Networks, Signals and Systems
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Lecture - 35
Fourier series

Good after noon friends, today we shall be taking up Fourier series. So far we have discussed about network functions and their behaviour under different conditions with different types of, for example ramp or a step or an impulse or a sinusoid like this.

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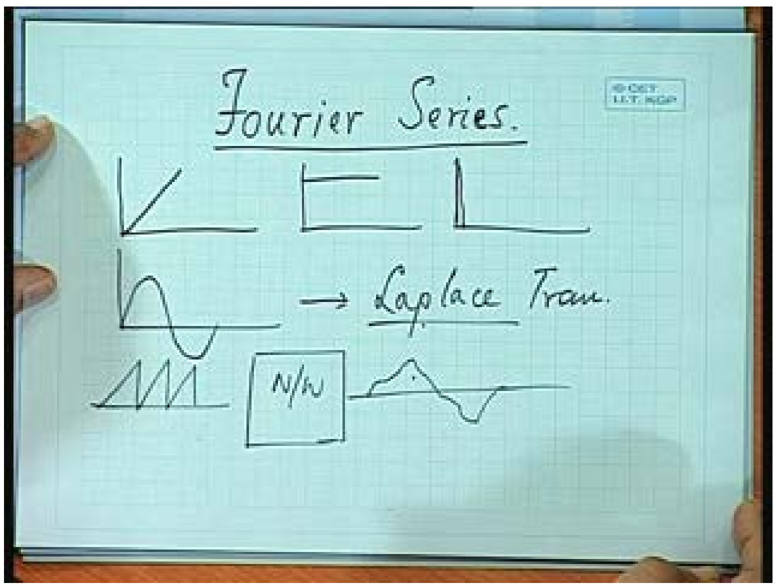

We have taken up different types of functions and you have seen the behaviour of networks and we have used so far in the time domain the response has been evaluated by using Laplace transform okay. Suppose you are given a network and you have a periodic function then you take the Laplace transform of this take the network function and then we can evaluate the response so that the response will also be periodic or something like this.

Now the response that we get taking Laplace transform of the function is giving me a total solution that is say total solution means including the transient. Suppose the response is like this after sometime it will be stabilizing to a steady value if you are interested in getting a steady, steady state solution. We are not interested in the transient part, we are interested only in the steady state solution then there is an advantage of using the frequency response technique we have studied so far the response due to a particular frequency okay. Now if you are giving a periodic function how say of this type if you are giving a periodic function as input, is it possible to resolve this into large number of sinusoidal functions.

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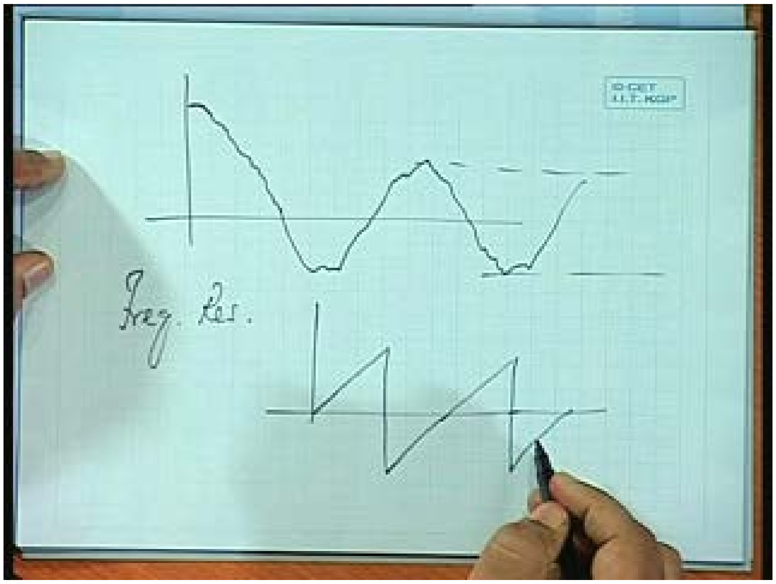

Our aim here is to resolve that and then find out the response corresponding to each elementary component of those functions, sinusoidal functions and the response due to each one of them you calculate independently then use superposition theorem take the total sum that will be the response due to any periodic function.

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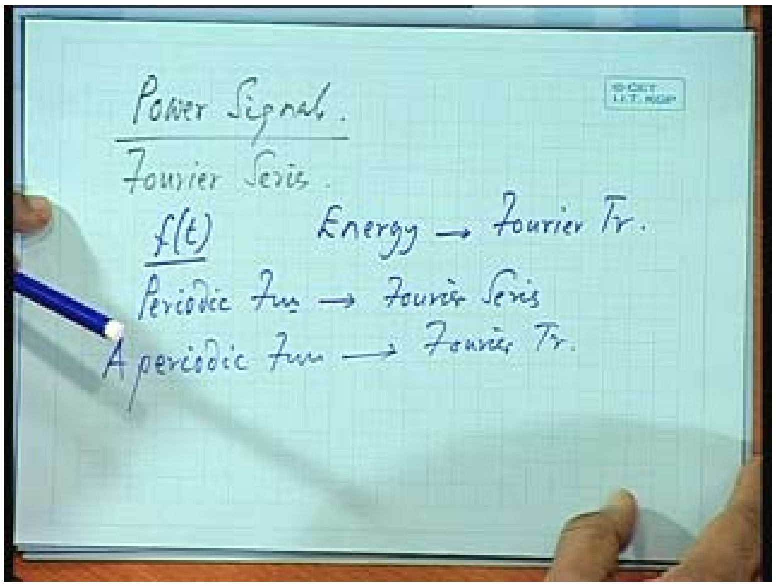

So for periodic power signals, we mean power signal means signals with finite average power. So with power signals this resolution results in Fourier series that is contain the function which is periodic has some average power finite average power and that can be decomposed into a Fourier series, if it is continuing if it is an energy signal that is finite energy but it is not periodic then we resolve it resolve this by Fourier transform. So for periodic function of finite average power, we go for Fourier series and for energy signals that is Aperiodic signal with finite energy, we go for Fourier transform.

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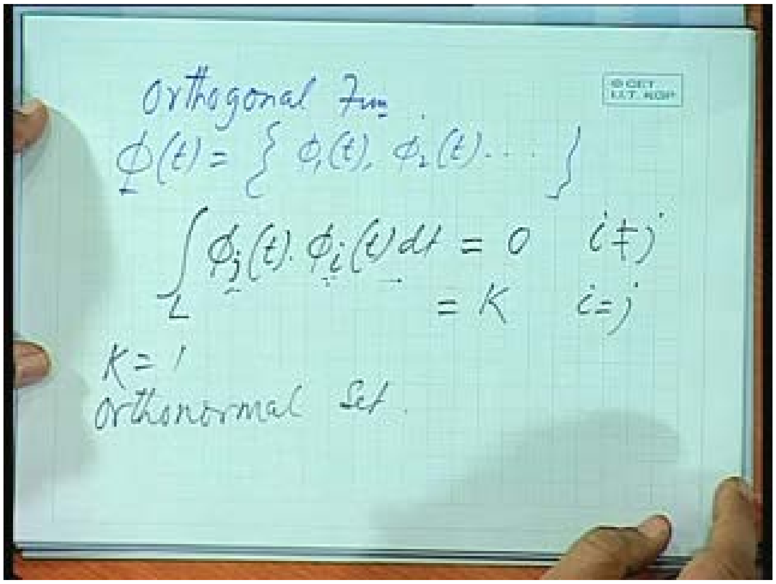

Now before we start, before we start resolving the periodic signal into Fourier series. Let us see some basic fundamental principles, fundamental properties of orthogonal functions, what are orthogonal functions? Orthogonal functions are set of functions say I call it phi t, it is a set phi 1 t, phi 2 t and so on, phi 1 t, phi 2 t and so on. Then this function phi j t, phi I t dt, over a range L should be equal to 0 for I not equal to j and is equal to some constant for I equal to j okay. If the function set phi the members of that function set obey this particular equation then it is an orthogonal set, if k is equal to 1 then we call it orthonormal set okay.

