just out of the blue there is actually an application for that. So very often when you do in Multirate we do encounter situations where the up sampling and down sampling are by the same factor. Now when you do sampling rate conversion they will always be different, but in normal Multirate applications input and output sampling rates invariably will be the same. So you can visualize the situation as follows.

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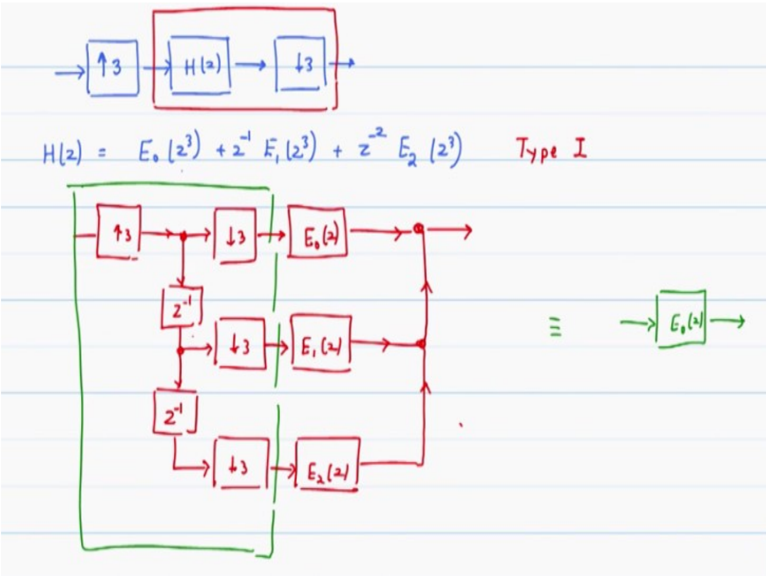

So I have up sampling by a factor of 3 just as one of the operations I want to filter it by H of z and then down sample by a factor of 3. This is more to test the application of the principle that we have just now developed. And I suggest that we do a polyphase decomposition of H of z E0 of Z cube+ Z inverse E1 of Z cube+ Z-2 E2 of Z cube and just as a suggestion I say that okay do the polyphase decomposition for the down sampling because for the up sampler you would have done type 2 polyphase decomposition.

So this is a type 1 polyphase decomposition for the down sampler. Now given the result that we have just discussed previously I would like you to think about the following structure. We have up sampling by a factor of 3 and I have done the polyphase decomposition I move the down sampler to the far left then I have down sampling by a factor of 3 polyphase decomposition of type 1 Z inverse down sampling by a factor of 3.

Z inverse down sampling by a factor of 3 and this one is E0 of Z it is no longer E0 of Z cube because I have applied the noble identity. Please draw the arrows so that we do not have any confusion. The second branch is E1 of Z the third branch is E2 of Z and all of these will get added up. Draw the arrows so that there is no confusion about which way are going which are the nodes which are adders and which are the nodes which are the ones where the signal get split.

So what have we done we have taken our down sampling part applied the polyphase decomposition and obtain the polyphase implemented structure and of course the up sampling still remains. Now if you focus in on this portion of the circuit and apply the result that we just now discussed where you have up sampling some delay down sampling how will you simplify this structure.

Up sampling by 3, down sampling by 3 it is a gain of 1 with delay of -1 that branch is 0, delay of -2 that branch is 0. So this is identically equal to E0 of Z. I hope you are able to see the simplification and how we have used the result that we have just obtained. So the result was up sampling by a factor of M Z power –M down sampling by a factor of M or Z power –K in general case invariably will come out to be a gain of 1 for some cases and it will be 0 for the others.

And we can always exploit that when we have the Multirate structure and I hope you can see one of the advantages of this particular result that we have shown. Let us move on to the topics of today.

