Estimation of Signals and Systems
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Lecture - 21
The Time-invariant Kalman Filter
KF for Coloured Noise
KF for Correlated Process and Measurement Noise

Good morning and happy DIWALI to all of you. Today we will discuss some of the special cases; for example, you know all the filters we have Kalman filters are the first filters that we are studying in this course.

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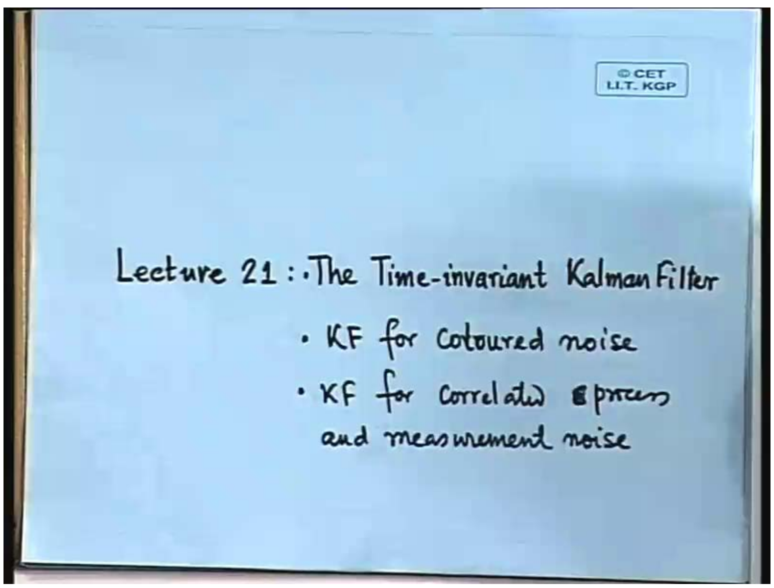

We have already studied other filters; but Kalman filter is distinguished because it we it was a time varying filter. Previously our filters were generally time invariant. So an important thing is to understand, that why, when does the does the Kalman filter at all become behave like a time invariant filter; which is simple to compute and analyze. I if I am if it does, is it stable? That is a that is a you know, you know I mean any dynamic system even Kalman filter is a dynamic system, so one of the important things is to know whether it is stable?

Analyzing the stability of linear time varying system is not that simple, but at least we can find out that whether under certain certain assumption at least; so that the Kalman filter is going to going to be stable by analyzing it's the time invariant fashion of the filter. We are also going to look at you know; we have derived the filter under certain assumptions, like this noise white, that noise white, now you know in a in a practical case, it may not be the situation that always this white this noise noise sequence is as so nicely white, because they may be having mainly because they may be coming from common source, right.

So one of the one of the things, that is mentioned is that is sometimes in their graphs, see these the sensor which senses its motion; for example ((00:02:50 min)) are mounted on the aircraft or the vehicle, if there is a if there is a disturbance coming on the on the air craft because because of wing gauges whatever, that disturbance is firstly firstly effecting it's motion. So it can be it can be interpreted as passes noise. Secondly it is it is also affecting the sensors which are on the air craft. So therefore it is also affecting the measurement.

So it can be so it also calling measurement noise and since they are coming from the same source; so they are obviously correlated. So so such situations are not uncommon, so we will have to see that, what what how we are going to tackle the tackle the problems when when the when such situations arise? That is the process and.. process and measurement noise becomes correlated or the or the process noise sequence and the measurement noise sequence are not fully white, you know getting getting getting fully white sequences are, fully white sequences are are are somewhat idealization.

So we will see how we can make you know easy transformations, so that the and get into some equivalent form which is again which again satisfies an an assumptions. So as usual we will obviously we I mean, we perhaps should not be need not be and and cannot be very mathematically rigorous in this class. So we will we will we will we will state the number of results, but we will I am intuitively try to see that; how these results have arrived at rather than you know getting through the equations, which is may not be very interesting or may be a bit depressing also, you know they get they get so complex.

So I have tried to avoid that, it in case if you want to find the equation; they are all there in the books, you can go through them.

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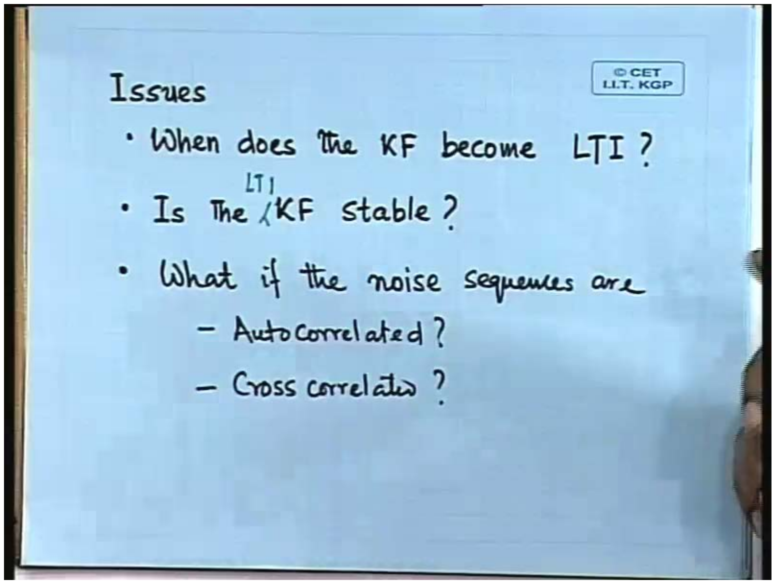

So so so here we are going to answer, specifically these are the question that we are going to take up that; when does the Kalman filter become linear time invariant? Is the is the we can say that, first we will we will examine whether is the LTI KF stable? And what is the noise sequences are, either auto correlated or that means that means that we each individual noise sequence is not white, the auto correlated I mean that the the auto correlation function is is not exactly a delta function. Or if they are crossed correlated, that is the processor may gave a noise sequence are correlated, okay. So let's take up things one by one, first of all let us realize, that this is the basic Kalman filter equation.

