Estimation of Signals and Systems
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Lecture - 13
Z – Transfrom Revisited
Eigen vector/ Eigen values Revisited

We are again back to basics because for some of the later chapters, again we need some background and I thought that it is better to cover; we also had some doubt about Z-transforms.

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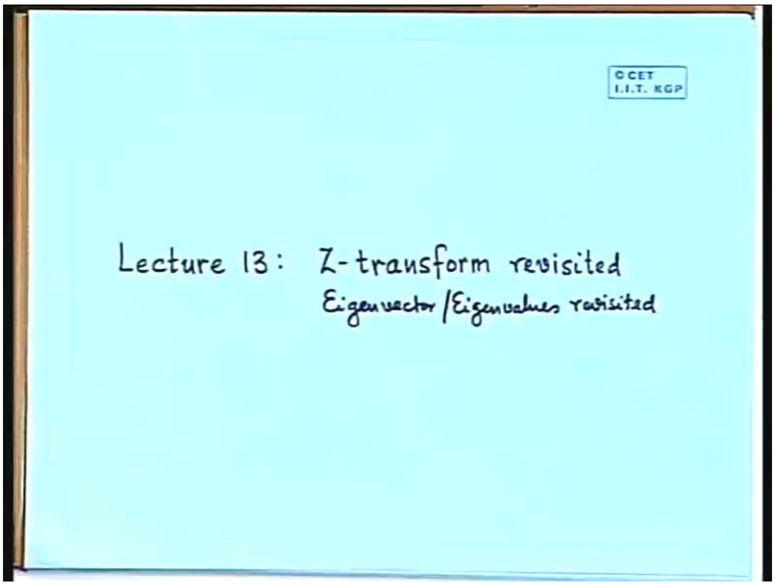

So I thought that we should cover that. So today we will look at some properties of the Z-transform, more on those properties which we usually do not see. We see the tables and other things and take a brief look at what are Eigen vectors and Eigen values; because that is going to be used very soon in defining quantity which is very common is called the innovations. So before we define start discussing the concept of innovations, it is important to know what are Eigen vectors and Eigen values of a matrix. So we start with the discrete Fourier transform.

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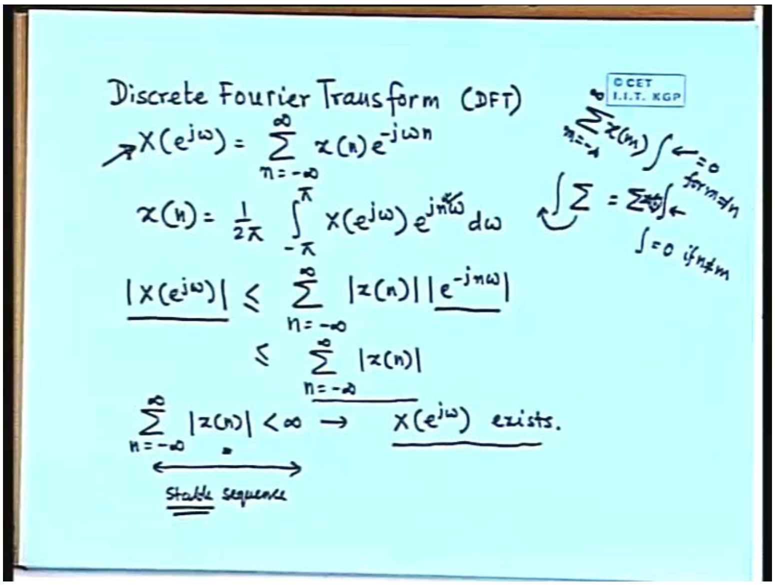

That is defined as we know, by this infinite series, right. And the inverse discrete Fourier transform is given by this integral; it is an integral, because the discrete Fourier transform is a continuous function of omega. This is discrete, but this is continuous function of omega and this is very easy easy to prove, if you just substitute this expression here. Take the summation out that is somewhat simple prove, you just have to you know substitute this this expression here. So then it will be it will look like an integral, inside a summation. It will look like this, if you substitute and you can take the summation out, then it will look like a summation and inside an integral. Are you able to follow? So here you will get x n; so you have to only evaluate that for, this integral is is is going to be, you can easily see, that this integral this integral will be equal to zero; if n is not equal to n, that is this side you have x n.