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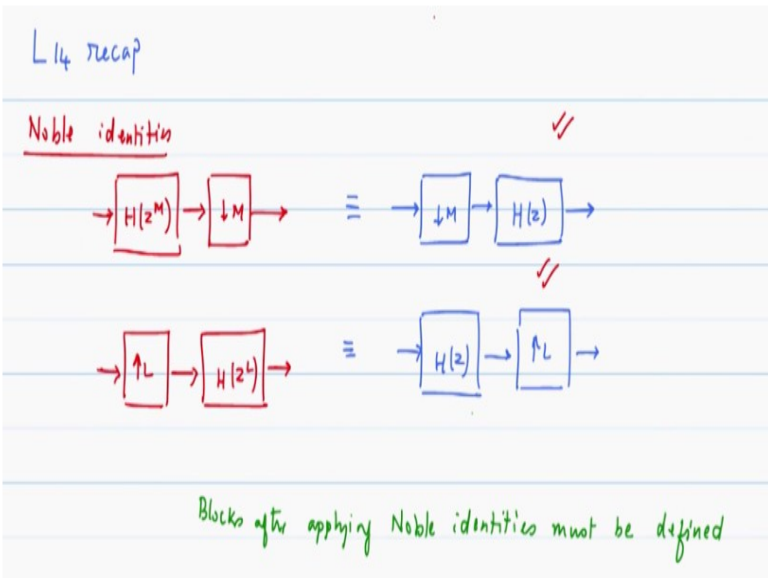

Let me just quickly mention the two key points. Lecture 14 the key results was the noble identities, but we must add a qualifier that blocks obtained after applying the noble identifies must be realizable they cannot be undefined blocks. Obtained after applying noble identifies must be realizable or must be defined cannot get things like Z power half and those things which we cannot define in a DSP context must be defined.

Keep that picture in mind so that way we do not get unrealizable systems and then actually get into results or conclusions that are not correct.

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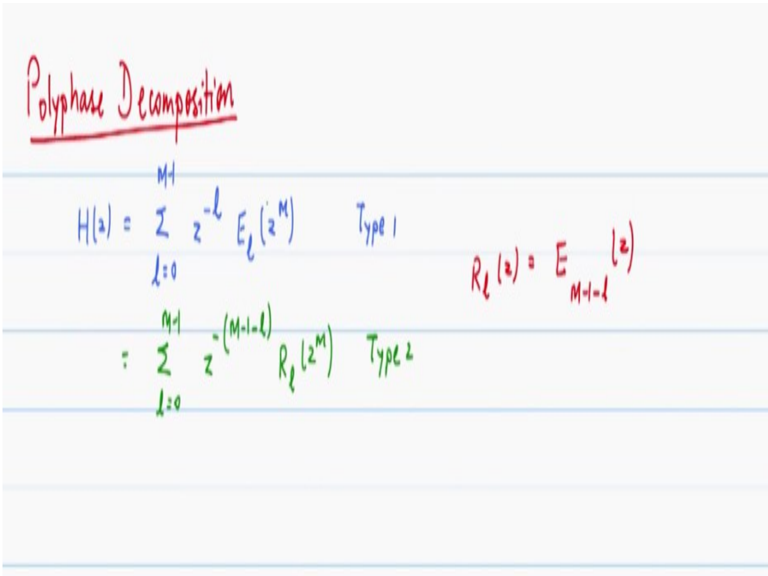

We have two types of polyphase decomposition usually the Type 1 polyphase decomposition associated with the down sampling or with the analysis filters. Remember we talked about when a signal gets split that is called the analysis filter. So usually Type 1 is associated with analysis filters Type 2 is associated with the synthesis filters. Again we will see several cases of these and then it will become familiar with that.

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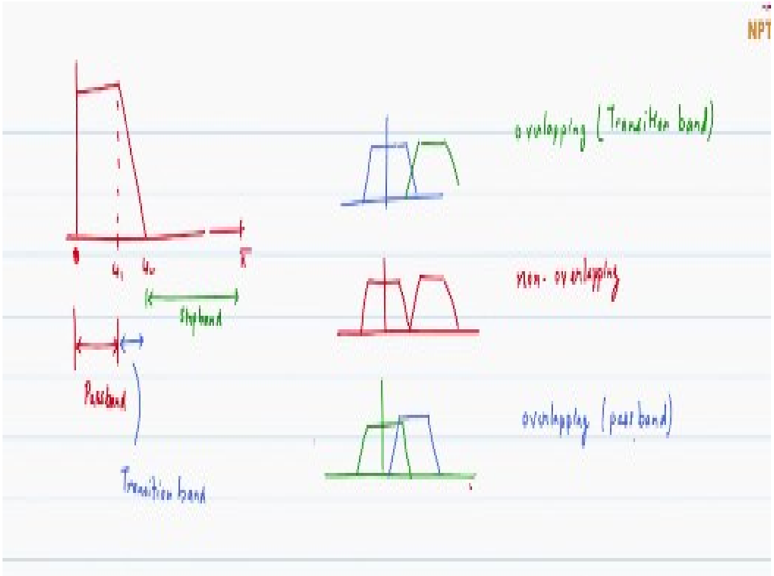