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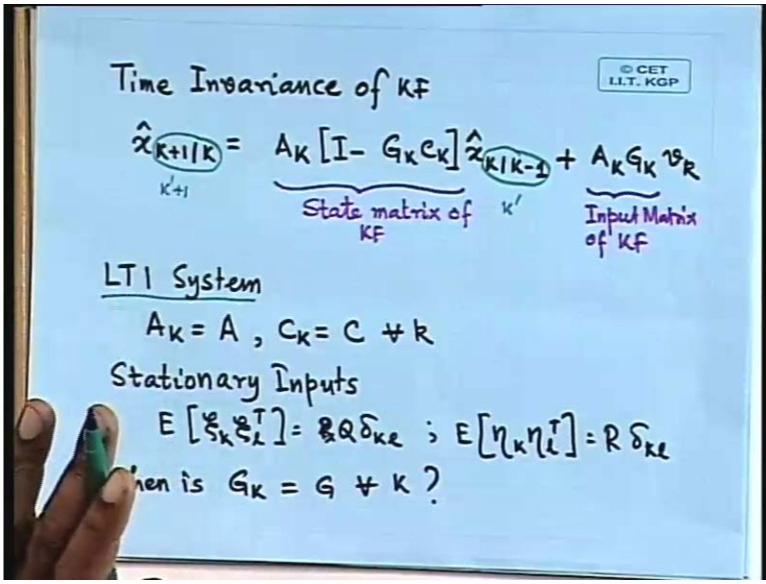

You can I have I have put in in you know in one step back prediction the form in the sense that, we have measurements up to K and we want to predict K plus1. And that I am have related that is in this equation, I have put both the time update and the measurement update, together. If you do the time update then from K K minus 1, you will go to KK; no you do, if you do a measurement update then from KK minus 1, you will go to KK .Then if you do a time update then you will go to K plus 1. So I have put both of them together and I have directly related K K given K minus 1 to K plus 1 given K, right.

So in that sense you know, it is like if you if you call this whole thing as you know let us say; K dash plus 1, so is this is called and then then this whole thing can be called K dash, right. So it is like a so it is like a state equation. This Kalman filter looks itself looks like, a dynamics I mean basically a discrete time state equation. So you have a x x k dash plus 1 is equal to something into x K dash plus something into another external sequence.

This something is the state term dimension matrix of the Kalman, as state matrix of the Kalman filter, right. So this is like A and this is like B. So obviously; if you are if you want now, if you want this filter to be we are we are trying to examine that, under what conditions this update time and value? So obviously you can you can understand that, at least if you want it for arbitrary A B C; then first thing is that this system should be LTI, that is system should be LTI means, that A k is equal to A. That is the state matrix of the system, does not change at every time instant; it is a constant thing which we normally consider.

So A k is equal to a and c k is equal to C, that is that is one of the possibly one of the requirement; otherwise you will require very you know very in some very peculiar cases of A and C, it may so happen that A Ks time varying and CK is time varying and G k is computed in that peculiar way through the Kalman filter, but the whole thing is time invariant. That we be rather here that that I mean I do not know whether it will ever occur, but that I mean it it suddenly will not hold for arbitrary EKCK.

So since we want a result which is applicable for any system, so we will assume that; so the first condition that we require for this system to be to be it linear time invariant, we will it seems into it if that will require that A, that is the that is system itself is linear time invariant. So let us first assume that, so A and C are linear time invariant, now what is remaining is this G. So if G is time invariant then the whole equation will become time invariant, correct.

So what is now G contains various things, G of all things it will it will contain P, it will contain q, r all these things will be contained. So first of all let us assume that, the inputs are also stationary. You know they if if the inputs are not stationary after all the inputs are coming here, so if the inputs are not stationary; I mean this system system can be of course system can be linear time invariant, but this this gain contains this Q and R, so therefore they will not be stationary.

So I am also so let us also assume that, that that these signals are stationary in the sense that; they are they are they are already assumed to be zero means, but their covariance depend do not I mean they are they are again constant, okay. Now if we assume this, then can we say that G k is equal to G, for all k?

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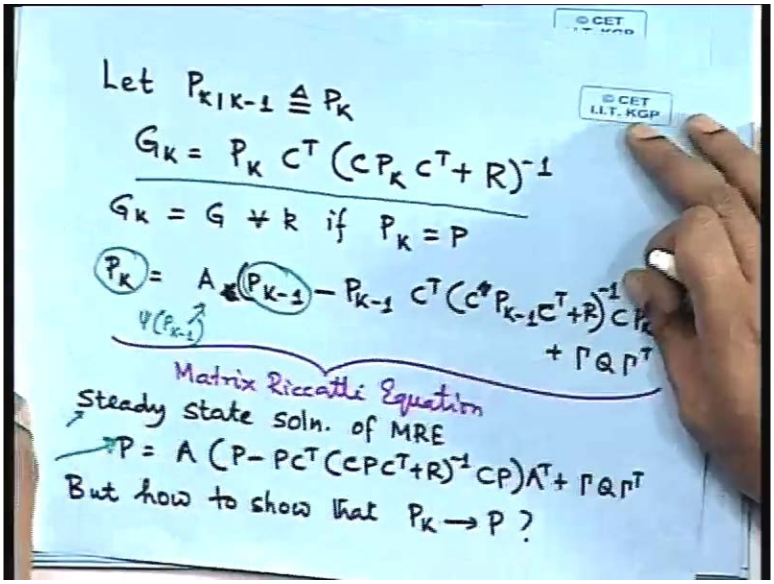

So let us look at the Gk equation. First of all let us understand that; this is the G k equation; after making the assumptions that this is the state covariance, I am calling P k given k minus 1 as P k just in a both both in this is a not necessary. So I have removed one. So so this is the formula for G k, these are all all derived. So we can look up so so you can find out that; the problem of determining whether G k is time invariant is the same as the problem of determining whether P k is time invariant. Whether P k is time invariant, G k is going to be time invariant in this in this all other things are time invariant, correct.

So basically the problem is to find out that whether P k is time invariant, now how does P k update? So that means that P k should not change which K, it should remain constant. Now P k is actually updated through the Kalman filter, again there are there are just like x k there are two updates like; you first go from P k given k minus one to p k given K and then go p two p k plus one, given k. So if you put club these two just like I did for x hat, you you get an equation like this, this is an algebraic.

This equation is is incidentally known as the matrix Riccati equation. Now if you want that in successive iterations of k, this be same matrix will be will remain; that means P k minus 1 is also equal to P, some constant matrix P and P k is also equal to some constant matrix P. Then I will put p in the P on the left hand side then I will put P on the right hand side, obviously. So this so if there is a steady state solution, if if P ever converges if P k ever does not change with K but remains steady, if if this has to happen then that matrix P must satisfy this equation; both both on the left and right hand side that means P will be is equal to A P this equation, in this equation I have just put in place of P k minus 1 I have put P, in place of P k I have put P then I get this equation.

So if if both of them have to remain constant and equal to some matrix P, then that matrix p must satisfy this equation. So this is this this is the this is known as the steady state solution of the matrix Riccati equation but the but if if it has so, if it converges then it will it will it will satisfy this equation. That is that is okay but that how do we know at all that it will converge? We have not yet we have not yet known known that, you have to follow the line of argument. We want to prove that the Kalman filter equation is LTI, so we have first made assumptions; that system is LTI, noise characteristics are LTI then we found out that, what remains to be proved is that G that the Kalman gain is LTI. Coming one step back back we have we have said, that G will be LTI if P is LTI, P does not change with time.