Here you have, this integral will look like sigma x m, m is equal to minus infinity to plus infinity, some integral, if you if you exchange the summation. So I am not doing the full thing; then it it will turn out, here you have n and when you replace this, you have to put m here, because otherwise they will get, they are is not the same n, you are evaluating this for a particular n. So n is fixed while m is varying. So m, you have to give a different index that is why I have given n. It will turn out that, this integral is equal to zero for m, not equal to n. So all that x n terms will go, only the x n term will remain and that will will be equal to one. Very simple to prove; just know ordinary integral knowledge will be sufficient.

So now there is always a question of when this is defined; remember whenever where we are defining anything by an infinite series, this we must remember, we must if which we are generally not conscious but, we should be conscious about when this is defined. That is when this sum of the series converges; okay, if it does not converge then, this has no meaning. So turn now, we so we have to find out, so we are trying to find that is why we try to find conditions under which this value, does not tend to infinity for any omega, okay.

So if you if you want to find such conditions; so one condition is that, if you if you take this, this is naturally less than equal to this, how I?, because magnitude of this is magnitude of this. Now magnitude of A B is always less than magnitude of A in than magnitude of B, right. So therefore I can write this this is less than or equal to this. What is the magnitude of this? It is one because, it is the e to the power j theta. So this is less than or equal to this, so if this sum is less than infinity, this sum will also be less than infinity. This says that, so this is called the known as the absolute summability condition. That is if you have a sequence x n and if you find that the sum of the magnitudes of x n, x n can go positive or negative of course; but if the sum of the magnitudes is less than infinity, infinite sum of the sequence, then this exists, such sequence is are sometimes called stable sequences.

If this condition is satisfied, this is called the absolute summability condition. This shows shows that then this exists, so and why is it called stable. It is why why why this terms stable has been coined, because of a particular result from stability and that results says that, suppose you have a filter; this is the convolution integral n now suppose we are talking about bounded input, bounded output stability.

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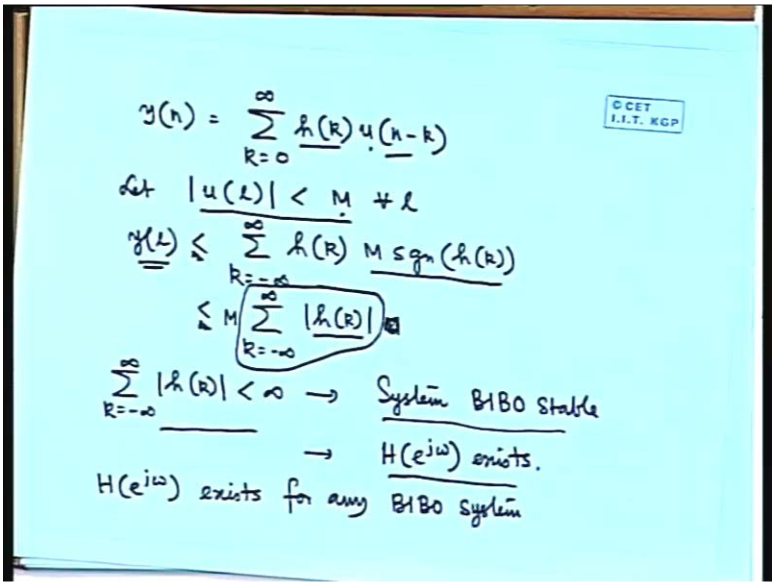

That is if the input is bounded, what is the condition that the output will also be bounded? So let let the input be bounded, so for all l magnitude of u l must be less than m, m is some fixed number however large.

So we are considering a bounded input. If m is bounded, u is bounded then; what is the how y is bounded? So you can put the maximum value of u, here m every time and not only that; you can just to make the sum, we are trying to see when the sum reaches the maximum under this condition. So it will be maximum, when all magnitudes will be m and not only that; when when when this will be minus this will be minus, when this will be plus this will be plus. Then then all the terms will be plus and it will add to the maximum, otherwise plus minus may cancel. So if so I mean just putting m will not do, just putting u is equal to m will not do. So we have to match the sign with this sign of h k such that, every time all the terms in the summation will be positive and it will add up to the maximum.

