Multirate Digital Signal Processing
Prof. David Koilpillai
Department of Electrical Engineering
Indian Institute of Technology – Madras

Lecture – 08 (Part-2)
Upsampler and Downsampler – Continued - Part 2

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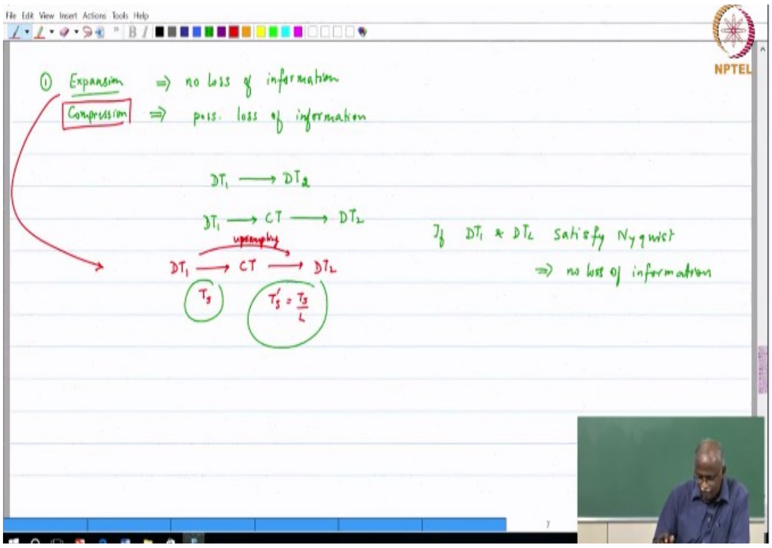

Okay so we now move to the downsampling part. So we have already looked at the expansion or the case where we have upsampled. The second case where we mentioned yesterday that when I do sampling rate reduction, there is a possibility of loss of information that is the one that we want to keep in mind and that is what we are going to focus on in today's lecture.

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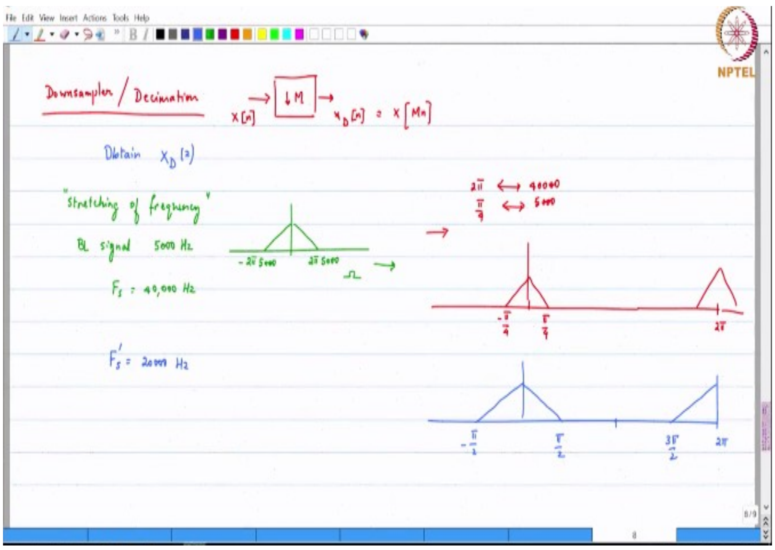

So the downsampler; opposite of the upsampler is the downsampler, opposite of the interpolation is decimation. So we will just call it either as a downsampler or as a decimator. The notation that we follow is a box with a down arrow with value=M. If this is X of n, we call this as XD of n that is=X of Mn okay and this we have already looked at in the time scaling part.

So nothing new or difficult in this one, the important thing that, what we would like to develop in today's class is the representation of XD of Z. Obtain the Z transform or the frequency domain representation of the down sampled signal or the decimated signal. So that is the goal for today's lecture okay. So before that what I would like to do is a notion of what is called stretching in frequency.

The word stretching is a layman's term but it is useful in terms of visualization. Stretching of frequency okay, again let me just explain what it is and then allow you to find a way to associate what it means when you say stretching of spectrum and what does it actually do okay. Now supposing I have a band limited signal, a band limited signal with the highest frequency component as 5000 hertz okay.

So if I were to show in terms of the spectrum, this is 2pi times 5,000, this is the continuous time frequency and this will be -2pi times 5,000 okay. Now I sample it at the rate of 40,000 hertz, sampling frequency is 40 kilohertz, 40,000 hertz, the sampling period is 1/40,000. So if I were to now translate this into the discrete time domain, basically 2pi corresponds to 40,000, so 5000 is 1/8 of that. So this will correspond to pi/4.