We are trying to gradually coming back and will will arrive at very nice conditions under which the the Kalman filter will become LTI. Then we are saying that, that when will P be LTI that is that is the question. So P will be LTI, if P at all converges to some constant matrix; it is it is true that it must satisfy this matrix Riccati equation, if it converges. But the question is will it converge? So how do you know that you know, how do you know that something converges?

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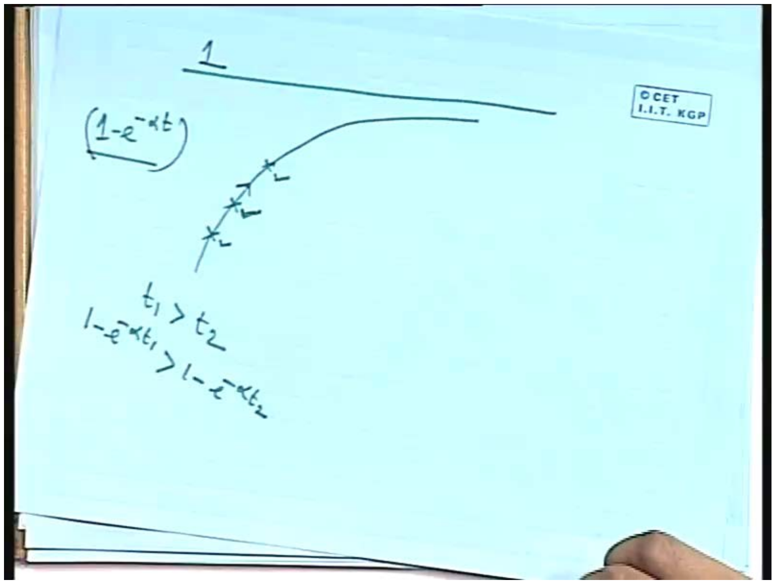

So for example, have you seen if you see for example this thing, if you are given let us let us talk about the number; see if you are given a number which actually increases like this, when can you say that I mean how can you argue that it is this sequence is this sequence or number say one value here, one value here, one value here, this sequential number is going to converge.

How do we argue? We we say that, first of all in this sequence every number this number is greater than this. This is greater than this, so it is an ever increasing sequence but although it is an ever increasing sequence; it none of its values will cross let us say 1 minus e to the power minus alpha t, this sequence looks like this. As t increases whether if you have t 1 greater than t 2; then you also have 1 minus e to the power minus alpha t1, greater than 1 minus e to the power minus alpha t2.

So this function is monotonically increasing with t. How do you know that it converges, because it is always bounded by 1, whatever is t its values can never be more than 1. So it is continuously increasing but it has a roof; it cannot shoot through the roof which means that it must stabilize at some point, if it did not stabilize it would have gone through the roof. So if you can prove that some function has an upper bound; so it cannot increase beyond the certain point but it is all the time increasing. That will show that it must converge, what else it can do, right. So so we are going to we are going to apply exactly that that logic. So you know this is the mathematically very just imagining that, somebody actually thought about it it is a it is actually; rather clever to show that, it converges. So we will first show that P k will find we have to find out, under what conditions these things happens?

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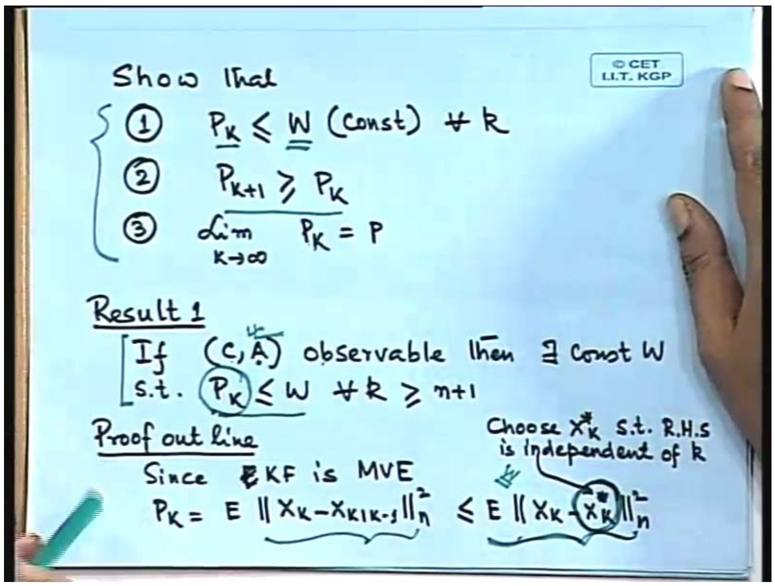

But if we can show that, p k is less than equal to some constant W for all k and simultaneously show that P k plus 1 is greater than or equal to k; and and also show that this limit will converge then we will find that the that the that the matrix will actually converge to another matrix. I mean of of course you will you will have to you know converge in some numbers are well defined.

What do I mean by what do I mean by a matrix converging? When the matrix converging in this case means; that that that they will converge element wise; that is the element P one one will actually converge to some constant number, p two two will converge to some number. So they will converge element by element to a to a to a to a constant matrix. So so we have to find out under what conditions this this happens? Then P will be LTI, then G will be LTI then Kalman filter will be LTI, okay.

So first of all we first result is this; that if you know this is because you I do not know whether any of you wondered, that in the first lecture we talked about deterministic state estimation, where there was no no none of these stochastic noise etcetera. And as we know that in a in a in a deterministic system, state can be estimated only if the system is observable; otherwise how can you how can you estimate the system?

Now here we are having an estimator, okay stochastic but still we are we are trying to estimate the state in the in the in a in a same kind of model, but never talked about observability; throughout the Kalman filter discussion you have never talked about observablity. So how is it that in one case in in both cases we are trying to construct, we are trying to construct an construct a state estimator, but in one case it is becoming I mean I mean observability is the key property.

In in a deterministic case you cannot solve for the state unless the system is observable, you will get no unique solution; it is that is all for x for a given set of measurements, but in the case of Kalman filter we however, we never even brought that concept, it never arose. So we will find that, so it is it is somewhat natural that somewhere it should come, it should play some role. So now you we we will find that observantly and controllability are going to play roles.

So first of all we we have the first result, that if C and A are are observable; then this matrix is bounded, after all what does it mean? This means that, what what is this P k? This P k is the covariance of the estimation error. So it says that, if the if the system is observable then the estimation error does not grow beyond bound.