So if we do that so. That is why, I have substituted m s gn, h k, s gn means; this signum function which is plus one, if h k is positive and minus one, if h k is negative. So then all this terms will so, that will be equal to this, because now plus one minus one multiplied by this will be magnitude of h k, this signum function when you debit here. So now now now m is bounded, so m is a constant. So it will come out of the summation actually. So if since m is bounded, y will be bounded if this is bounded, naturally. So the condition there under all, so if this is bounded this, this system is bounded input, bounded output stable, right. So the the

the this called the absolute summability condition, absolute summability; because you are taking absolute, you are taking magnitude and then you are summing and that sum should converge to a bounded value, though it is an infinite sum.

So if if this happens then, that is why such sequences are called stable sequences. So stable sequence means; if you using that sequence have impulse response coefficients, if we implement a filter, then that filter is going to represent a bounded input bounded output linear system, right. That is why it is called stable. So if this is less than, infinity then system is bounded input, bounded output stable and, then H e to the power j omega exists. See otherwise H e to the power j omega does not exist. We are sometimes we are you know; we deal with we first learn in our courses, we first learn Laplace transform. Now Laplace transform existence condition and Fourier transform existence condition is not same. Laplace transform exist for many values of alpha, that is an e to the power minus alpha t there, which is not here, right.

So so so the existence condition, e to the power j omega is more stringent; that is in many cases, e to the power j omega will not exist, but the Laplace transform will exist, Z transform will exist as, we shall see, okay. So I mean just putting for for all functions we cannot talk about a Fourier transform, unless it converges. So now to to to generalise that question, let us talk about the Z-transform.

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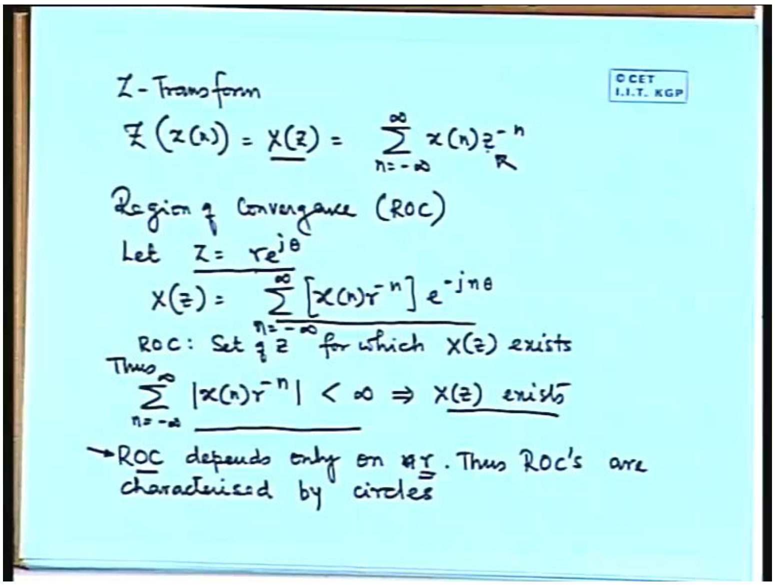

So now we have taken a, this is a more generalised transform; in the sense that, Z is any complex number. See what is the transform? Transform is that basically; it is like a, it is like a reflected domain, just like you are seeing something through a mirror. So rather than studying the function x t; which may be difficult to study, you are you are transforming x t to another domain called Z and you are studying the behaviour of x t, through the behaviour of x Z, because it is mathematically more convenient, nothing else. I mean typically, because various kinds of time domain properties will become an algebraic properties. So time shifting property will be Z times something, I mean; all your differential equation kind of operations will all become an algebraic operation, under suitable conditions.

So you can very easily deal with it. That is why we take the function, x t to the transform domain and then study it there, okay. This is the sole purpose; x Z as such does not have any much of physical significance. Sometimes it has with respect to energy and all but as such the real quantity is the function x t or in this case x n, okay. So here we are trying to study it, defining it in terms of an arbitrary complex variable Z. See the see in the previous case, we were studying it by through a function omega and e to the power j omega n, okay. So now we are not doing that, now we are you we are further generalised, it so obviously here also; there will be a there will be a region of convergence, so for some value of Z this will converge, and for some values of Z it will not converge.