So the discrete-time spectrum for this signal is –pi/4 to pi/4 and then this spectrum will repeat at 2pi alright. That is a straightforward sampling representation. Now do the following. If you now change the sampling frequency to FS, FS dash=20 kilohertz okay. What happens to this one? The drawing now becomes basically it becomes twice as wide, it goes from –pi/2 to pi/2 and the spectrum here also is 2pi.

This is 3pi/2 and pi okay. Now the signal did not change, the same input signal I changed the sampling frequency, I reduced the sampling frequency by a factor of 1/2. The discrete time representation, seems like the spectrum has been stretched, whatever was the input spectrum at when I did the sampling at 40 kilohertz, when I do it at 20 kilohertz the spectrum is wider.

Now why is it, why does it look wider because the sampling frequency is now smaller with respect to the sampling frequency my highest frequency component now looks a larger fraction and therefore it looks. So if I we always tend to draw 0 to 2pi, when you draw 0 to 2pi, it looks like my spectrum has stretched. So this is what we mean by stretching of the spectrum.

The signal did not change, nothing has changed about the actual signal or the signal contents but the representation looks different, one looks like there is a stretching and this is what will come into play whenever we do the changing of the sampling rate and that is where it is very important for us to keep in mind. So the stretching of spectrum I hope you will if you have a doubt you will be able to just come back and take a look at this example.

And then say okay now what exactly is happening and why is it happening and be able to explain it in that fashion.

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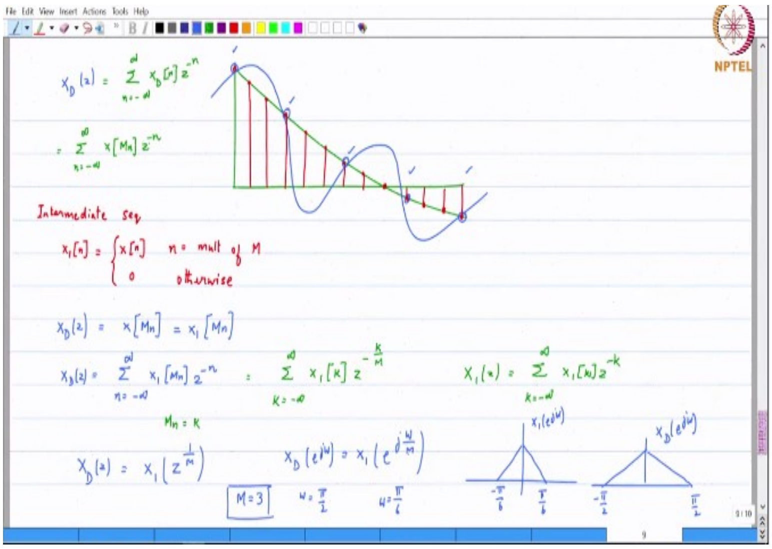

Now we move into the actual derivation of the XD of z. Again, the underlying fact is that I am reducing my sampling rate, reducing my sampling rate, I already know a couple of things, my spectrum will stretch, that is a stretching in frequency, the other element is there is a possibility I am losing, in the process I may lose information and therefore introduce some ambiguity.

So there is one example that I would like to show you and it is a standard example when we say that what happens when you have aliasing as present. So if this is the continuous-time signal and I sample it at the following points and I get the following expression okay. These are the samples that are obtained. Again, this is the sampling process and this is something that we have well understood.

This is uniform sampling at a given sampling rate. Now if I have down sampled this signal by a factor of 3. So that means this sample remains, this remains, this remains, this remains and this remains okay, those samples, the others are not there in my, so in other words my samples now are this, this, this, this and this okay. Now here is the example and its illustration. Now if I had, just as a case study, now if I had a continuous-time signal that was given by this blue graph and it had been sampled, I would have gotten the same samples.

As if I had had the green line and sampled it at this rate. Now when I am given only the samples and asked to reconstruct, what I am most likely to reconstruct is something that looks like the green line because I do not have enough information to reconstruct a blue line. So in other words the minute I have thrown away samples, there is a risk that I may not be able to recover the original signal.

And here is a classic case where if I had had all of the samples, I would have reconstructed even the blue line I mean assuming that the samples were present but the minute I go down to a lower sampling rate, I may have lost that information. So this is where the underlying notions of aliasing come in. Aliasing basically means that I am not able to reconstruct the original signal.

I do not have enough information to reconstruct that. No aliasing means that I have satisfied Nyquist which means that I have sufficient information to reconstruct the original signal. So with this in mind, basically I would like to write down XD of z is summation n=-infinity to infinity, XD of n z power –n. Basically, the standard z-transform representation. So this is n=-infinity to infinity X of Mn Z power –n okay.