Remember that, we are not requiring that the that the original system is stable; they may be an unstable matrix, that we are not require. We are talking about the filter, try to distinguish between their filter and their system. The system has state matrix A and the and the filter has state matrix A into i minus G c that we have shown, filter is filter is also a system, okay.

So here we are saying that, if the if C i is observable, then this state estimation error does not grow unbounded. So it is it is good, it we we we at least found an evidence that observability of the system helps in estimation; even in this case just like it helps in the deterministic case, right.

So so how do you prove that, so the basic argument is this I rather clever very clever algebraic manipulations. So what they are trying to say is that, since the Kalman filter is the minimum variance estimater, so this estimation norm is going to be less than any other estimation norm; that you have you have any other estimator you construct and and and put here, the corresponding estimation norm is going to be more than the estimation none of these because the Kalman filter is minimum variance, okay.

And then they will choose this estimator in such a manner that, this thing becomes becomes independent of k. So then this will become a constant matrix. So they will prove that, that that that there exist a constant matrix, such that the Kalman filter P k will be lesser that than this, that is all that you need to prove. So the whole idea is in a in a in a very tricky manner choose this estimator X star k, and and you can choose it when when the system is observable.

That is how observabiliity comes in. So you can you can show that, you can construct some x star k in a clever manner; such that if the system is observable then this whole expression become a becomes a constant matrix W. That is what is that is the way it is proved has lot of manipulations. So it turns out, so first result is that the so so we have found that; if C i is observable then that roof we have established P k cannot go beyond that, okay. So now the next one, so now we say for example; we have let us define this this this right hand side of this thing is we call psi P k minus1, this this function is we are calling psi just giving it a name. This right hand side function it is a function of P k minus 1, we are calling it a function psi P k minus 1.

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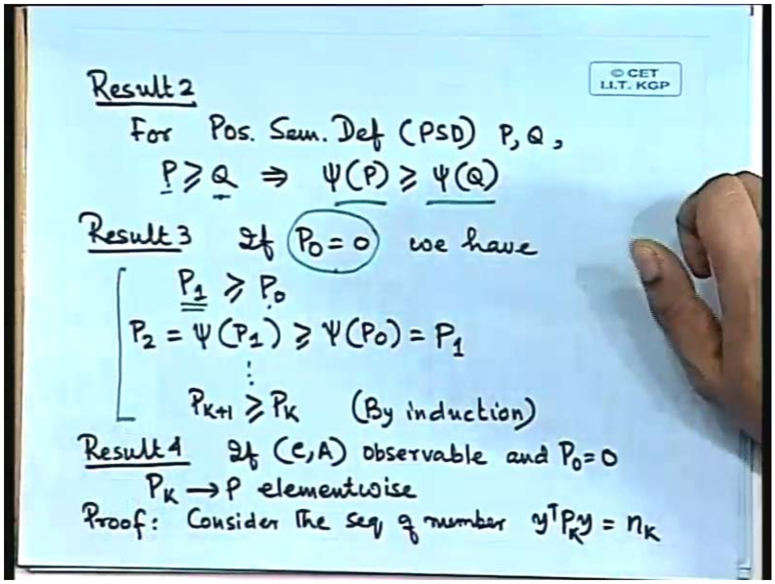

If that happens then the then there is another result, that we want to prove; that if P is greater than or equal to Q, take any two matrices take a a matrix P, take another matrix Q, if P is greater than Q remember that every time remember that when we are saying P is greater than Q, we mean that for all x, x transpose P x greater than x transpose Q x, it is right. These these are all positive semi definite matrices.

So x transpose P x is greater than or equal to zero for all x and x transpose Q x is is also always because greater or less about two matrices; how can you say, a matrix that has many numbers. So when whenever we say P is greater than or equal to Q, it is in it is in that sense, okay this you must, remember.

So P greater than or equal to Q implies that, psi of P is greater than or equal to psi of Q. In other words, the psi function is monotony with P, if you take greater Q; you will you will get a greater psi, it is like log x, okay. If if if if x one is greater than x two, log x one is greater than log x two like that. So now we can prove that, now let us assume that P0 equal to 0 just arbitrary, actually P zero is not zero; if P0 is 0 then obviously P1 is greater than P0 because P1 is positive semi definite, so it cannot be less than zero. All P matrices be symmetry are positive semi definite matrices; any symmetric matrix must be positive semi definite, okay because because becasue it is I mean because it can be written as g g transpose; so if it if it can be written as rather if it can be written as g g transpose, then it cannot be then then it must be positive semi definite.

So this is positive semi definite, therefore it is greater than equal to P0 because P0 is 0, correct. So this is this is the another kind of proof; you know there there are there are we have seen proof by contradiction, we have seen proof by construction, this kind of proof is called proof by inductions. So I have to proof that P k plus one is greater than or equal to P k for all K. So I am first showing the the base case, that is at least for K is equal to zero, it holds.

Then I am showing that, if k if for K is equal to 0 it holds then for K is equal to 1 it holds, because P2 is equal to psi P1. See the previous equation was left hand side was P k, right hand side was P k minus one; so we had P k is equal to psi P k minus one. That is the.. I mean covariance of that equation. So we have P2 is equal to psi P1, but since P1 is greater than P2; therefore psi P1 must be greater than psi P0, but then what is psi P0? psi P0 is again P1, if P2 is psi P1 then P1 is psi P0, correct. So P2 is greater than P1.

So if P0 is greater than if P1 is greater than P0 then P2 is greater than P1, it follows. So obviously now, if P two is greater than P one then P three will be greater than P two, in this way we can go on and then we can proves in this way continuing we can say that P k plus one is greater than P. So now we have established both of them, that it is bounded but it is increasing, correct.

So actually if okay now if now we can now just just to know that, it will it will actually converge element wise. You can think at you can think about it like this, imagine this number; some for some vector y. Take an arbitrary vector y then y transpose P y will be a number, it will be a scalar number. So if for since P k is continuously increasing, so y transpose for a for a for a for a for a given y; y transpose p y is also increasing of course that is the definition, how do we say that P k is increasing, if for a given vector y, y transpose P y keeps on increasing then only we will say that P k is increasing, okay.

So so this is this is an increasing sequence of number. So now we have come from matrix to number. So we have a increasing sequence of numbers which is bounded on the top because y transpose w y is constant. So so therefore this this this number sequence must converge, there is no other way, now now I can now this should happen for any y. So so now I can choose y to y wish. So so so first of all I choose y is it y as an as an unit vector as one zero zero zero zero. If I choose y as one only first element one and all others are zero then this y transpose p k y will be what? It will be the first element p one one of the matrix p, right.