So we must remember this this this region of convergence, suppose we just evaluate when it will converge? Let let us choose huh Z is equal to r, e to the power Z, r e to the power j theta, that is we are I am trying to represent a complex number, in the polar notation, okay. So then now, if I put this here I will get this. See now that this looks like a Fourier transform, right because, this e to the power j n o j n theta, which was previously like j n omega, so this says it is the same thing.

So now when will this converge? We now we already know the full, we already know the Fourier transform condition. So so if this is less than infinity, then then this will converge, correct. So so so this defines the region of convergence for X z, there are there are two two interesting things to note; first of all that the region of convergence depends only on r. So so so whether the radius is going to be smaller or bigger, so all regions of convergence are essentially characterised by circles in the Z plane. So it is going to be either less than r or; it is going to be greater than r, but it is always going to be on a circle, because it is characterised only by r, right. So so so that is an is an interesting thing to note.

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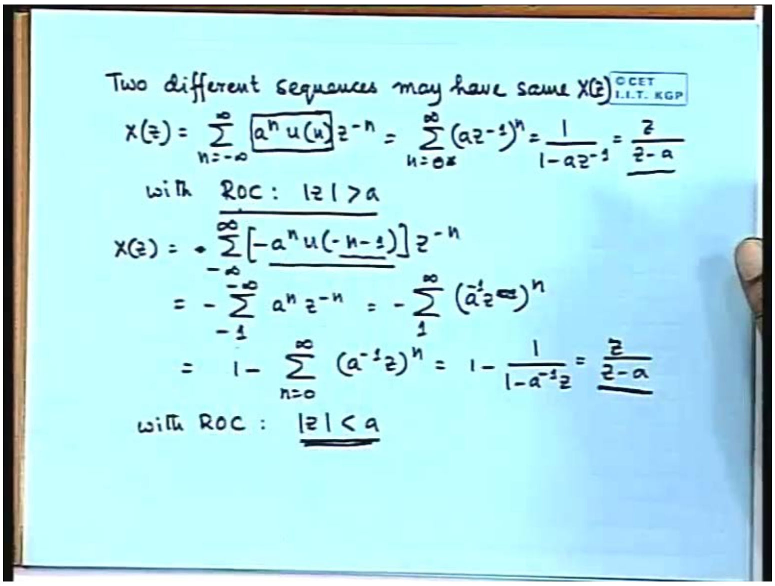

Now this we had seen previously also; just re-mentioning it again that, two difference sequences may have the same X z , so why so this is that this is the old example I think which we did. So if this is the sequence x n, un is the unit step function; then I have this as a Z transform, simple just just by putting minus infinity to plus infinity, becomes zero to infinity because un is zero, for all n less than zero. So then you then then this will sum, if the magnitude of this is less than one. So and it will sum to this, so you have z by z minus, a as a Z transform and the region of convergence becomes these, this is very important. So this function has, this zth transform, over this region of convergence, that is the complete statement.

This function has this Z transform is not a complete statement to make; I mean the function cannot be uniquely obtained, unless you specify also the region of convergence, which we do not do usually. So on the other hand, this function, which is a totally different function, also has the same Z transform. This is just just by mechanically working, see minus n, minus one is of u this will also exist for for positive arguments of this. So this will this function will be non-zero or one only for these arguments, minus one to minus infinity. So now if you just do elementary transformations, you will get that; that that is also equal to z by z minus a, but now the region of convergence is modulus z less than a. So these two are totally different functions; give the same Z transform but, with different regions of convergence. So this is a fact which we should remember.

Yes? Student< sir how do we assess what is the region of converge>
[Conversation between Student and Professor – Not audible (16:22)]
comes from by by, you have to you have to use different properties. For example, in this case we are trying to use the property, that that one plus x plus. We are trying to take the take the sequence, trying to case it in a particular form and then using our result that that that that one plus x, plus x square up to infinity is sums to one by one minus x, provided modulus x less than one, we are using that property.

Student<Sir but in these two cases; the formula is the it should not it be giving the same result?>

No no no obviously the formula’s same so what? But formula will always be same, but these two are different sequences. See this sequence with n, this is a to the power n and this un, so so for n greater than zero, it will be a one, a a square, like that. This is going to be a minus a to the power n first of all. So it is a totally different function; but it so happens that they yield the same when they when over the regions where they sum, you get the same sum. But this z values have to be different for in both cases, that is all right.