Now any orthogonal function can be scaled to an orthonormal set, if I divide, if I divide phi I by root k define it as some phi I dashed then phi dashed set which will be phi 1 dashed, phi 2 dashed and so on, this will be an orthonormal set because when I take the product it will become root k into root k, so that will give me k, so the end product will be 1. Now let us use this in our analysis of Fourier series any function say v(t) call it which is periodic v(t) is v(t) plus minus kT, capital T is a period then we are resolving v(t) as a trigonometric series a_0 plus a_1 cosine omega naught t plus a_2 cosine twice omega naught t and so on plus b_1 sin omega naught t plus b_2 sin 2 omega naught t and so on okay.

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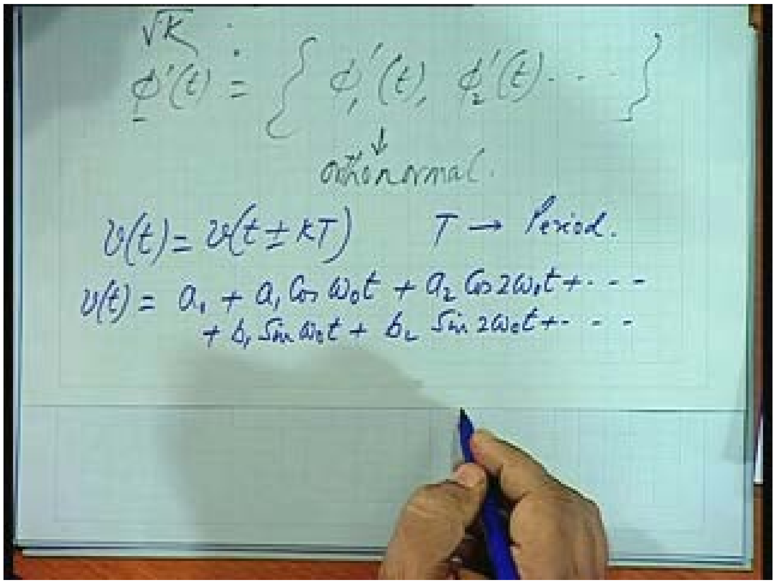

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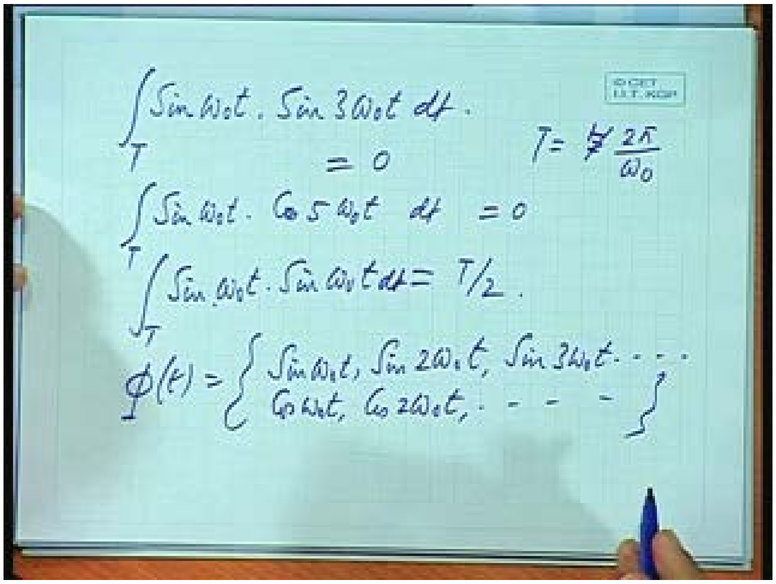

Now let us see this sin and cosine function set sin omega naught t for example into sin say 3 omega naught 2 if I integrate over the period T, T is the period corresponding to 2 phi by omega naught okay. So over this period T if you integrate this it will turn out to be 0, you take any function okay like phi 1 t, phi 2 t, 2 different functions we will find sin omega naught t into cosine 5 omega naught t say that will also be equal to 0 over 1 period, sin omega naught t into sin omega naught t if you integrate which is sin squared it will turn out to be T by 2, you can work it out yourself.

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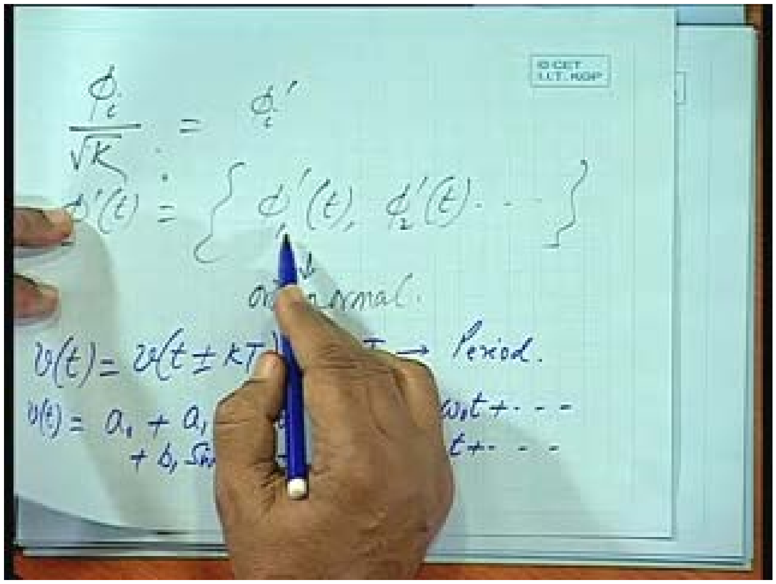

Similarly, you take sin 5 omega naught t into sin 5 omega naught t that will also give you T by 2. So the set phi whose members are sin omega naught t, sin 2 omega naught t, sin 3 omega naught t and so on and cosine omega naught t, cosine 2 omega naught t etcetera, they are all members of this orthogonal set okay. You should be using this property in resolving these values before we start resolving this let us come to a simple example.

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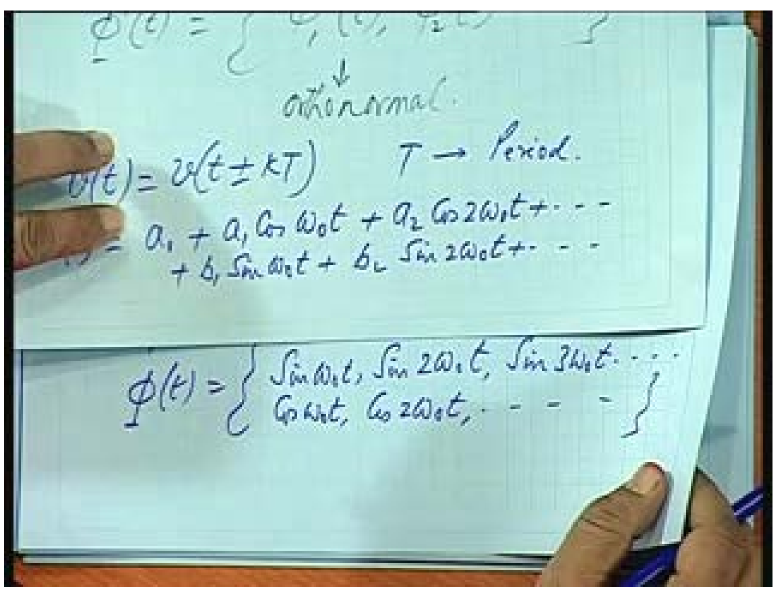

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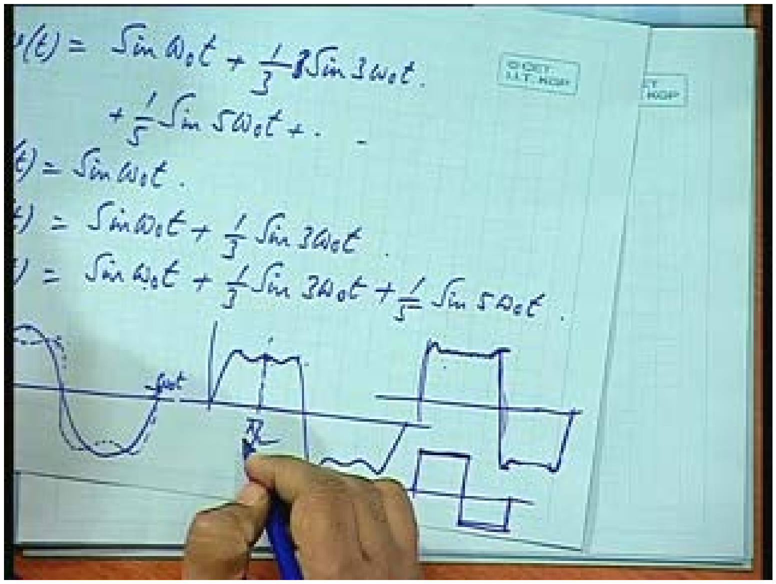

Let us have v(t) equal to sin omega naught t, let us take just a trigonometric series sin 3 omega naught t no sorry, sin 3 omega naught t into 1 third plus 1 by 5 sin 5 omega naught t and so on. Let us take a series like this, let us take only part of this function and gradually increase the number of terms $v_2(t)$ is sin omega naught t plus the second term $v_3(t)$ is again the third term added and so on. They are all called harmonics of omega naught, 3 omega naught t, 5 omega naught t. So that means they have a time period which are integral integral well frequencies are integral multiples of the fundamental frequency is called the fundamental frequency or integral submultiples of the time periods and integral submultiples of the original time period, what will be sin omega naught t like the first function, the first function looks like this.

