Multirate Digital Signal Processing
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Lecture - 14 (Part-1)
Polyphase Decomposition Continued

Good morning. We begin lecture 14. We will quickly review some of the concepts in lecture 13 and then build on that. So the flow of today's topics, the last lecture one of the key results that we have obtained is the noble identities. We will describe that; we will look at an alternate way of describing. We have described the operations of the down sampling and up sampling using the frequency domain.

There is a very interesting way of looking at the multirate, not the sampling rate conversion, but the multirate filtering point of view. So that is the point that we want to highlight then of course polyphase decomposition, we gave the introduction.

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We will now show that there are several advantages with the polyphase filtering and what we will develop towards the end of the lecture, is the application of the multirate structures, for what we refer to as filter banks, very useful, very powerful concept. It is related quite extensively to communications. So just to sort of give you a feel for where all do we see filter banks. Whenever you compute an FFT or an IFFT, you are actually doing a filter bank operation.

Today in the cellular world, we use OFDM for most of our 4G systems and looking ahead it looks like it will be part of the 5G systems as well. OFDM is also another flavor of the filter banks that we are going to be studying, wavelength analysis, filter bank. So basically there is a whole range of applications that come under the umbrella of the filter banks. So we will look at the first of the filter banks, which we call as the DFT filter banks.

Filter banks that use the DFT as the underlying computational tool. So this is a starting point to understand and apply DFT filter banks and of course what is DFT used for? Primarily for spectral analysis. So we will make a comment about using DFT for filter for spectral analysis, as a tool for spectral analysis. Once you see the filter bank relationship, you will probably be able to see the reasons, why you would either do a DFT and a spectral analysis or not to do that.

So again these are the insights and that is precisely the reason for studying a course like this, because it has got its links. It already builds on the framework that you have or you have developed and importantly it shows you, gives you insights that help you understand the tools that we work with in communications and signal processing. So that we, towards the end, we will get to that or probably in the next lecture, okay.

So a lot of very interesting things to discuss today. So we will move quickly through some of the points that we want to do.

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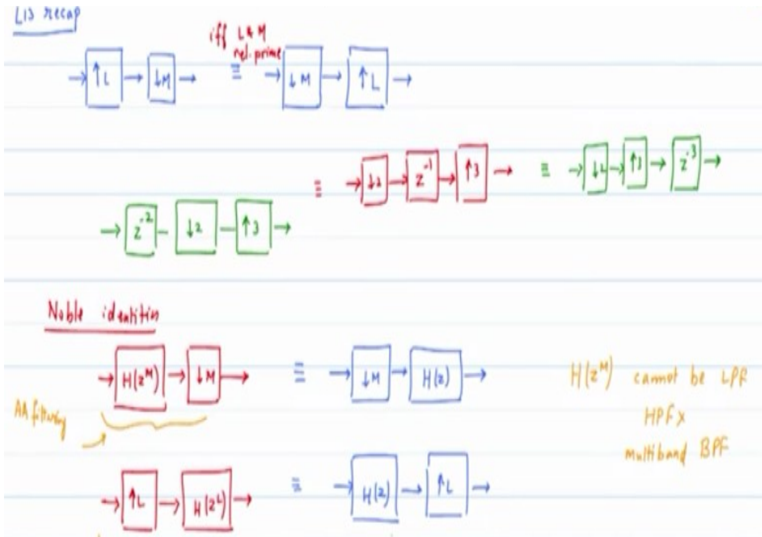

So lecture 13 recap or a review, of course with a little bit of some new elements avoiding complete repetition of the things that we have discussed. One of the key results is the ability to interchange the up-sampling and the down-sampling. Again, I write this multiple times because this is one of the key results that help us in developing efficient multirate structures. So L and M are equivalent to the interchanged order, if L and M are prime.

Down sampling can come first followed by the up sampling okay. So this is if and only if L and M are relatively prime. Relatively prime is enough; they do not have to be prime numbers themselves okay. So an application or an example for this. Let us quickly look at it. So I have Z power -2. I have down sampling by a factor of 2, up sampling by a factor of 3. Of course I can interchange these two if I want to because they are relatively prime.

Now the problem itself at this point is not the thing the fact that you can interchange, but keep this example with you. We will come back to in a minute. So the noble identities, and then we will come back to this example. Noble identities tell us that if you have a filter of the form H of Z power M followed by a down sampling. This is typically the structure that you would have for anti-aliasing filtering. So this is the structure for anti-aliasing filtering.

This is the structure for that, this combination. So anti-aliasing filtering followed by the down stamp and this would be a correct way or the more efficient way of doing this will be to do the down sampling first, followed by the filter H of Z, now. It is no longer H of Z, Z power L. It is not just the interchange of the two. These are time varying systems. So therefore we cannot just interchange them.