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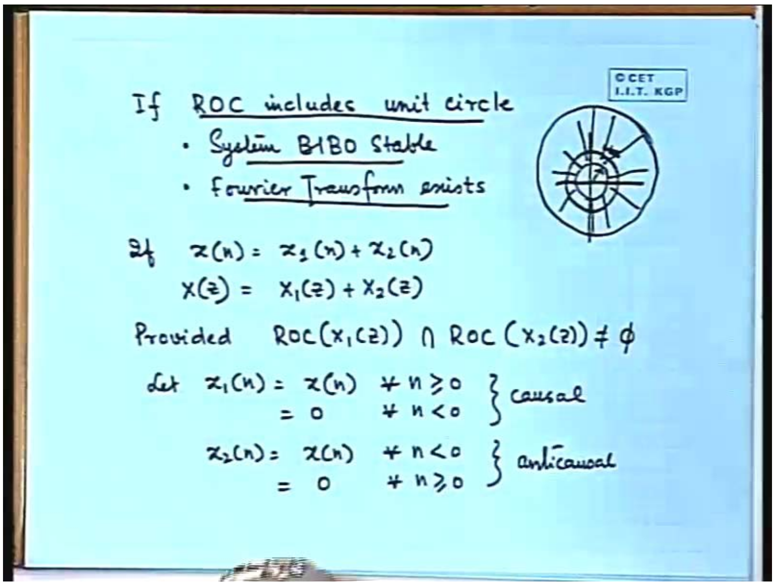

Now again we know that, if the ROC or the region of convergence includes the.. or rather if the system is bound input, bound output(BIBO) stable, now we are putting the other result. Now when does the when does the Fourier transform exist? Fourier transform exist means, the what is the Fourier transform? Fourier transform is is nothing but the Z transforms, evaluated over the unit circle because; over the unit circle z is equal to e to the power minus g omega, right. So in the Fourier transform you put z into z is equal to e to the power minus j omega, so you are evaluating the the the Z transform; on the unit circle.

Now and if the Fourier transform exists, then that sum converges; which means that, the unit circle must be inside the region of convergence of the Z transform, otherwise it will not converge on the unit circle. So if thus system is bound input, bound output stable, then the Fourier transform exists and the region of convergence of the Z transform includes the unit circle. So it will be something like this; the probably the region of convergence will be something like this, the that is the outside of this sum circle, it make be like this, it it may be between two circles also that depends on.

So this is the first property which we can get and the second thing is now, here in many cases you know we will get we will get a function in terms of that is; we we will we will get a sequence in terms of two sequences, right. In many cases we can, we can express a sequence as a sum of two sequences. It can happen, now it can it can also happen that x one n has a Z transform, X one z over some region of convergence and x two z, x two n has another Z transform X two z over some other region of convergence; it is possible simply, if you if you add these two up.

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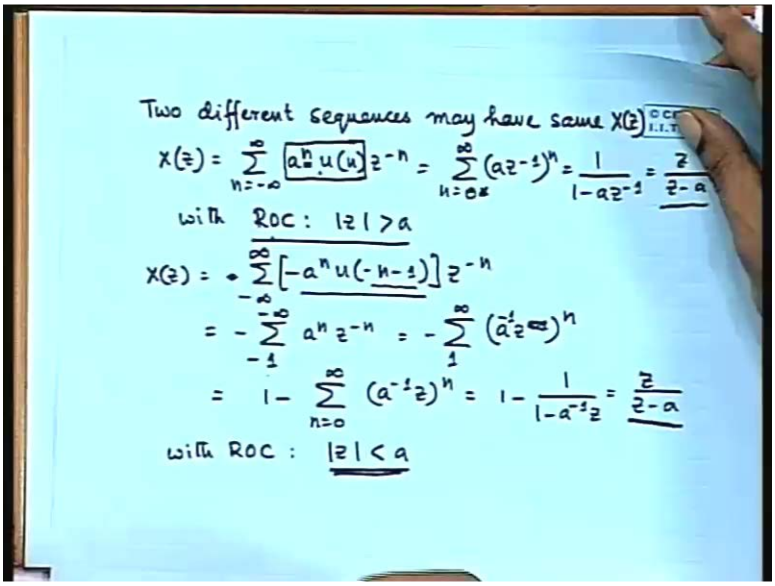

If you define another function which is like this, which is this plus this you can define, right.