Okay then the second function when I add like this, the third function to avoid jumbling up I will show here separately. So like this, so you can see there are 3 humps, there are 2 humps here when I am taking the second term then I am taking the third term there are 3 humps and so on okay. So as you keep on increasing this, this gradually approximates to a function like this and finally it will tend to when I take it up to infinity it will tend to a rectangular function that means as you keep on increasing the number of terms this gets more or less flattened to a constant value but the edges are having where such changes are taking place, there are higher ripples here. This is known as Gibbs phenomena, we will come to that later on. So if I take an infinite number of components then only it will tend to it will be ideally matching with a rectangular function, what would be the value of this at say 90 degrees, this is say omega naught t that is theta so when omega naught t is phi by 2, what will be this value? What will be its magnitude tending?

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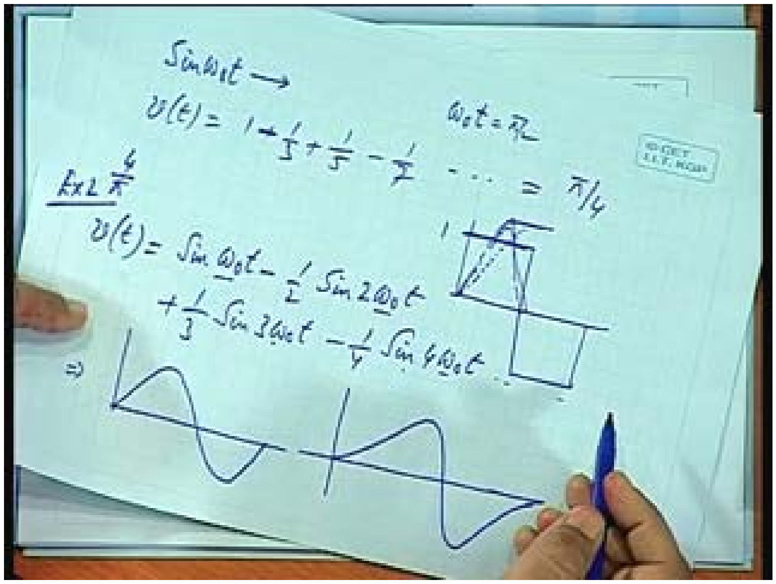

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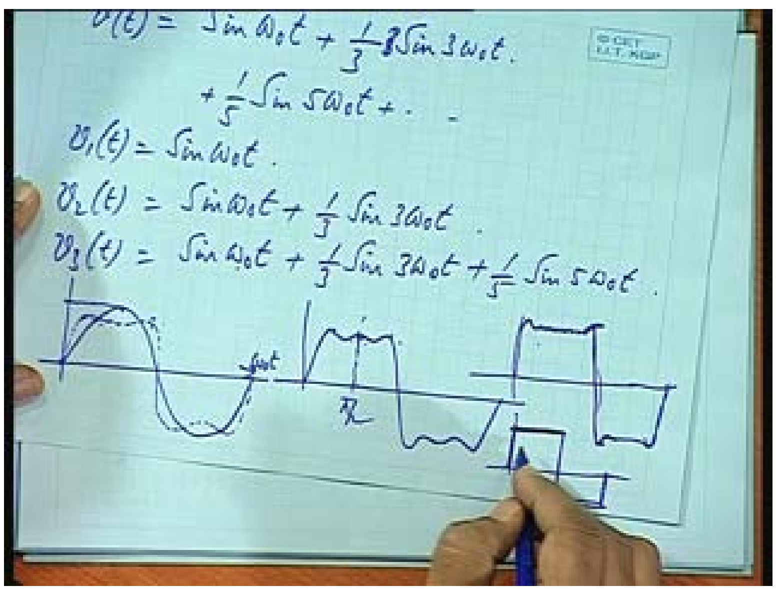

Now sin omega naught t at omega naught t equal to phi by 2, this will be sin phi by 2. So the series will be tending to 1 plus sorry minus 1 third plus 1 by 5 minus 1 by 7, this is a convergence series it will give you phi by 4, this sum is phi by 4 that means if I have a magnitude of 1 here the resultant will be coming to phi by 4 approximately, phi by 4 is approximately 80 percentage .8 very close to .8, so if you normalise it where 4 by phi, if I multiply by 4 by phi that height will be raised to, if this is 1, if this is 1 then the first component will be push here of magnitude 4 by phi, if this is of 1 then this will be of magnitude 4 by phi okay. If we have another example I will take, if I have a function like this v(t) equal to sin omega naught t minus half sin 2 omega naught t plus 1 third sin 3 omega naught t minus1 by 4 sin 4 omega naught t and so on okay.

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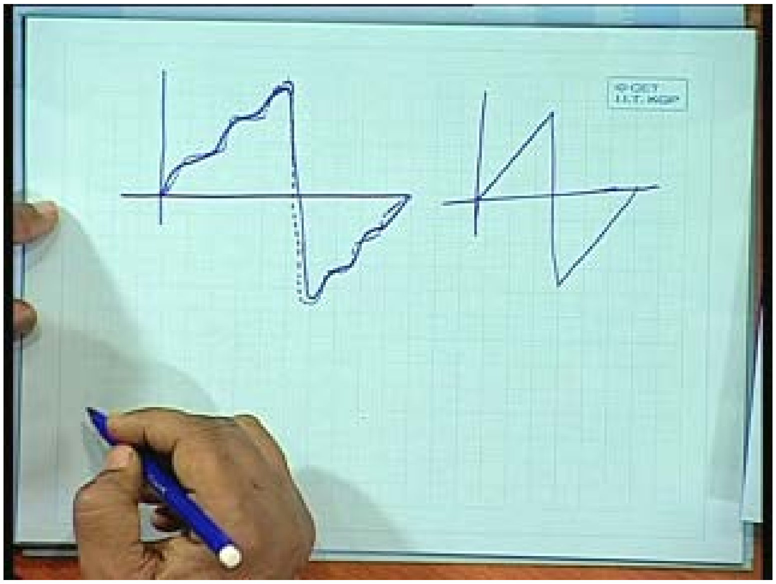

Now for this type of function I have added this even terms, we will find if I take only the first term it will be a pure sinusoid, if I take first and second term you see what kind of distortion comes, it will be somewhat like this. Can you guess what it will tend to yes, triangular, it will come to a triangular that is very good. If you take the third term something like this, if you take the fifth term there will be more variations this will become sharper. So this will tend to finally a function like this as we go to an infinite number of terms okay. So basically what I wanted to drive at is if you take so many harmonic terms, harmonic frequencies then the overall function is again periodic with the same period, with the same period. You take any combination of periodic functions and their harmonics that means compare to the fundamental period these are all these periods are submultiples okay integral submultiples of this period, if you take any combination the overall function will also be having the same period.