So now y transpose p y convergence means what? P one one convergence. So in this way we can prove that, first if I choose y as the first element one all other zero, then I choose second element one all others zero, third element. So I can first prove that all the diagonal elements will converge, then I can choose y as first element one, second element one, then I will get p one one and p one two; now p one one has already converged, so therefore p one two will also converge. In this way, we can prove that the whole matrix of y, whole matrix p will converge element wise.

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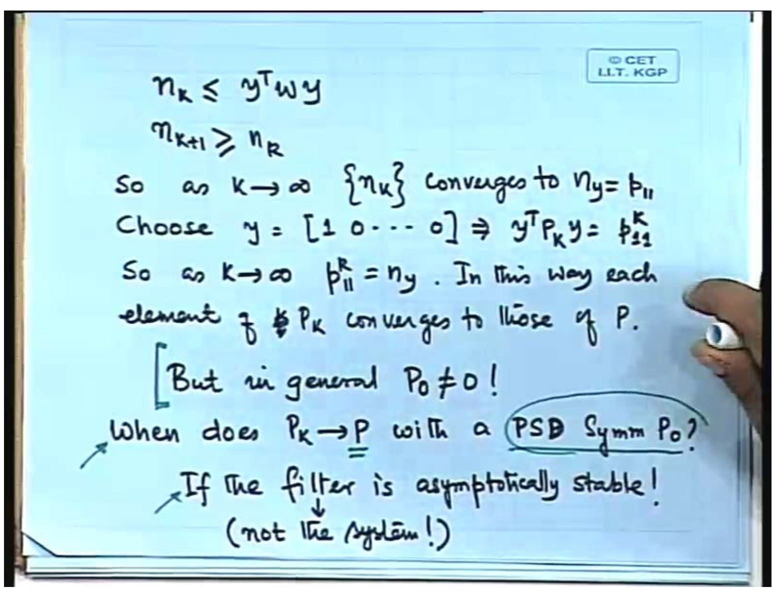

Nice argument, see how they are in in steps here and there there gradually proving one thing or the other; that is rather nice but what is the catch now? We are we are not yet reached our destination the catch is here; that we assume that p zero equal to zero, p zero is not zero, p zero is expectation x zero minus x zero, y as x zero, y where x zero y is x zero, y transpose how can it be zero? So just we are so this is not enough we have to prove more. So the question is when does P k tends to P, with a positive symmetric non-zero P? Zero already it is semi definite then I said PSD I mean positive semi definite; when I say PSDS I mean positive semi definite symmetry. So well so this is the question that is of interest to me, because P zero is never going to be zero. So the answer is that, if the filter is asymptotically stable then it happens. This you can you can probably understand that; P zero is zero means, P zero is like an initial condition, right.

So for for zero initial condition, we have proved that the matrix will converge; because the because the see the elements are we have we have considered to be I mean, A B C, we have considered time invariant, input properties v etcetera have we have considered time invariant. But when does the when does the effect of initial condition die down in a system, eventually; if the system is asymptotically stable, that me know that the initial condition responds of any linear system dies down to zero; if it is if if the Eigen values of the system are strictly within the unit circle in the in in the discrete case then only it will die down know. So so here also it will turn turn out that, if the filter is asymptotically stable then P Q will tend to P, so how do you look at that?

This this is again a lot of equal amount of algebra but in but interestingly it turns out that, various reachability conditions etcetera come out. Actually you know we are trying to say that, for arbitrary A B C, we are not requiring any condition of stability of asymptotic stability of the system; we are require a asymptotic stability of the filter, so the system may be unstable. So if and you can understand I mean, if you I am just trying to relate these with you know various kinds of results, look at this;

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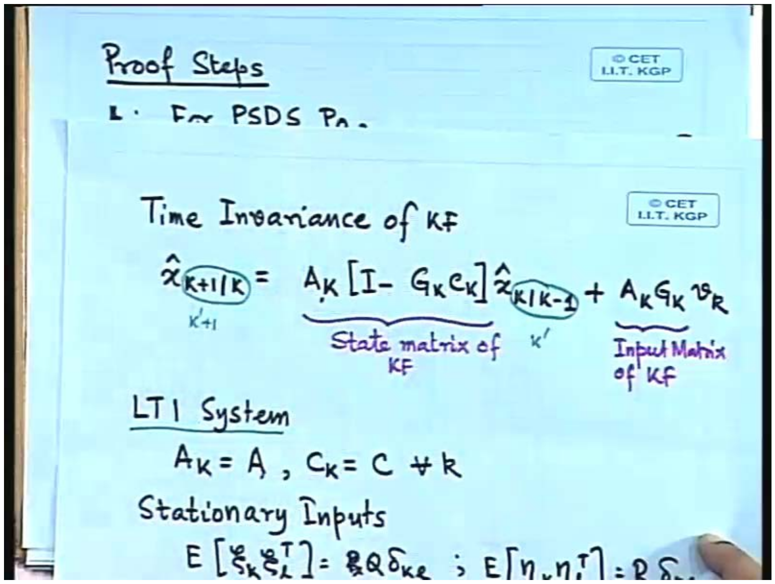

this looks like a somewhat looks like a pole placement result know, A is the original matrix minus some A k, G k, C k pole pet pole placement clause matrix is what ? A minus K c, if you have if you have output feedback here, we are having output feedback. So this closed this is this is somewhat like a closed loop system of I means arising out out of some pole placement in estimator and we are require that the estimator be stable; irrespective of the system being stable or unstable. So when can you stabilize by pole placement, any system by state feedback?

When the system is controllable, when the system is controllable you can always find the gain k which will give you the stable close loop system. So obvious now we are requiring that that that whatever is the whatever is the whatever is the original system; the filter should be stable. So obviously it should it it seems natural that, it should translate to the controllability of the the A gamma matrix; in this case B is gamma, is it not? So so so this result is it is not intuitively unnatural. So it will turn out so first of all

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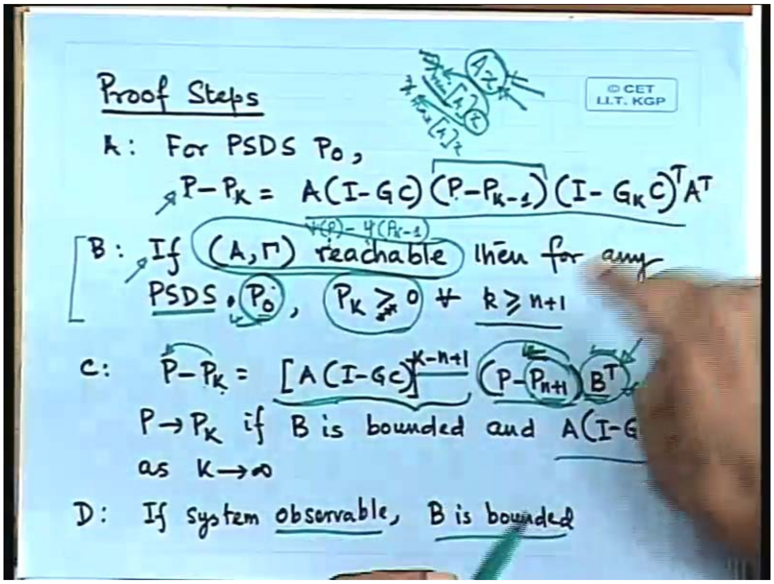

they will prove some technical result that, if it converges then P minus P K. This is you know just you know put put P is equal that lengthier solution, P minus P K will be equal to psi P is they will write that P minus P K is equal to psi P minus psi P K minus 1. And then they will put you know obviously P is equal to psi P; that is the steady state solution you have already said that, if if a steady state solution converge exists as it will we have just now proved it then then then that must satisfy P is equal to psi P.