In the last lecture we introduced the notion of filter banks and we said that we would like to get several filters which span the range from 0 to 2 pi and in that context we showed that there are types of filter banks if the adjacent filters have some overlap that is the most traditional or most commonly encountered. Occasionally you may want them to be non overlapping or you may want to have significant amount of overlap.

Both of these are possible and based on the application you may design it as follows. Let me just answer one of the questions that came up after class and that probably will answer questions maybe others also are having.

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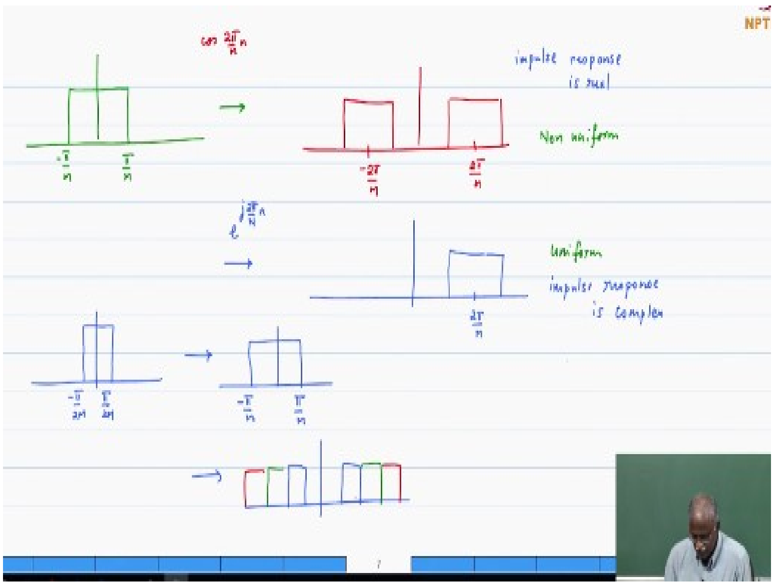

Now supposing I had a filter H0 which has the following form- Pi over M to Pi over M. Now by modulating the impulse response with a cosine function you can actually get a filter whose response looks like this. Let us say we modulate it by cosine 2 pi over M times n. Let us say you just want to use the so then what will happen you will get 2 copies of the spectrum basically you will do modulation.

This will be 2 pi over M there will be a copy of the spectrum at -2 pi over M. So this is obtained by using so this is a case where the impulse response is real. So if the original filter had real impulse response will still remain real because you have done, obtained the new impulse response by multiplying by a cosine function impulse response is real. On the other hand the method that we talked about was using a modulation which was e power j 2 over N times n.

And that gave us a different set up where you basically got a single spectrum it was not plus and minus there was only one centered at 2 pi over M. Now there is a difference between these two which of these are we talking about and why are we talking about one and not the other. If you look at the top line when you take the filter multiplied by cosine 2 pi over M then what happens is the derived filter has got twice the bandwidth of the original filter.

So it does not satisfy the uniform criteria so this is a case where it becomes non uniform because one of your filter the prototypes or the basic low-pass filter has got a certain bandwidth all of the derived filter have twice the bandwidth of course there is no problem with that, but it does not satisfy the requirements of uniform. Whereas when you do the multiplication by e power j 2 pi over N the complex exponential then we actually get the case of uniform.

Now if you insist that no I do not want to have to deal with by the way these have impulse responses that are complex even if the original impulse response was real. So here you find that the impulse response is complex. Now if for any reason you prefer or you want to constrain your impulse response to be real and you want to have uniform it is possible. In that case you will have 2 basic filters.

One of them with the green type the other one will be –Pi over 2M to Pi over 2M half the bandwidth and now what we will do is you will have the basic filter going from –Pi over M to Pi over M that means total bandwidth of 2 Pi over M. The derived filters will be derived from not by modulating the basic filter, but by the auxiliary filter. So what you will get is filters of this type so you still have the bandwidth of 2 Pi over M.