Yes, so Fourier series is v(t) as I had written earlier we can write this as a_n cosine n omega naught t plus b_n sin n omega naught t, n varying from 0 to infinity okay 1 to infinity, 0 is included here all right. Now the question is what will be the values of these constants a_i's and b_i's. Now to evaluate these we take records to the properties that we discussed just now for orthonormal functions, orthogonal functions. Now if v(t) is this if I integrate this both sides if I integrate I will declare by dt, multiply by dt what will be the result on the left hand side you get over the period T on the left hand side you get integral v(t) dt over T, here you get a_0 into T and all these orthonormal functions orthogonal functions integrated over 1 time period will be giving you 0, we just now derive that okay. So a_0 comes out as 1 by T integral v(t) dt, so given function if you integrate over 1 period and then take the average divided by the time period T will give you a_0 okay.

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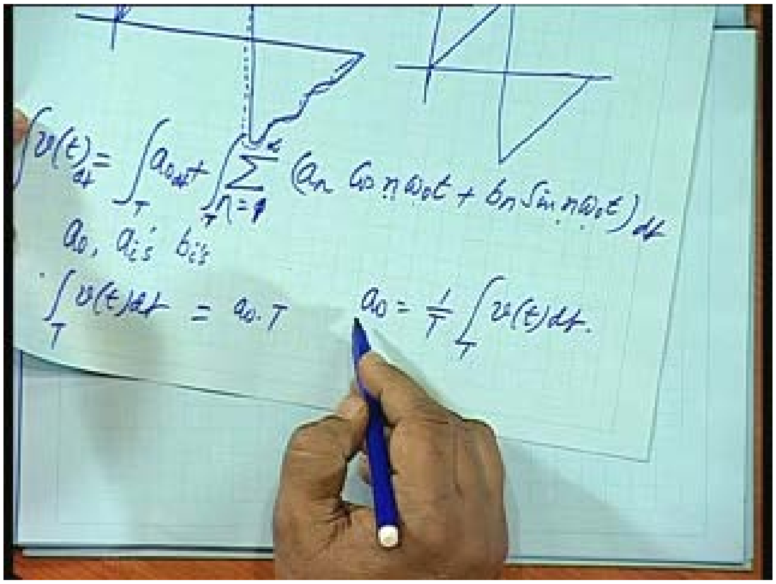

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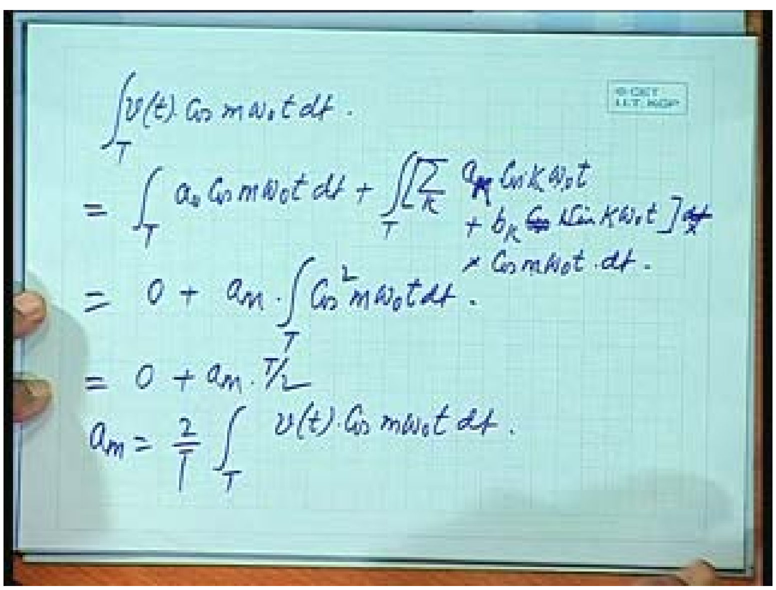

Next v(t) if I multiplied by cosine m omega naught t dt and then take the integral over period T what would be the result, so on the right hand side you have got a_0 into cosine m omega naught t dt plus sigma a_m okay I will write a_k, a_k cosine k omega naught t plus b_k cosine sin sorry, sin k omega naught t into dt okay summated over k. Now what will be this, this is a cosine function integrated over 1 period, so it is an orthonormal function, orthogonal function. So this is 0 now cosine k omega naught t I forgot to put cosine m omega naught t and then put dt this multiplied by cosine m omega naught. Now in these products when k is not equal to m all other terms in this will become 0 similarly for this it will be 0, only cosine m omega naught t that is present here will give me corresponding co-efficient a_m.

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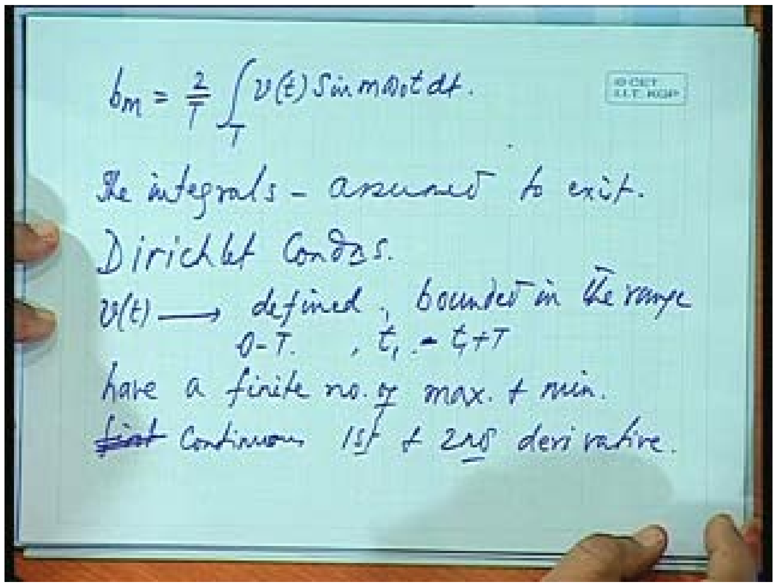

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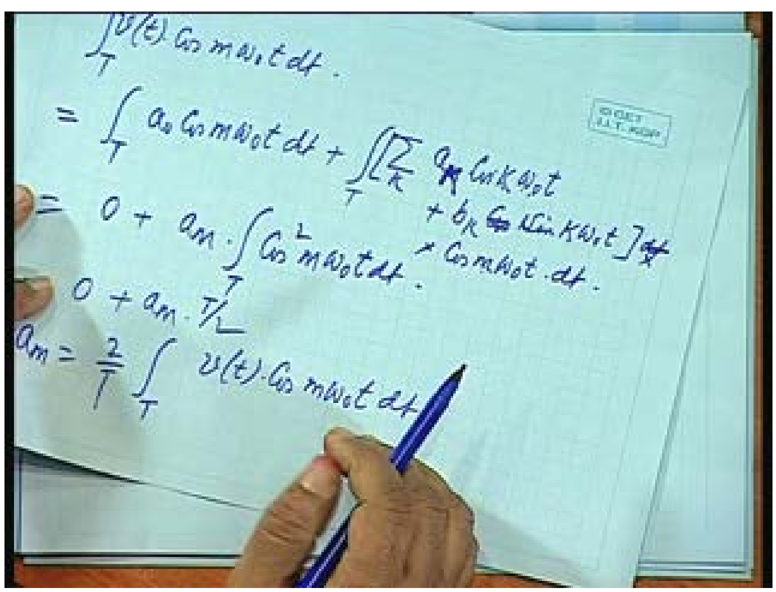

So it will be a_m integral cosine squared m omega naught t dt and that is what that we have already evaluated as T by 2. so a_m will be 2 by T integral v(t) cosine m omega naught t dt okay similarly, you can also multiply by sin m omega naught t dt integrate both sides and you get b_m as 2 by T, v(t) integrated over T, v(t) sin m omega naught t dt okay. Now we have made certain assumptions here, what are those assumptions the assumptions was yes it is a periodic function all right but the assumption was all these are integrable, there are definite values of integration we have ruled out the possibility of a function going up to infinity all right.

So the functions may have finite number of discontinuities but the values must be, must not go to infinity. Number 1, so the integrals where assumed to exist the integrals assumed to exist that means they will exist, these integrals will exist provided these conditions are made, they are called Dirichlet conditions that means v(t) must be defined okay bounded in the range 0 to T or you can say any t_1 to t_1 plus T okay, t_1 to t_1 plus T that is over 1 period and it will have a finite number of maxima and minima, maxima and minima okay must have also first, continuous first and second derivative, continuous first and second derivative, most of our functions obey these MS physical systems. If you have for example a periodic function having jumps like this then at this point if I evaluate the value of the function from the Fourier series.