And normal matrix Riccati equation satisfies P K equal to P K is equal to psi P K minus 1. So obviously P minus P K is equal to this. See if you just put, psi and psi P and psi P K minus 1; those those function subtract and take common, do some manipulation you will get this form. So now now now now you see that, gradually you will find that observability conditions we have we have we have we have already got in our analysis in our steps dual argument. Now we will start getting controllability conditions.

The moment we will come require that the that the filter converges for non-zero initial conditions; we will start getting controllability conditions because of this reason that I that is discussed. Then we will prove that, if A gamma is reachable reachable means what? That the that the actually controllable and reachable you must be knowing, that it is that is one say some technical difference; the mean I it is too origin from origin KS, I mean if if if the A matrix is zero then also it may be controllable. I mean if if I mean that just imagine that, that if a if if A and B are both zero then in one step you can reach the origin form.

So so reaching the origin is not a problem, reaching the origin does not require that the matrix B A B A the point minus one, B that matrix has rank n. Reaching them origin can be given for given for degenerate cases of the systems. It is actually reaching an arbitrary point formed origin which is more important or more practical, so such things I can reachable. So basically when I say this reachable I mean that that that that corresponding matrix that is gamma, A gamma, A square gamma, A to the power n minus one gamma, that that matrix we have should be full, right; as you all as we standard result I mean the the controllability matrix. Then it can be shown that for all P K will be P K will be continuously grater, not positive semi definite P K will be positive definite; for K greater than n plus1.

Wherever wherever P0 you the whole sequence, if you start with a positive positive semi definite symmetric P zero; then after n plus 1 iteration n is constant n is the order of the system. All other P K will be greater than zero. This is this is this is the intermediate result not of much significance only thing is that, only thing why I wrote it; it actually a in a intermediate manipulation result but why I wrote it is is is to just show you that the reach ability condition is coming, okay. Then you can show that this by by further manipulation; this is P minus P K is equal to this into P minus P K minus 1.

So now you again say that, P minus P K minus 1; that is this one you now expand, you can expand this. You can continuously go on expanding this and then finally come to P minus P n plus 1. So every time on the left hand side you will get another a into i minus G C. So here you will get this term, can you can you follow this? If P minus P K is this then again P minus P K minus 1 will be another A A into I minus G C. Here i minus G k minus 1 C and in between P minus k, P K minus 2. In this way you can go on expanding. So this is on the left hand side, this is obviously bounded because P is a constant matrix P n plus 1 is again some constant matrix, and now this p n plus 1 will actually depend on P0 obviously, it will depend on P0. And this B, I have basically by B it is it is it is a long matrix it is i minus G k minus one C into i minus G k minus two C into n and so on. So this is just a symbol.

So what do what do I want? I want to know when P k converges to P even if there is a non-zero P. Now I am working with non-zero, p zero, okay. So I want to know when will it happen, it will happen if this goes to zero. See if this is going to zero, this is bounded; if I can prove that this is also bounded for arbitrary K, let K tend to infinity. So K tend tend to infinity means this this this will be something like x x I mean, x to the power k. See K is on the top and this is like this is like a it a number of things, that is I minus G k minus 1; so it is also the product of infinite number of terms. So if this has to go to zero then this even if it is a product of infinite number of terms, it should be finite.

If this is finite, this is finite and this goes to zero then this goes to one that is true; two finite numbers just treat them as numbers to understand, right. So now it can be shown that, if system is observable then then then this is bounded, then that will will step by step prove this results. And now this is obviously bounded, because is this is it is at a fixed point, it is at n plus 1, n is fixed. So you have started from p zero, so within p n plus one you cannot grow up, okay. So so this is a constant matrix, this is, so this is bounded definitely. So that means that this will be zero, this is bounded this is bounded. So this will be zero, only if this goes to zero; when will this goes to zero? When will a matrix A to the power K go to zero? When it is Eigen values are less than one, it will contract, right.

Any matrix A, if if you take any vector Ax, this vector Ax is going to be bounded by lambda min of A into x and lambda max of A into x. So the length of this vector is always going to be greater than lambda min A; if you multiply lambda min is a number if you multiply the original vector by lambda min, if you multiply the original vector by lambda max, you will get two vectors of two different lengths. And this vector is is is always going to be somewhere in between. So the length of the vector by by linear transformation is always bounded between the minimum and the maximum Eigen value. So if this has to contract, gradually shrink to zero, I am saying that as K goes; it will slowly shrink to zero and p will converge to P K.

So then obviously it should have less than unit to Eigen values which means that, the system should be asymptotically stable, filter should be asymptotically stable not system; this is the state matrix of the filter. So now we have the now so you can now prove also now, how do you know that this has? So so the so P will converge to P k, if this Eigen values less than one. We do not yet know whether it is less than one, but if it is less than one then even for non-zero P, zero P k will converge to k and then finally you can prove that; if A C and gamma are controllable and observable then this is less than one.