So the key point that, is one of the identities and of course the second one addresses the structure that we will have for interpolation. Interpolation says you do the up sampling followed by filtering and if you have a filter of the form H of Z power L, then the noble identities tell us that you can do the filtering first. Again it is H of Z followed by the up sampling factor and we have shown this both through the transform domain as well as the time domain.

We have looked at it and demonstrated. So this is the structure that we that we get for interpolation, after you do the up sampling interpolation okay. Now notice I did not say. This is an anti-aliasing filtering or this is only the structure or the form or the sequence in which the filtering and the multirate structure. Why do I specifically say that this is just the structure, not actually anti-aliasing filtering?

Because of the following : H of Z power M cannot be a low-pass filter and why is that it will have multiple copies of these spectral. So can it be a high pass filter? So low pass filter is not possible. Can it be a high pass filter? No, okay. What it can be or what it has to be, is a multiband band-pass filter okay. So definitely that is not what we need for either anti-aliasing or for interpolation. So therefore this is only the form of the noble identities okay.

Now please apply the noble identities and simplify this. So this is equivalent to usually read at the first stage. I can interchange the, so the delay and the down sampling can be swapped. If you swap it, then you will get the delay for down sampling followed by Z inverse followed by up sampling by a factor of three. Can you shift the filtering that delay, to the right of this I? I want, task is to do that.

So this gives me, if I want to have the Z inverse followed by the up sampling and I want to move the delay to the to the right of the up sampling, it will become down sampling by a factor of two sampling by a factor of three and this becomes Z power -3. Again these multi rate structures do give us certain results or tools, which may not be very obvious unless you actually sit down and write it but if you are able to keep the multi rate identities in mind, then it is easy for you to do these changes okay.

So we will come back to looking at several of these type of applications because multirate is all about developing tools of this type.

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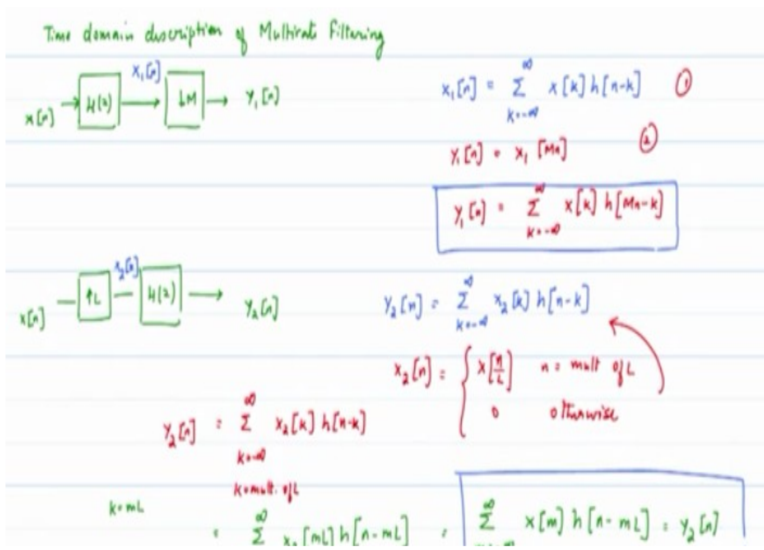

Now I would like to introduce to you the description, time domain description of multirate filters, time domain description of multirate filtering okay. So the two forms of multirate filtering that we encounter very often; are the anti-aliasing filtering and the interpolation filtering and we will write down expressions for both of those. So the first one that we look at is the anti-aliasing filtering.

I have H of Z, if my input is X(n) followed by the down sampling by a factor of M and I write down the output I call this as Y1(n), I have 1 intermediate signal which is helpful for me to get an expression for, let me call that as X1 of n between X and X1, it is an LTI system, because it is a digital filter. So I can write down the input-output relationship between X1(n) and X2(n) as a convolution.

K=-infinity to infinity X(k) h(n- k), so input-output relationship, LTI system first stage. The input as X(n) output as Y1 is not an LTI system. So therefore in general difficult for me to write that relationship; however, we do have the following; Y1(n) is X1 (Mn) that is the down sample. So nothing prevents me from substituting 2 in 1, if this is equation number 1, this is equation 2. I can substitute 2 in 1 and actually get an input-output relationship that looks like convolution, but it is not because it is not the sliding convolution.

It is an operation that, it is an operation that is similar to convolution, but it is it is different in terms of the indexing. So this would be K=-infinity to infinity X (k) h (Mn-k). This would be Y1(n). So in other words you do not take all the impulses, the entire impulse response. You take some shifts of the impulse response, not all of them and therefore that is like saying, okay some shifts when you think of it in terms of the convolution, I do not need all the shifts.