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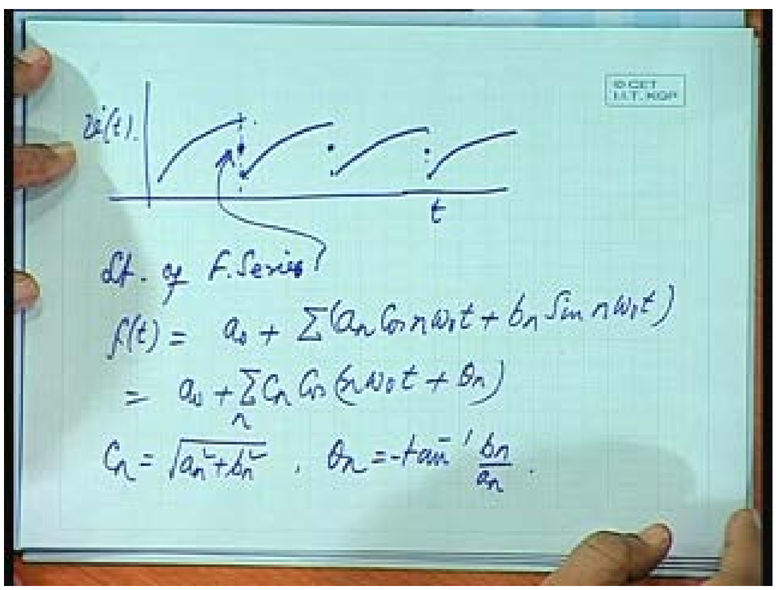

So the limit of the Fourier series say at this point will be in the middle that is half of these 2 extreme values. It will result in these values, so these are the limits of a Fourier series this is t and this is v(t) okay f(t) is equal to a_0 plus sigma a_n cosine n omega naught t plus b_n sin n omega naught t all right. I can write this as c_n cosine n omega naught t plus theta n all right summated over n, where c_n will be nothing but a_n squared plus b_n square and theta n is equal to minus tan inverse b_n by a_n mind you it is minus because in a cosine expression cosine a plus b becomes minus.

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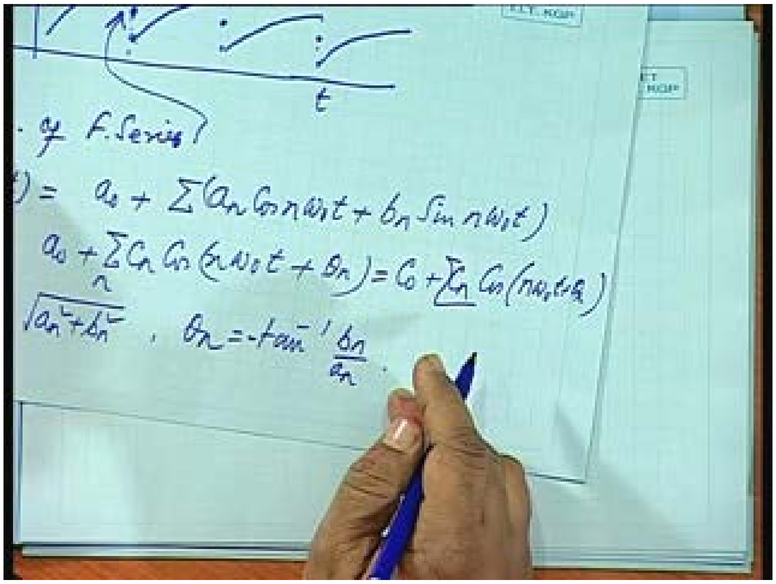

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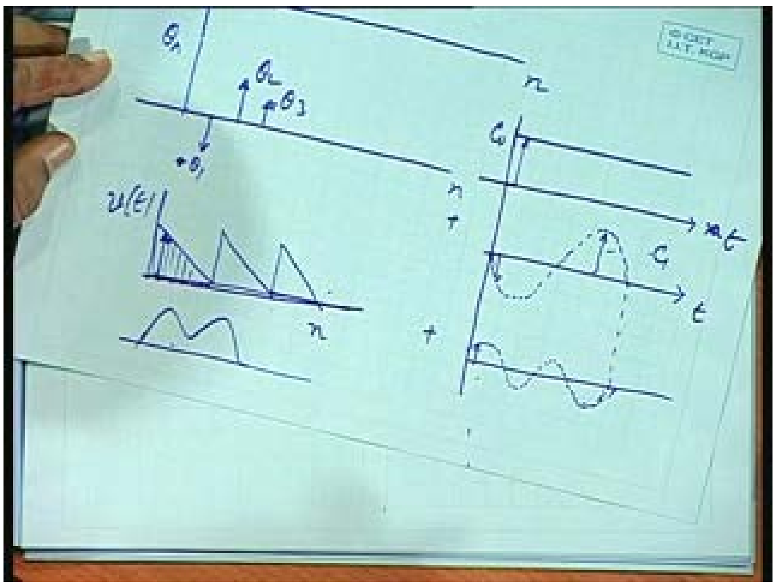

So f(t) can be written therefore I can put this as c_0 plus c_n cosine n omega naught t plus theta n summation all right, what will be the values of c_n like, this may be c_0, this may be c_1, this may be c_2, c_3 and so on, normally progressively coming down and correspondingly theta n for the first 1 there is no frequency 0 frequency that means something like your average value or dc value it does not have a phase. Here, it may be negative minus theta 1, I should not say minus theta 1, theta 1 is in the negative direction, theta 2 may be in this side, theta 3 may be here and so on.

So the representation of the Fourier series components in the time domain will look like this. For the function, for which we have say evaluated c_0, c_1 etcetera it will be this much c_0 plus I will write t, t this is having c_1 with a negative angle, so it will be may be like this depending on the value of theta 1, it may be like this of magnitude c_1 plus this may be positive. This is of frequency twice that frequency c_2, twice the frequency of c_1 and so on. So sum total of all these components at each and every point will give me the original periodic function whatever be that okay that function could have been may be like this. We are not talk about any specific function here, any periodic function whose components have been resolved is realised at every point by adding up all these values say at this point this plus, this plus, this and so on will give me at that corresponding interval this magnitude okay.

Now I have to match it with an infinite number of points here that means all these should be giving me the values at all these intervals that means there are infinite number of points where it is to be matched so they will be infinite number of components. To perfectly match this particular wave form I need an infinite number of components. So that is why Fourier series extends up to infinity it may so happen a particular periodic function which may consist of 2 or 3 such functions, a regular function that we are discussing say for example this it contains just first and third harmonic but in general any periodic functions especially when there are sub changes will contain a large number of harmonics, theoretically infinite number of harmonics okay.

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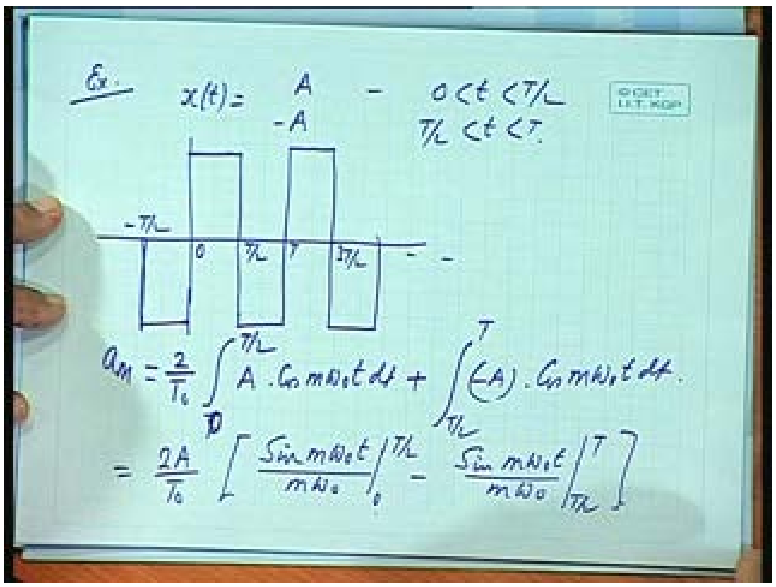

Next let us take 1 or 2 examples, suppose x(t) is a function which is given as A in this range and T by 2 t, T that means this is the kind of function to define periodic functions are only in 1 period that is good enough like this okay. This is 0, T by 2, T, 3 T by 2 and so on, this is minus T by 2 and so on, what will be the Fourier coefficients for such a function a_m will be 2 by T_0, a 0 to okay we can write 0 to T by 2, A, its magnitude is A into cosine m omega naught t dt minus sorry plus T by 2 to T that this becomes minus A cosine m omega naught t dt if you integrate 2 A by T naught will come as a constant and here you get sin of m omega naught t by m omega naught 0 to T by 2 then with a negative sign, sin m omega naught t by m omega naught T by 2 to T.