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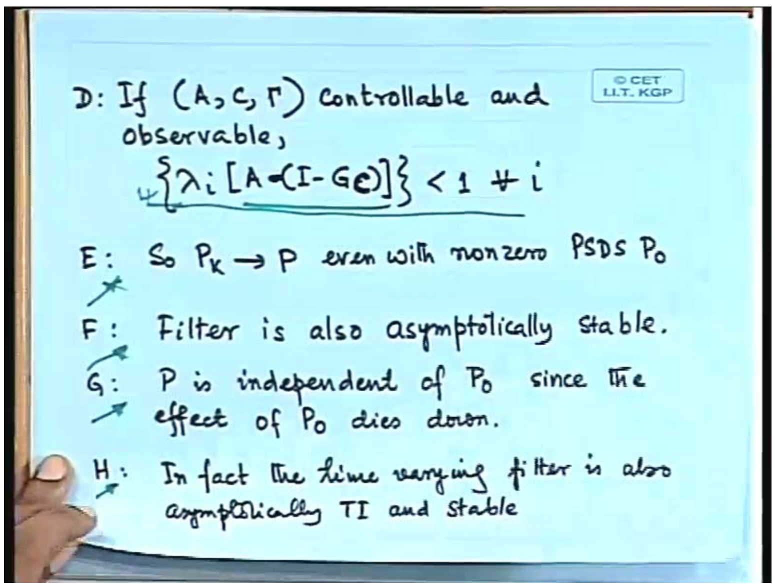

That is like a pole place, that means the the Kalman filter by its optimizing algorithm, always finds the state feedback gain k. See the environment is such that, there will exist a state feedback gain K, since it is controllable and observable. And then the Kalman filter by its algorithm will always compute a gain k such that the closed loop system will be asymptotically such that the filter will be asymptotically stable; I mean if you see that the filter is like a like a closed loop like like something you obtain by state feedback from the close loop system, from the open loop system.

So finally we have arrived that P k tends to P even with non-zero P S D filter is also asymptotically stable, obviously and naturally P is independent of P0; because if the filter is stable then the effect of P0 will die down, asymptotic stability. So whatever initial state error, P zero you start with eventually P will converge P k will converge to the same P, right. Now question is that, so after it converges what does it show? Shows that after it converges it acts like a stable filter, see we have proved the stability of the converged filter, okay. So in fact the in fact the time varying filter is you can also prove with with with lot of manipulations; I do not know how exactly, I have not read that proof then. The time varying filter is is also asymptotically time invariant and stable then you can do the analysis actually for the time varying filter for stability, that will be of much more complicated.

So I think we have five minutes time and this this theses two topics are very simple.

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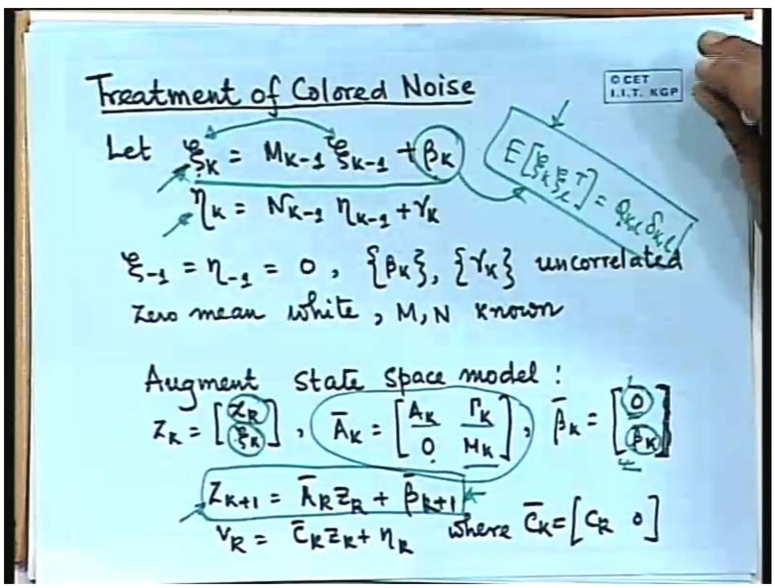

So we will you have to just understand the concept. So now you want to see that; if you have colored noise, what do you mean by colored noise? Colored noise means that expectation of xi K, xi l transpose is equal to Qkl, delta kl; this assumption that we made that the process noise sequence itself is uncorrelated with its own samples, that assumption is I am not that I am not assuming any more.

So I am assuming now xi K is rather generated in this fashion. So obviously xi K and xi k minus 1 are correlated, because it is because i k is generated from the i k minus 1; but now I am assuming that this is a sequence which satisfies this property. This is an uncorrelated sequence which which I am feeding through a filter to xi; so the process noise sequence is correlated but it is modeled as the output of a linear filter, I mean linear linear filter whose input is a white sequence. That is the standard way of modeling correlated sequences because we have seen in a previous thing that, if you have any correlation pattern or if you have any prospectual density; you can always does do spectral factorization and then assume that, that that has been generated by a white input coming to a filter.

So here also we are assuming that, that the xi K which is correlated with itself auto correlated is is generated from a white sequence through a filter. That is a very standard way of modulating correlated sequence, and eta K is similarly being generated like this. So now what is the optimal filter, because our our now our now our assumption is gone. So then how do you make make an optimal filter minimum variance? So so so that is I mean the approach is very simple, the approach says that just just model these also as part of the system. So now your states vector will become previous state vector plus, xi k. That is as if xi K is also a state, we assume like that; you augment the state vector. So now you write the write the update equations for X k and xi K both.

So the update equation for for x K is the old one, A K gamma K. And for xi K is this one so M K. So now this becomes your state matrix, new corresponding to this state the state matrix and this will become your input matrix. So now you have defined a an augmented system; first one containing the original system then the continuing the noise dynamics, that also you are trying to estimate. So now you are over all augmented state equation is this, correct. And interestingly this satisfies Kalman filter assumptions, because this has, what does it have? It has zero B K, it is it is the new new process noise sequence. In this so the vector is still so the vector is still zero min because, this is zero min and this is zero.

What is its variance? For for this components components variance is zero, for this components, it is still white. So this is also a white sequence, so this obeys the original Kalman filtering box, right. Only the output equation is a bit of a problem. So and and and you can easily write the output equation again in this form.

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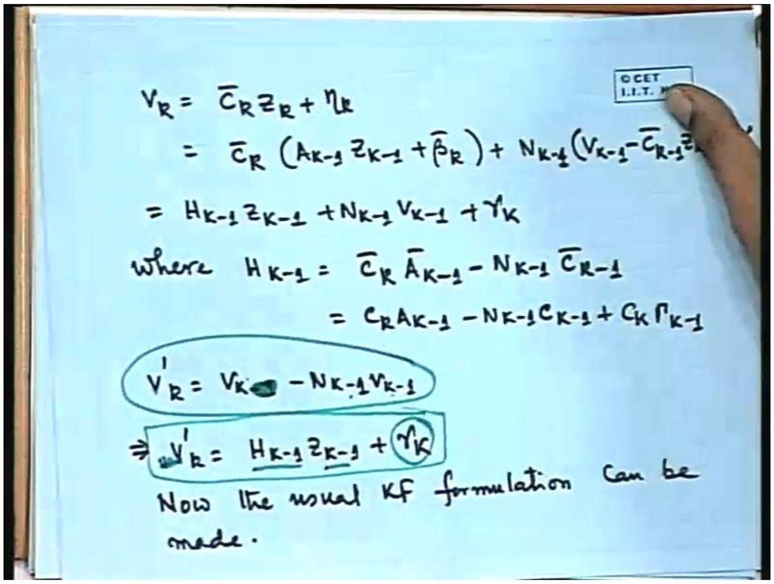