I have just substituted the expression for the time scaling in the discrete time. Now it is actually difficult for us to proceed beyond this point if we do not define some other sequences that will help us. So we are going to define an intermediate sequence, and you will find that this is probably one of the most interesting and useful insights in the whole discussion today. So there is an intermediate sequence which may not seem like very useful but it actually turns out to be very, very valuable.

The intermediate sequence, we will label it as x1 of n this is=X of n if n. Is equal to a multiple of M +-0 ,or 0 +-M =+2m and so on equal to 0 otherwise okay. It actually has less information than X of n, why not might as well have worked with X of Mn because that has the same information but actually this sequence has a lot of value and benefit. Now I would like to go back and look at XD, so XD of z=X of Mn that is=X1 of Mn correct.

Now you may see that well that is even less we cannot may looks like it is even less useful. So XD of z is=summation n=-infinity to infinity X1 of Mn, z power –n. Probably not seeing any benefit at all of having defined X1 of n except the following. Now I would like to do a change of variable Mn=K. Please note I am going to change the variable and this is going to be a very, very important step.

So K going from -infinity to infinity, X1 of K, now you may say wait a minute because the previous summation took only the multiples of M. Now you are saying I am going to run over all the values. I did not lose any information because notice that X1 of n has got zeros at the other points. I can run my summation over all the indices over the entire set of indices. So this is very, very important.

This step is important, so if I am running my summation over this variable K which is=Mn. Then, z power -n will come out to be –K/M. It is a modification of that and this is if X1 of z is=summation K=-infinity to infinity X1 of K z power -K if this is a general definition of X1 of z then I now have an expression for XD of z. XD of z is nothing but X1 of z of 1/M. It is a very interesting and an important result.

Because this is like the counterpart of the zero insertion, because what happened in the zero insertion, the upsampled signal had a scaled version of the spectrum of the input. Now this one says okay it is not the input itself but it is some intermediate sequence which has got a factor of 1/M. Let us quickly take a look at what does this scaling by a factor of 1/M. Previously we looked at what happens to a scaling when you have a scaling of L.

So if XD of e of j omega we said is=X1 e of j omega/M okay, so now this is where you will see the stretching of the frequencies because if I take an omega=pi/6 okay, omega=pi/2 for XD e of j omega, this will correspond to X1 e power j pi/6, right? Because whatever was the original frequency I have to divide by 1, and if I take the case where M=3. If I take M=3, then the relationship between the frequencies of the XD and this one are.

So if I were to show it in terms of a spectrum, I am going to show only from 0 to pi, so again do not be confused by what we are drawing. So if I am going to draw a signal that goes from –pi/6 to pi/6 okay. This is for X1 e of j omega; this is purely for illustrative purposes. If I have a spectrum that looks like this, then XD is going to be the same thing stretched by a factor of 2.

It will go from –pi/2 to pi/2, this will be XD e of j omega, so basically this notion of stretching in frequency. When I reduce the sampling rate, there is a stretching in frequency that will come that is very important for us. It is linked to the stretching in frequency that I described at the start of the lecture. Again, this is an intermediate step. I mean to define a new sequence, this sequence has got lots of 0 valued samples and it is related to the down sampled signal XD of n using the same relationship.

And it gives us a closed form expression for the z transform of XD of z okay. This is still not the final answer. **“Professor - student conversation starts.”** Yeah, exactly so very, very important point. When you have XD e of j omega/M okay what is the periodicity of that? It is M times 2pi. So now can you have a discrete-time signal which is not periodic with 2pi? No, that is why I said you have to be very, very careful interpreting this expression okay. **“Professor - student conversation ends.”**

So this was shown only to tell you that in the range from 0 to pi/6 you will actually have a stretching of frequency. So if I actually had to draw a e power X1 e of j omega/M or XD e of j omega, I would actually have to draw from -M times 2pi to M times 2pi okay but the periodicity of XD is still 2pi, so that is why I did not want you to get confused with the scaling, but yes whenever you have a scaling like this the original function had a frequency period=2pi.

Now by design it will have a period M times 2pi okay but we will not violate any of the fundamental principles and that will come out in the next portion the rest of the lecture okay. I am glad you asked that question because that is a very important one and it is something that actually is one of the important elements we should not miss in this discussion okay.