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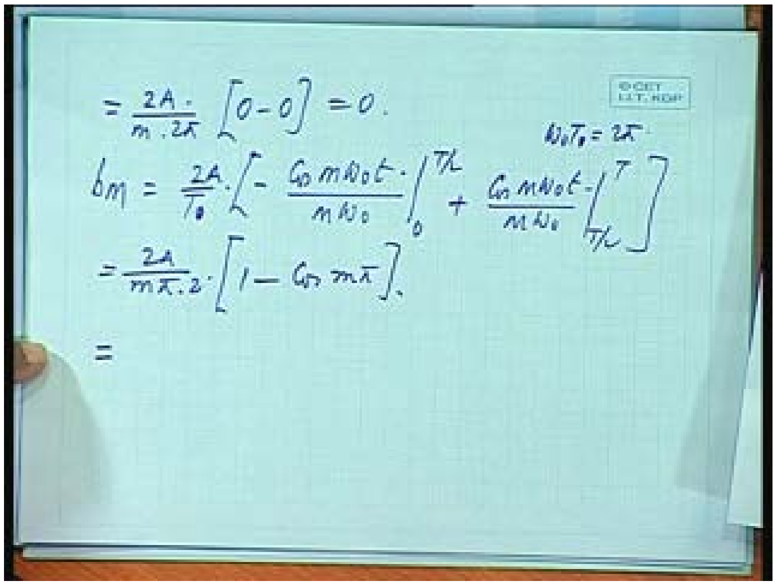

Then if you work it out with 2A by m into omega naught into T naught will give you 2 phi, is it not omega naught into T naught is 2 phi t need not write T naught, T means that, so this will be m into 2 phi and inside the bracket it is sine m, n omega m omega naught t if I put T by 2. So sin of m phi which will be always 0 so both the limits will be give me 0. So all a_m's are 0 and b_m it will be once again 2 by T naught, 2 by T minus cosine m omega naught t by m omega naught 0 to T by 2 plus minus of minus there is a minus A and when you integrate you get another minus, so that gives me plus then there is an A, I can take that out okay cosine m omega naught t by m omega naught T by 2 to T.

Now that gives me 2A by m phi, correct me if I am wrong this is all right 2A by m phi m into 2 phi, is it all right 2 check when sorry 1 minus cosine m phi okay sorry, there is some slips somewhere 2A by T, 2 by T naught okay. So 2A by T naught I should not have written T naught, this was T omega naught and T, check it then this one is 2A by T cosine m omega naught t, m omega naught. So you are putting T by 2, so m T by 2 that will give you phi naught 2 phi okay actually there and it is a function of t m and omega naught omega naught into T sorry omega naught into T not this t, so it will be 2 okay, okay, omega naught into T is 2 phi and T naught be must be there okay.

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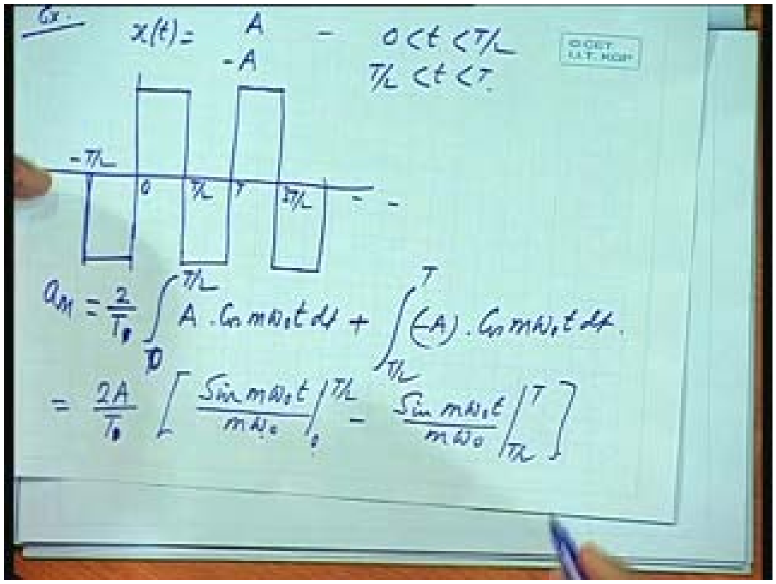

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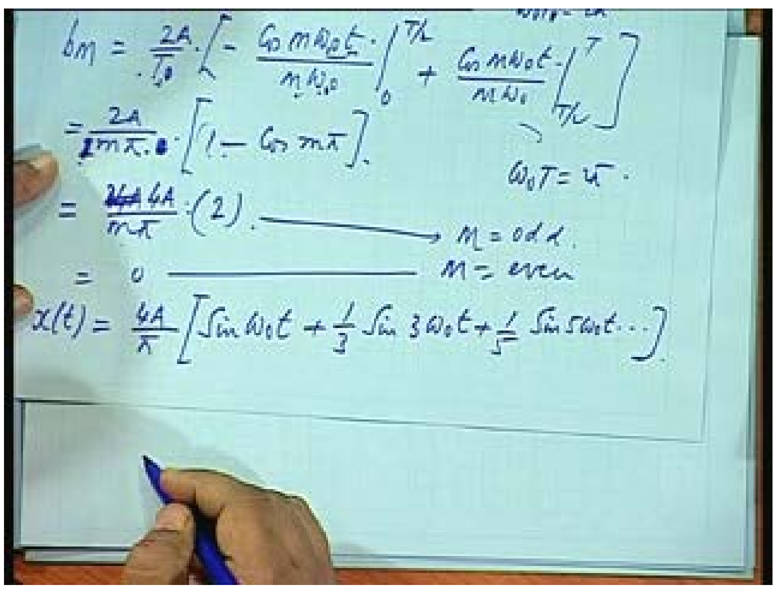

So 2A by m phi into 1 minus cosine m phi all right I have already put these 2 values okay so that give me 1 minus cosine m phi, when m is odd 2 then it will be 2 and when m is even it is 0 there was some slips somewhere, I mean that T by T naught the T naught is in the second step 2A by T no is this all right, 2A by m phi first term first 2A by T naught is there 2A by T first then T is not continuing afterwards no, there is no T here, there is no T here, so T gets multiplied by omega naught that is given me 2 phi but this 2 should not have been there where is a Pelosi, I understand when you are adding this there is a 2 coming out from here.