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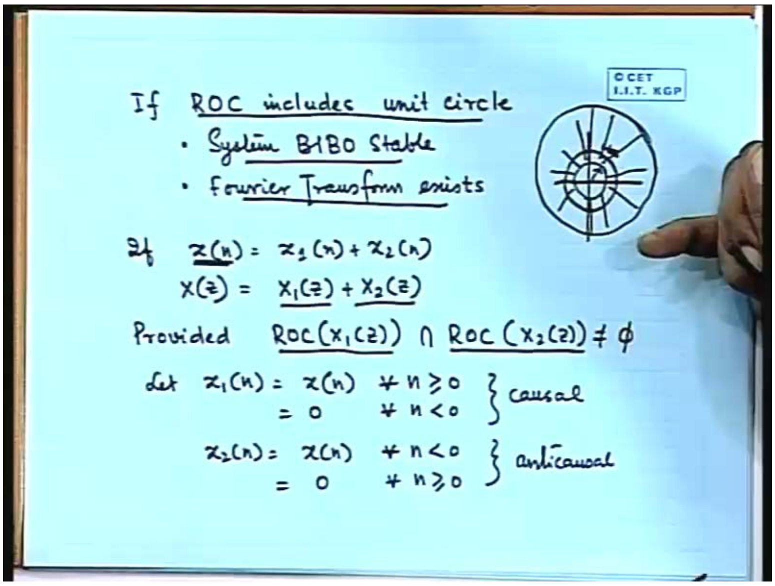

And then you can, now remember that; if the so then what is what is the Z transform of x n, that is the question. So the Z transform x n will be simply this, sum of these, when only what will be its region of convergence? It will be the intersection of the region of convergence of this and the region of convergence of this. Because whenever this will be defined, whenever this has to exist, this has to exist and this has to exist both have to exist. So it must exist on points, where both exist, so therefore it must be on an intersection of the region of convergence, for example in this case.

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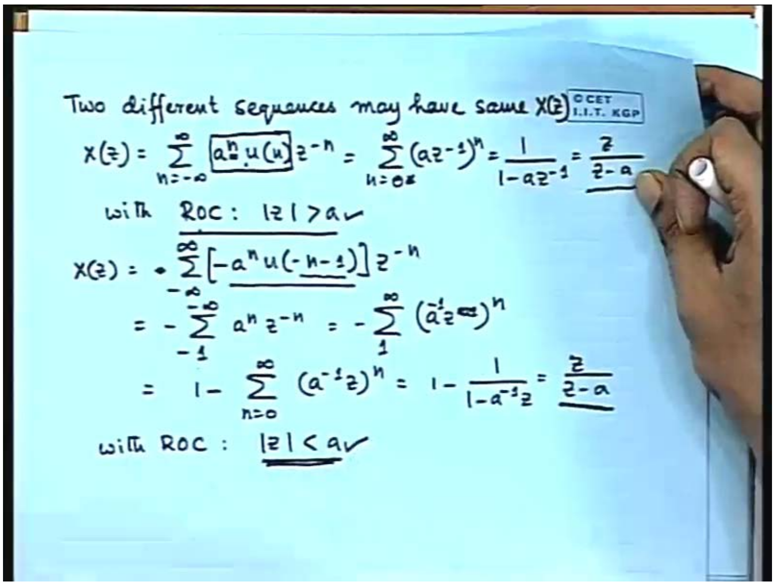

There is no intersection, so if you sum these two sequences, its Z transform does not exist, okay.