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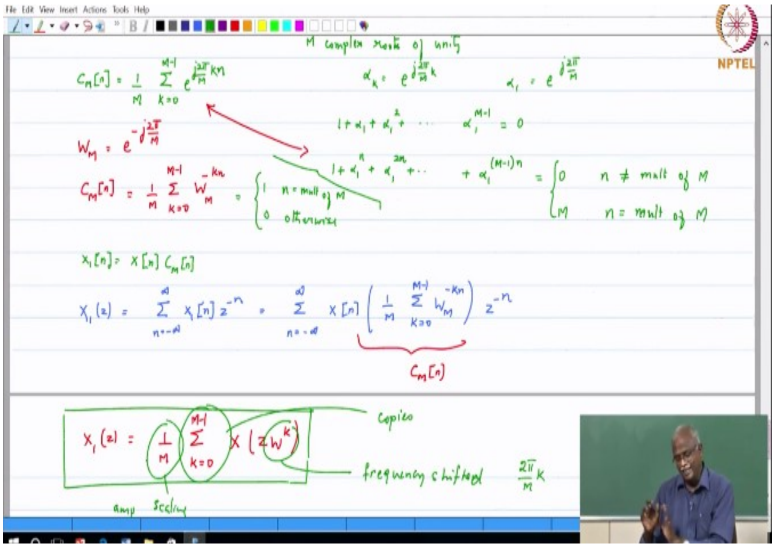

Now how do I relate X1 of n and X of n? What is the relationship between these two? Basically, I have set some of the values to 0 okay and we can think of the following. We can define a comb sequence, C stands for comb, n is a time index, M stands for where the comb occurs. So the comb frequency, comb sequence is 0 then it has got lots of 0 valued samples, then at index M there has got 1 and then some 0 valued samples at 2M it has got.

So basically this is a comb sequence right with the subscript index 1. So these are samples of height=1. So if I multiply X of n with this comb sequence, then I will get X1 of n okay. So that is an interesting observation. So in other words, the way I set the sum of the samples of X of n to 0 is by multiplying the original sequence with a comb sequence which will take care of it, so comb sequence definition this is=1 if n is=a multiple of M, =0 otherwise.

So it basically sets all of the other samples to 0 okay. Now how do I construct comb sequences? Very easy, complex variable theory gives us a very, very useful result. I am sure you would have used it. The M complex roots of unity, M complex roots of unity. What are they? Alpha K is e power j 2pi/M times K correct and so basically I can take any of the roots of unity.

Let me take alpha 1 that is e power j 2pi/M and then I can write down the following equation 1+alpha 1+alpha 1 squared all the way to alpha 1 raised to the power of M-1 that is=roots of unity when I take the powers of root is equal to 0, thank you. Basically, you will do a geometric series and show that numerator goes to 0, so therefore you get this expression okay. Now if I look at the following, 1+alpha 1 raised to the power 2n, +alpha 1 raised to the power of M-1 times n.

This is still equal to 0 if n != multiple of M; however, if n is=multiple of M then all of these roots of unity will be=1 and therefore then it becomes=M. If n= a multiple of M okay. So now you can see how the comb sequence more or less emerges from this discussion and so we say that the comb sequence C M of n I do not want the summation to be=1, so I add 1/M to normalize it.

Summation K=0 to M-1 e power j 2pi/M times K times n okay. That is basically relating it to this equation okay. So that will tell me that I will effectively get a comb sequence. Now when you study DFT, you would have introduced the following notation, W subscript M that is e power minus j 2pi/M that is a very useful notation from there. So the comb sequence can be written in terms of the DFT coefficient W in the following form=1/M summation K=0 to M-1 W subscript M-Kn okay.

Because of the definition of WM there is a -sign that comes in the exponent of this okay and this you can verify, this is=1 if n=a multiple of M and equal to 0 otherwise. So this is the comb sequence or a very compact representation. So X1 of n is=X of n multiplied by the comb sequence okay. So basically multiplication by the comb sequence, now I want to evaluate X1 of z.

So far I did not do that, let me evaluate X1 of z, this is basically summation n=-infinity to infinity X1 of n times z power -n. X1 of n, I substitute the other expression. So this is summation n=-infinity to infinity X of n, I have to multiply by the comb sequence. The comb sequence is 1/M summation K=0 through M-1 WM - Kn times z power -n okay. This is nothing but the comb sequence okay.

This is CM of n okay. Now just the last step and I will pick it up from here in in tomorrow's lecture but just wanted to leave this as the last point. So X1 of z, if I interchange the summations, I would like you to show that this comes out to be 1/M summation K=0 through M-1 X of z WK. So this equation I would like you to interpret using the following; one : there is a scaling.

This is a scaling, amplitude scaling okay. There is an amplitude scaling, the second one : there are copies of the signal, multiple copies, how many copies? M-1 additional copies and the third : is that this basically means that they are frequency shifted. The copies are frequency shifted and frequency shifted by what? They are shifted by 2pi over M times K okay. So there is a scale factor and then there are shifted versions of that.

So please interpret this equation and we are one step away from writing the expression for XD of z. That we will complete in tomorrow's lecture. Thank you.