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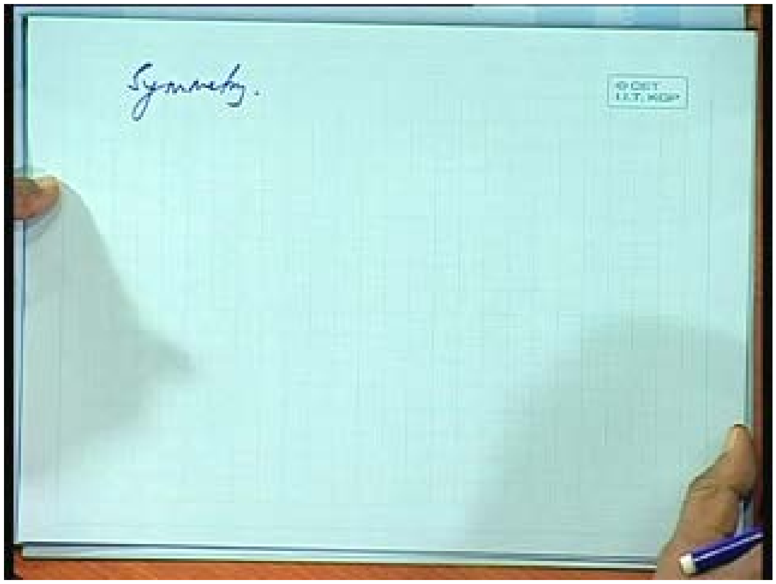

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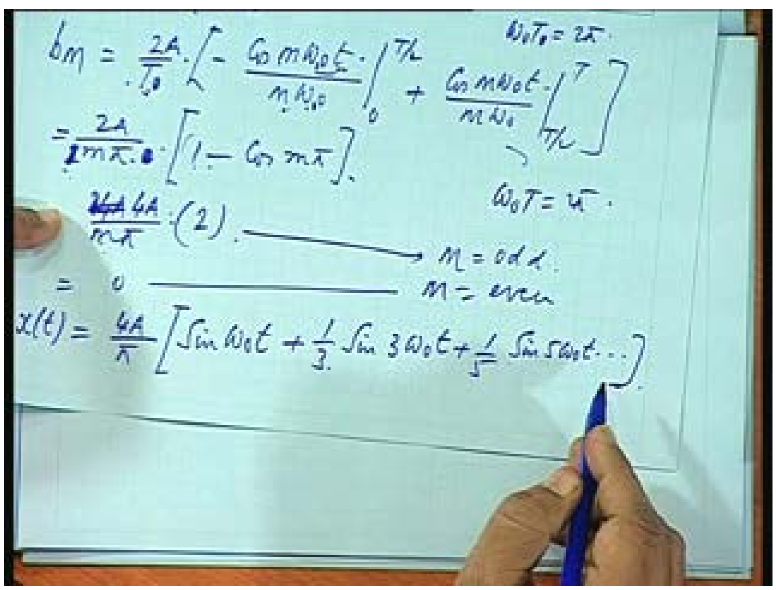

We put m omega naught into t omega naught into t by 2, so that gives you phi. So minus 1 plus 1, so this is 1 minus cosine m phi again here 1 minus cosine m phi, so there is a 2 here, so that will get cancelled with this 2 okay. So it will be 4, 2 into2, 4A by m phi all right, there actually 4 terms which are condensed in this, so x(t) will be 4A by phi sin omega naught t plus 1 third sin 3 omega naught t plus 1 by 5 sin 5 omega naught t and so on okay m is equal to 3, 5 etcetera odd values because of the symmetry, it is the symmetry which causes sum of the sin and cosine terms to vanish they are not present because of the symmetry.

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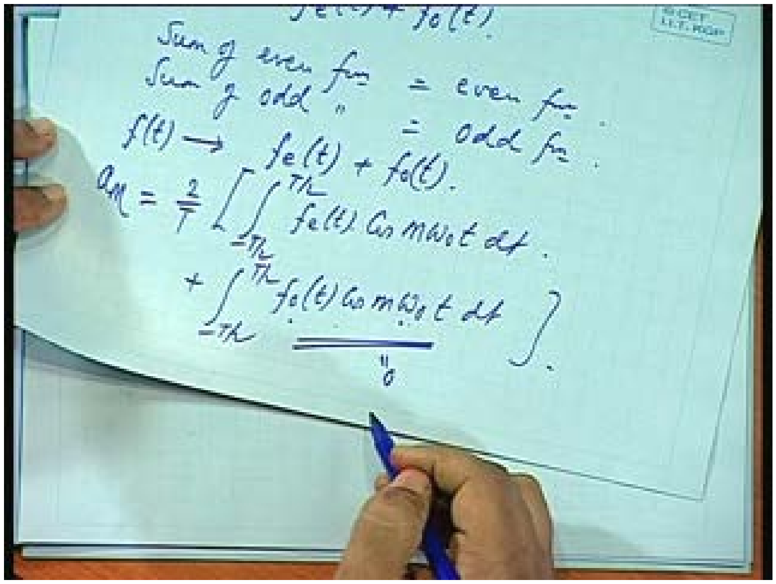

Now let us see some of the important properties of even and odd functions that we can use here. Any function f(t) we can resolve into an even part and an odd part okay. Now sum of even functions is equal to an even function all right, sum of odd functions is equal to an odd function okay. Now if f(t) the given f(t) is even or odd then how do I evaluate see f(t) I can write as $f_e(t)$ plus f naught t. Now I want to evaluate a_m, so a_m is 2 by T okay say minus T by 2 to plus T by 2 and instead of f(t) I will write $f_e(t)$ cosine m omega naught t dt plus again T by 2 $f_o(t)$, odd t cosine m omega naught t into dt that is f(t) I have written into in terms of these 2 components all right. Now this is an odd function this is an even function so the product is an odd function.

So for an odd function if you integrate from minus T by 2 plus T by 2 between any minus T dashed to plus T dashed it will be 0 for an odd function, is it not. So this term will be reducing to 0 okay so what you have with this even term. So it will be 2 by T integral minus T by 2 to plus T by 2 f even part into cosine m omega naught t dt all right. So let us see first if f(t) itself is even then what happens then only a_n will exist, is it not. The odd part cannot exist that is a_m is equal to 2 by T minus T by 2 to plus T by 2, f(t) itself is $f_e(t)$, so f(t) into cosine m omega naught t dt.

Now for an odd function, for an even function this product is even, for an even function minus T by 2 to plus T by 2 is twice the integral between 0 to 2 T by 2. So I can write this as 4 by T_0 to T by 2, f(t) cosine m omega naught t dt is it not for any even function even functions are like this. So the total area integral means total area is same as twice this area that is what I have done similarly, for an odd function by the same logic, for an odd function what will be the values for an odd function a's will be 0 only b's will be present and the integral will be once again evaluated between 0 to T by 2 and multiplied by 2, f(t) sin m omega naught t into dt okay.

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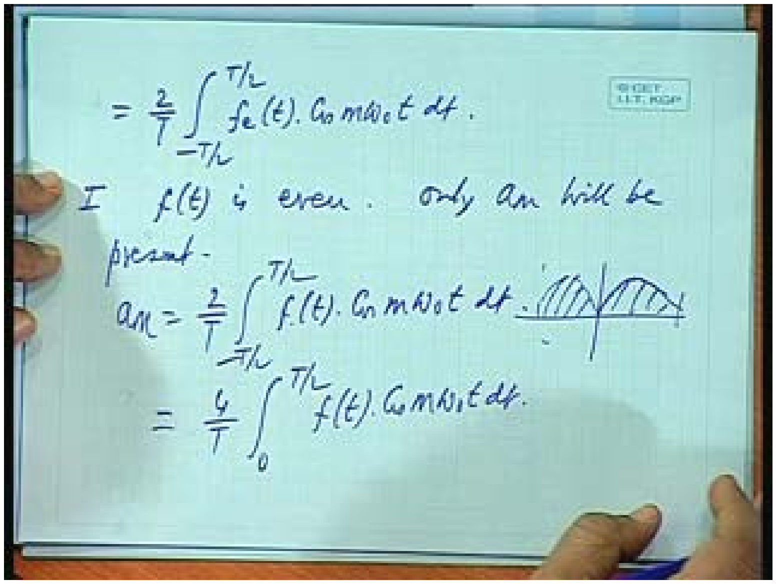

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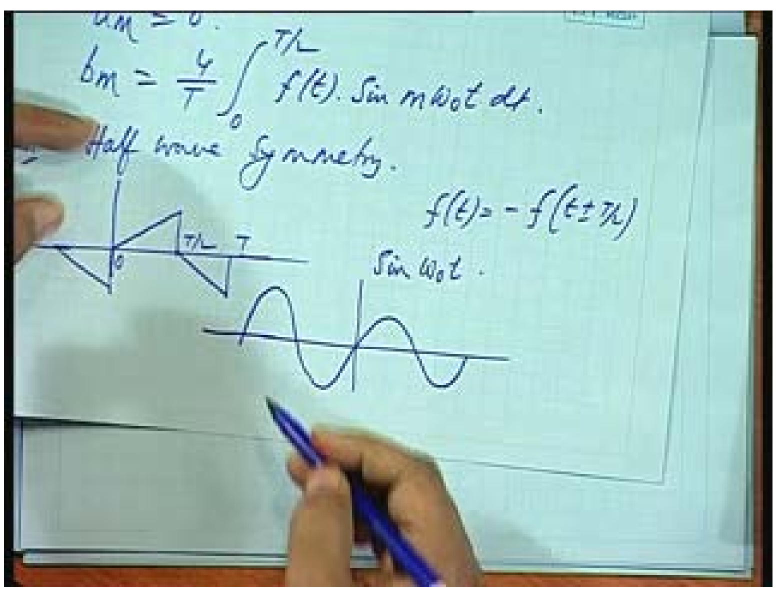

Next suppose we we have half wave symmetry, what is half wave symmetry if you have a function f(t) is equal to minus f(t) plus minus T by 2 that is say function like this, this is 0, this is T, this is T by 2, it is like a saw tooth wave, no, saw tooth function alternately you twist the teeth okay. You get a function of this kind, now what are the types of components that may be present in such a function. Let us see, let us multiply this function by a sinusoid, if I multiply this by a sinusoid what be the result okay. Let me draw it little more clearly.