But at any given shift you will take all the inputs for doing the computation. So this is you can find interesting ways of interpreting this result. I will leave that as an exercise, or you know something that you can write down as an additional note to this one. The second, we have the second operation is when I have the up sampling, up sampling by a factor of L followed by the interpolation filter H(z) and let us, you call this as X(n), this as Y2(n) and the intermediate signal let me call that as X2(n).

We need to be a little bit careful with the interpolation, but it turns out that there is a very simple way of writing the expression for this as well. So the input-output relationship where the LTI part is concerned is easy for us to write down Y2 (n) is summation K=- infinity to infinity, X2 (k) h (n-k) okay. Now write down the input-output relationship for the up sampler X2(n)= X(n)/L. You probably are very familiar with this by now.

If n is equal to a multiple of L equal to 0 otherwise, 0 otherwise. Now let me just impose this condition on the previous equation. So basically what it says is that there are lots of 0 valued samples in X2(n). So basically this summation can be re-written as Y2(n) instead of, of course K will go from minus infinity to infinity, but the only things are that are of interest, are of actual computation will be when k equal to a multiple of L.

So if you impose additional constraint, multiple of L, you do not change anything because you are only looking at those values of X2 that are non-zero. So this would be X2(k) H (n-k) and whenever you have such a constraint, it is basically it is of the form K = M * L, K goes from minus to infinity, that means M goes from minus infinity to infinity. So this will be summation M = minus infinity to infinity X2 (mL).

Wherever there is a K, I am replacing it with mL, h(n –mL). So of course X2 (mL) is nothing but X (m). So this can be written as summation m = minus infinity to infinity X (m) h (n-mL). What this says is, the input-output relationship, all these samples of X are involved, but in terms of the filters I use not all the impulse response coefficients. For a particular shift, I use a subset, a grid of the impulse response. Again that is another way of visualizing what happens when you have a lot of 0 valued samples.

The filtering actually looks like you are applying some kind of a grid onto your data system okay. So this is another result that you can note down. So this is equal to Y2 (n). Now the third of these, which are also a useful and interesting result is when I do a fractional sampling rate change.

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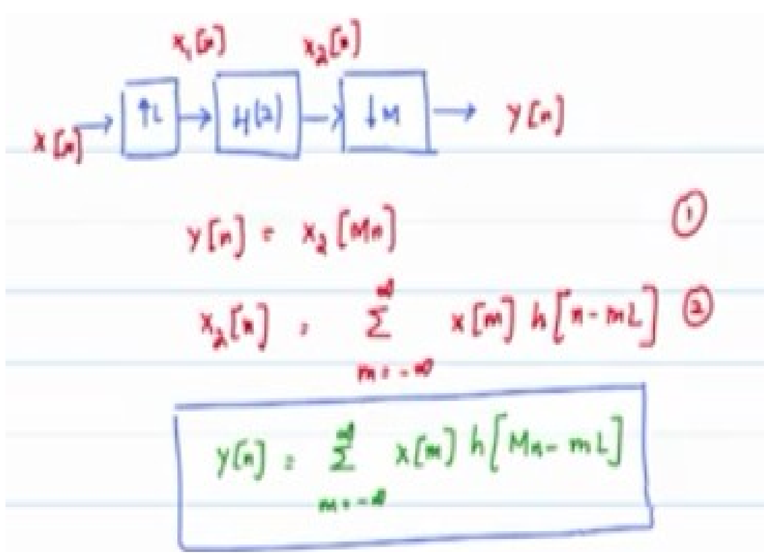

Up sample by a factor of L, filter by H(z) that it could be an ideal low-pass filter with a cut-off which is minimum of L and M. So I am not writing down all those things. Please fill in those details. Up and down sampling by a factor of M, so two intermediate signals. There is X1(n), there is X2 (n) and output is Y(n), input X (n). I would like to relate the input and the output in the following way. So Y (n) is X2 (Mn), X2 (n).

We just noted the up sampling, explanation for the up sampling. So I will just use the expression from the last derived result. So X2 of n can be written as summation m = minus infinity to infinity X (m) h (n–mL), right. That is the last expression that we have just written down for the up sampler. So combining equations 1 & 2, we get the final result which says Y (n) has, also has got a very nice compact expression m = minus infinity to infinity.

This is X (Mn) wherever there is n replace it with Mn. So this would be X (m) no change. It does not have the variable n. H of, there is n here I write as Mn - mL and this is the final result. Interesting to visualize what exactly this equation is telling us. It does throw away some samples I mean it will, so it does not compute all samples, but it has a very interesting structure in terms of how the filtering is done, which samples are retained, which samples are discarded.

So think about it, it is not difficult once you spend a little bit of time. Again it is an interesting way of looking at it more for insight.