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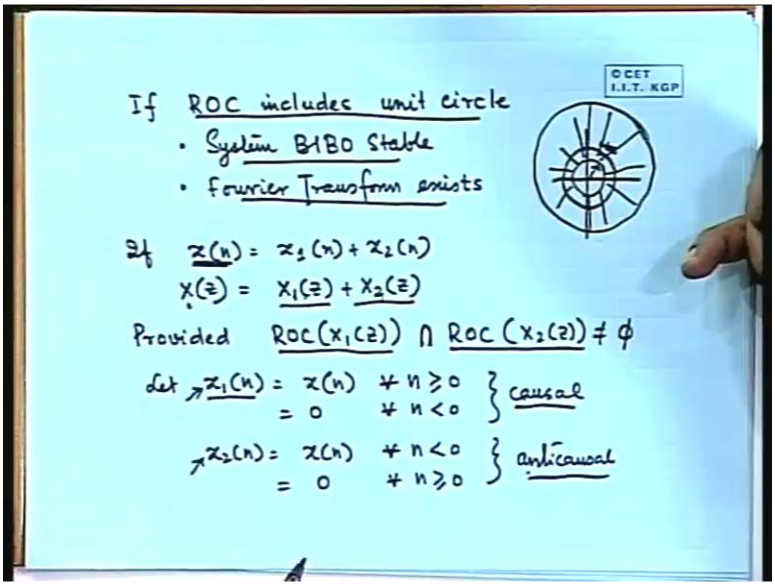

So we have to remember this. Now we have known that, such sequences for example, suppose; x 1 n is x n for all n greater than equal to zero, and zero for n less than zero. And similarly suppose; x 2n is just opposite, that is it it is x n if that is x n means, has some non-zero value, for n less than zero and it is always zero for n greater than equal to zero, this sequences we call causal anti-causal, why? Because of the convolution property, that is if you convolve with this, we will always find that the output is affected only by past values. If we convolve with this, we will we will always find that, the output is is only affected by future values, true.

This we have discussed many time that h n minus k, gets affected by u k and all. So these that is why, so if you they are where called causal and anti-causal, because if you use these sequences as impulse responses, then the in the corresponding filters will be causal or anti-causal. That is why I mean, otherwise there is as such no meaning of a sequence being causal. Causality always comes, when there is a cause and there is a effect, so so there has to be two sequences. So so when we say, when we say sequence is causal; we mean that if we use that sequence as an impulse response of some filter, and that filter will be causal or anti-causal, that is what we mean.

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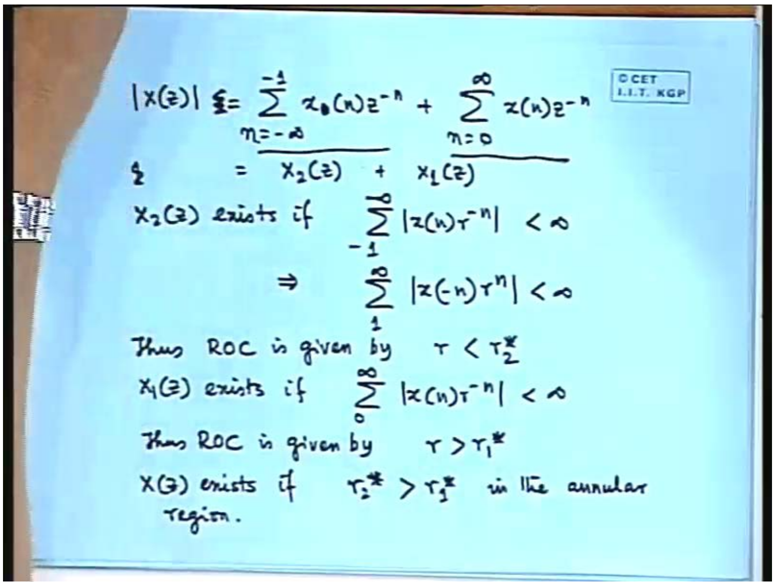

So interestingly now if you take any sequence x n, it can always be divided into this this this whole sum; which is from minus infinity to plus infinity, can be broken up like this, one is from minus infinity to minus one, another is from zero to infinity. I just separated the terms, right. So now if I if if if I if I make a definition of x1 n and x2 n like this. That is all take all the values of x n for n less than zero, and and make into a make into x 2 n; take all the values of n, for n greater than zero and take make into x1 n.