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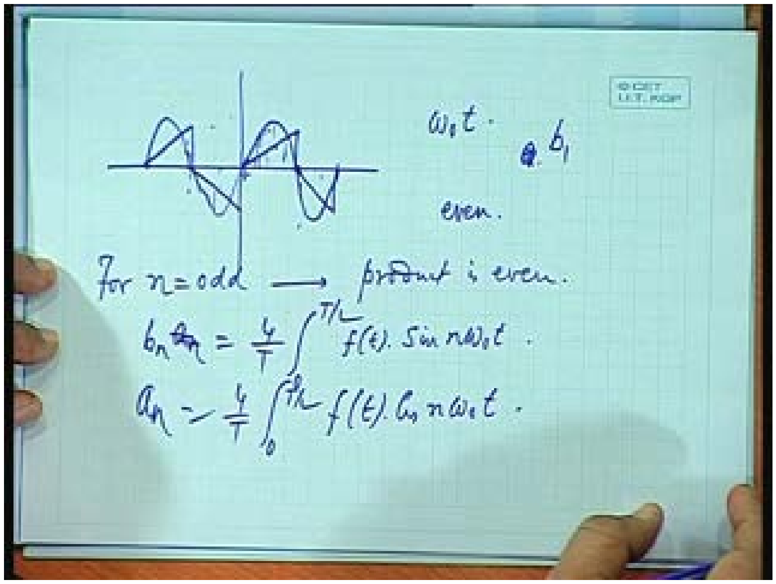

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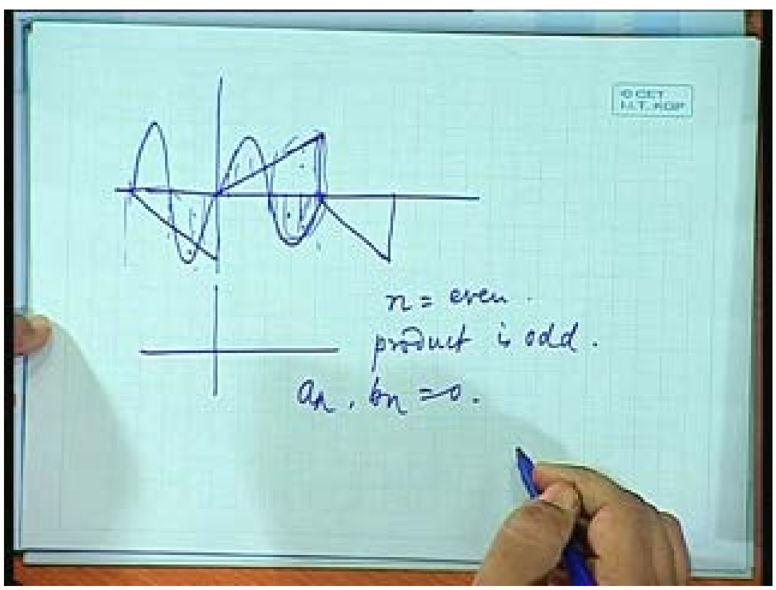

Similarly here, so this is one period okay now if I multiply by a sinusoid all right and so on. Here the product is positive into positive, positive into positive and so on all right, is it all right. Here, so for odd terms I have taken a sinusoid, sin omega naught t, so this an odd term all right a sorry b_1 I have taken okay. So if I take an odd function the product is even is it not here it is positive into positive in this zone again it is positive into positive all right.

So it is on this side also it is negative into negative is positive, so it is basically the product is even okay. So for n odd the product is even and hence the integral will have some finite value non-zero value okay. So these coefficients will exist and they are identical, so you can take between 0 and T by 2 integral between 0 to, so it will be 4 by T, 0 to T by 2 that function f(t) and sin n omega naught t where, n is odd and similarly the this is b_n and a_n will be similarly 4 by T, 0 to T by 2 f(t) cosine n omega naught t, okay if you take n even then what happens, let us see.

Now you are having between here this and this point, say something like double harmonic. Here you see this is one period, this is positive and positive but here it is negative and positive and the this is negative but this is a large value and this is this much value its counterpart is here where both are negative. See this component the product of this part this zone is equal in magnitude to the product of this but here it is positive into negative that is negative into negative. So they will be cancelling, so this is an odd function, so if n is even, product is odd, this product is odd and hence the integral is 0. So a_n and b_n will be 0 for n even for this type half wave symmetry function all right. So thank you very much in the next class we shall be continuing with this a few more interesting features of Fourier series and some standard functions we will take for evaluating the Fourier series thank you.

Preview of next Lecture

Lecture - 36

Fourier series (Contd…)

Good afternoon friends, we were discussing last time Fourier series. Fourier series represents basically is a representation of a periodic function in terms of cosine and sin functions of different frequencies, the frequencies are multiples of one fundamental frequency.

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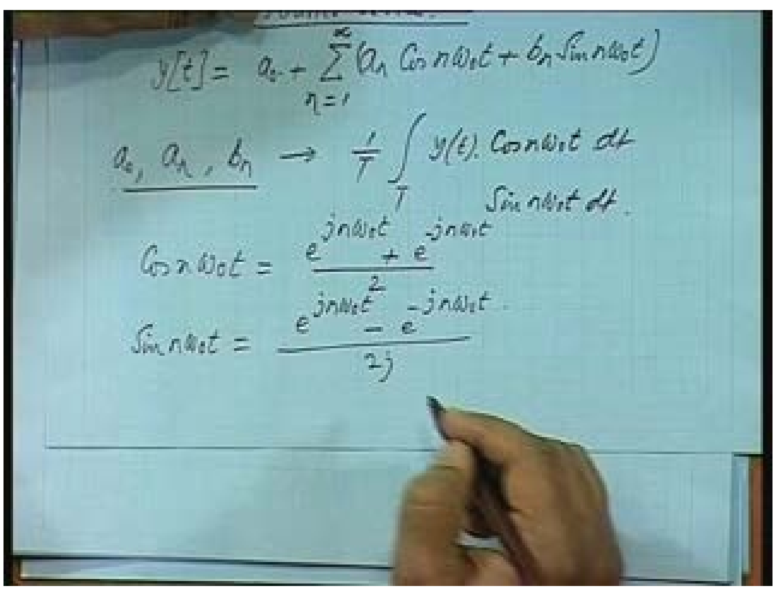

So we wrote y(t) as a_0 plus summation a_n cosine n omega naught t plus b_n sin n omega naught t, where n varies from 1 to infinity. Now this representation gives me the coefficients a_n and b_n including a_0 as 1 by T integration over T this function y(t) multiplied by the corresponding orthogonal function say, if I want to evaluate a_n it is to be multiplied by cosine n omega naught t if it is b_n then to be multiplied by sin n omega naught t and then integrate it over this period and if it a naught then a naught is multiplied by just a constant so you have to just integrate this and that gives me the coefficients a_0, a_n and b_n, we have used a properties of orthogonal functions.

Now one may express cosine n omega naught t and sin n omega naught t in terms of exponential functions. Similarly, sin n omega naught t as e to the power jn omega naught t minus e to the power minus jn omega naught t by 2j. If you express the functions sin and cosine functions in terms of the exponential functions and substitute here, what we get is y(t) as a_0 plus a_n for cosine n omega naught.

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