The next filter would be of this type and so on. The color show the pairs so K=0 is your basic filter that came from a different set the all of the derived filter or the modulated filters came from prototype filter which was half the bandwidth and of course you would satisfy this condition. Again there are applications where this type are however our focus is not this we want to have uniform, but we do not mind having a complex responses.

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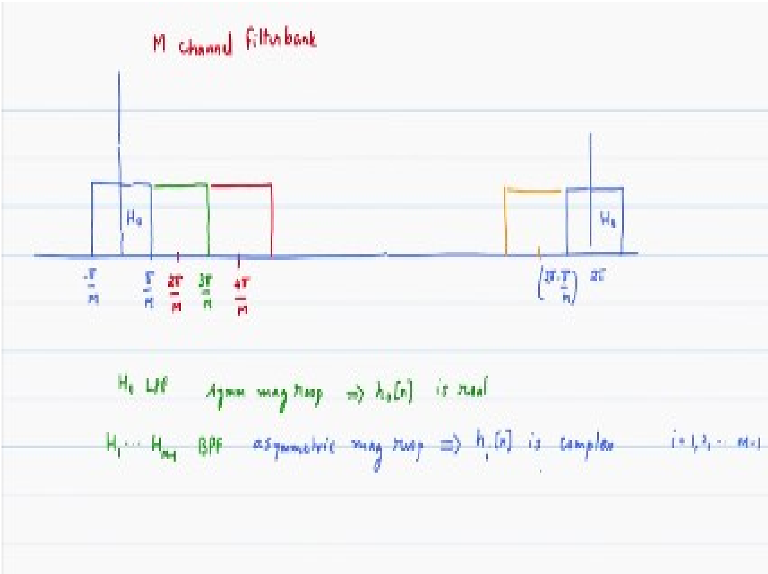

So therefore the uniform filter bank that we are talking about has the following structure. If it is a M channel Filter bank that means the range from 0 to 2 pi will be divided into M filter of equal bandwidth and they are obtained by the shifts using the complex exponentials M channel Filter bank. And this turns out to be very, very interesting and very rich area and we will spend quite a bit of time studying and analyzing this particular case.

But before that one more time we will look at the notion of complexity reduction using the polyphase decomposition.

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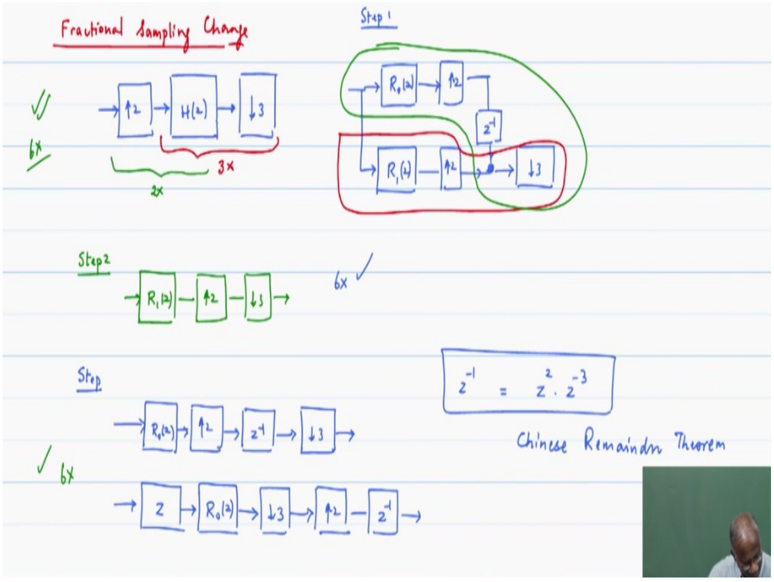

So we would look at the case of a fractional sampling rate change, fractional sampling rate conversion and again think through with me and even if you do not have to draw write down everything this particular thing is more for you to engage with the discussion point being presented. So I have the scenario of up sampling by a factor of 2, H of Z which will be my interpolation filter it will also serve as my anti-aliasing filter.