So finally you can get an out, if you can you have to define a new output which is V k minus 1 minus N k minus 1 into V k minus 1. So now two two output but finally you will get output equation, this form which is again our old form because here this is our output; equivalent output this can be computed, this is V k. So at v dash k, V k is known, so therefore this can be computed. So it is an equivalent measurement. So so now your new measurement equation is this with a new matrix state and gamma K, which is again a white sequence. So with this state equation

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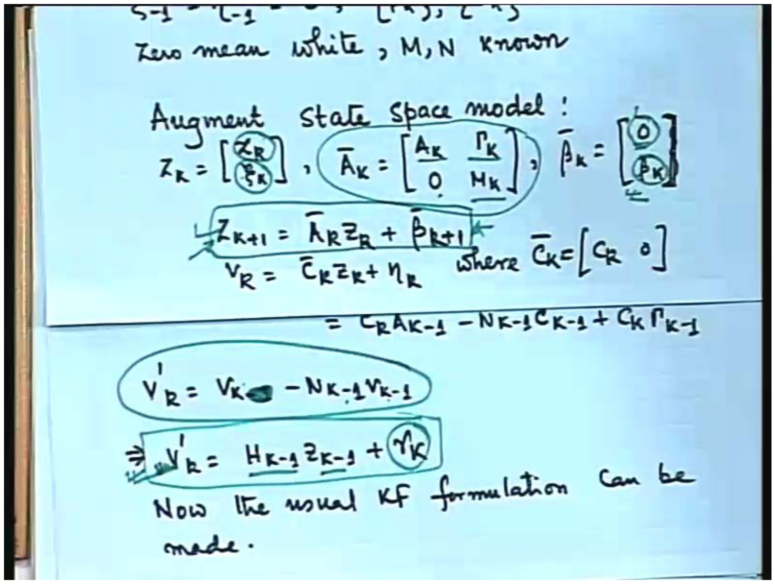

and this measurement equation these two are in the are in the normal Kalman filter form. So you can estimate the Kalman filter for these two, right. That will give you the optimal filter from, there from there if you take the top path, you will get the state. So the so the trick is simple, trick is to define an augmented system, model the noise dynamics also within the system dynamics, right.

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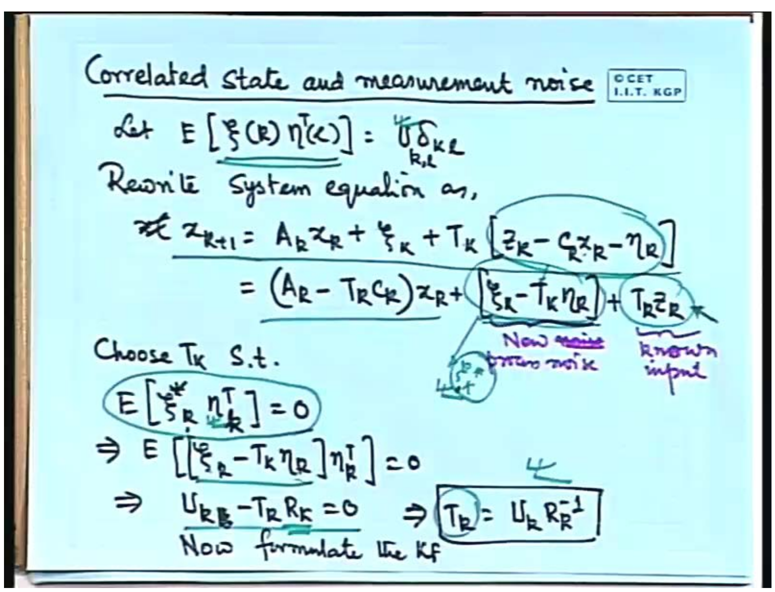

And lastly if what happens, if colored if correlated state and measurement noise. So I am assuming in this case I am as this from this correlation can occur with with some delays also, but I am assuming that the that is this is the most common case that; xi K and eta l are now correlated. Previously what was my assumption? That xi K and eta l is equal to zero for all K and l; even if this is k this is k because these are difference this is zero that was my assumption. Now I am assuming that, if this this is k and this k then it is non-zero this is u k l delta, k l. So at some instant same instant xi k and eta k are correlated, okay.

So then again I have rewrite the system equations, I have to do some clever transformations and such that; the transform system has in all all those nice property the previous Kalman filter maintained. So what I am doing is I am writing this transform system, I am writing a new state equation, why this is correct? Because this equation is this is identically zero, see z k equal to c k x k plus eta k, z k means z k actually; I have written it from a different book. So my notation has got changed, this is z k. z k is equal c k, x k plus eta k, so that means this term is identically zero; so I can always add it, okay. If I then I then I am rearranging in terms, so this t k, z k, x k, I am put it here; I have this term and I have this term. Now you see interestingly; this term is known because z k is a measurement, so it is like a known input case.

Previously even for Kalman filter we we can have known input and we can have unknown input, the unknown input we had to assume to be random; known input is fine known input we will we will again partition, this is a state. So this is like a known input, this is like a like like like the new noise process noise. Now the new process noise is xi k minus T k eta, this is my new process noise in this transform system. Now so I have to choose now, I have to ensure nice properties of these by choosing this T k, appropriately. So so now what is my what is my condition? That choose T k such that, this must be equal to zero; that is the expectation of this is zero, that is that is obvious but expectation of this is this I am calling as xi star k oaky, so this is the new process noise.

Now I want that T k should be such, that the new process noise and the measurement noise should be now be uncorrelated. I started with the assumption that process noise and measurement noise are correlated, I am now defined the new process noise and I must define this process noise in such a manner; that this becomes uncorrelated. So I will now put what is this equal to this and then I will simply solve now; what is xi k, eta k transpose? That is U k and what is eta k, eta k transpose? That is R k. So I want this to be zero.

So I will choose T k is equal to U k, Rk inverse. That will make it zero. See if I choose such a T k and then if I write this this thing, again I have a new state equation in which I have a new state matrix, I have a I have a new noise sequence; which is uncorrelated with the measurement noise sequence and I have a known input, known input I know how to handle. So I can now write for this system for this system and the and the and the old measurement equation C k, X k, I can write the Kalman filter.

So that is all today, we have seen an important property of time invariants and we have seen two of the you know common cases, because these assumptions have to be sometimes the assumption appear restrictive; but in in in it turns out that in many cases we indeed have by you know by by manipulation, by some augmentation, by some transformation we can define an equivalent problem on which we can define a Kalman filter, thank you very much.