And down sampling by a factor of 3 it is a 2 by 3 rate conversion. And of course the anti-aliasing filter requirements will dominate so this will be more like a anti-aliasing filter because it will be cut off with the Pi over 3 as my. Now observation number one I know that if I do polyphase decomposition I will get a factor of reduction in terms of computational complexity. If I combine it with this side I will get a 3 x complexity reduction because that is what I get a polyphase decomposition and then I do that.

Of course if I do it this way I will get a 2x. Now the question is I am a little bit greedy I want 6 x. I want to get why not ask the question can you give me a structure that does 6 x and this is a classic problem and you will say well just now we spend a lot of time remember this delays once these delays are present you cannot shift the polyphase down sampler and up sampler.

So let us take a quick look at what is possible in this structure. So step 1 and again I will leave several steps for you to complete but here is where. So actually I am going to combine it with the up sampling by a factor of 2. So the up sampling by a factor of 2 remember I said we will do Type 2 polyphase decomposition whenever you are doing the up sampling so it is

R0 of Z up sampled by a factor of 2.

A delay element and the lower branch is R1 of Z up sample by a factor of 2 you add these 2 signals and then there will be a down sampling by a factor of 3. Please draw the arrows these are the polyphase decomposition of order 2. So there is R0 and R1. So polyphase decomposition is done for the Filter H of Z into 2 polyphase components and we get this structure.

Now the goal is if I want to get a factor of 6 that means the decimator has to move all the way to the left. So let us take 2 cases let take the case where you take the lower branch as step 2 and then you take the upper branch as step 3. So step 2 is straight forward let me just request you to fill in the details of step 2. Step 2 is I have R1 of Z followed by up sampling by a factor of 2 down sampling by a factor of 3.

I want to get maximum computational advantage up sampling by a factor of 2 down sampling by a factor of 3 can be interchanged they are relatively prime and then I have a filter R1 followed by down sampler which means that I can do polyphase decomposition move the down sampler all the way to the other end and get the factor of 6 advantage for the lower branch.

Now upper branch is when we may have to spend a little bit of effort, but again it is interesting it is just the way of thinking about it and the upper branch is step 3. Step 3 says the structure that we have to simplify and apply to. So 6 x is obtained for here I will assume you are able to convince yourself for the lower branch 6 x is obtained. Upper branch it is R0 of Z followed by up sampling by a factor of 2. Z inverse down sample by a factor of 3 that is the upper branch.

Now if both had been the by the same factor then the answer would have been 0, but of course they are up sampling by 2 and down sampling by 3 so that result does not help us. Now the problem is we know that the up sampling and down sampling are time variant systems and Z inverse is an LTI block we cannot shift these things around actually it will mess up and I cannot even apply the noble identities because I know that I cannot get any unrealizable blocks in that.

Now is there any way to simplify, is there any way that you see that we could basically I want a factor of 6 gain which means down sampling by a factor of 3 must move all the way to the left up sample by 6, but then you have down sample by 3 Z inverse down sample by 3 that would not help. Okay think about this Z inverse is a same as Z square times Z power -3 then what you do is you swap these 2 around.

So rewriting it what you should eventually get is you get Z upfront R0 of Z down sample by a factor of 3 then up sample by a factor of 2 times Z inverse. If you can get to this point, then of course you can get another factor already we got a factor of 2 so you can get the factor of 3 savings and get the factor of 6. So this particular result particularly if L and M are relatively prime what is that result called Chinese remainder theorem look it up.

So basically any integer can be written as a combination or a linear combination of 2 prime and this is referred to as Chinese remainder theorem. Again our goal is not the mathematical aspect of it basically it is a very well known results. So we will just use that Z inverse is simple enough that we do not have to work too hard to get it Chinese remainder theorem. Of course there are several other applications where this actually comes into.

So this structure if you do this you will get the factor of 6x so the lower branch also get a 6x advantage and we will put a tick on that. So overall you did get a 6x. So that is a very, very good question so if you have up sample by L down sample by M they are relatively prime only then we will able to interchange them then these delays if they are in the range 0 to M-1 let us say the Chinese remainder theorem says that I can always write it as a combination of some multiple of L and some multiple of M.

And you will always be able to interchange that is a very good question. So do try it for the 3 branches I am sure it will be interesting to see. In fact, you can try that if you did the down sampling first then you can actually try it out you will get the Z power 0, Z power -1 and Z power -2 you will get all the 3 terms.