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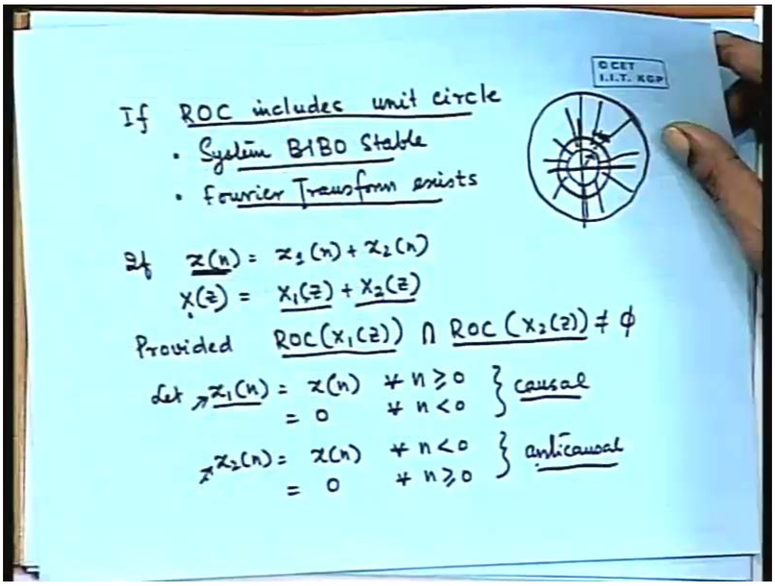

I can always define and they if I add them, I will get the overall function from from minus infinity, to overall total sequence I will get.

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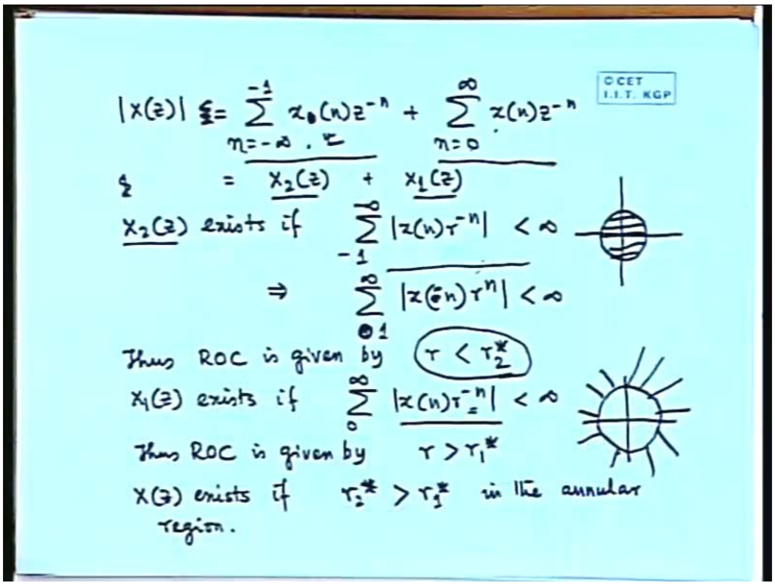

So what will be the X2 z corresponding to x2 n and X1 z corresponding to x1 n? They will be these two for, this will be the those terms. So now, when when does X2 z exists? If this sum is infinity, this is the this is the X2 z, so if this is infinity, this will exist and if this is less than infinity, and if this is less than infinity, this will exist. Same formula we are using, previous previous argument we are using; only thing is that these two may not exist for the same value of r, that that that is what we have to see.

Now if you see this condition, typically what will you know? See n is negative here, which means that, the power of r are actually positive. So which means that; if this has to be less than infinity, r has to be less than a quantity, if r becomes greater than something, then this sum will diverge. This sum generally increases, with increasing values of r because it is positive powers of r. So so so obviously, the the region of convergence is going to be inside some circle, for this set of terms, r has to be less than something, then it will converge that sort of condition we will get.

So the condition for this, will obviously be some radius and the inside of a radius will be the region of convergence, for this to be true. On the other hand for this to be true on the other hand for this to be true; this is not infinity, this is zero to infinity for this to be true obviously r has to be less than some quantity ,this here… one second, one second. X2 z, okay. This this is okay, so this is minus n square okay, for x1. Now so so this is the condition, some there is some r two star and r has to be less than this. Now X1 z will exits, what is the term X1 z? X 1 z is this. So here n is positive and r to the power minus n, that means r has to be greater than a quantity.

So the so the convergence region for X1 z will will typically be, the outside of a circle. So this gives us an idea; that if we have an anti-causal sequence, which is this, then the region of convergence is generally the inside of a circle. If we have a causal sequence, which we usually deal with, the region of convergence is always outside some circle, right. This is this is this is to be remembered, because that is why we we always say that that that, for a stable sequence poles; should be unite inside unit circle, as we shall we will see this result because then the poles are not covered by the region of convergence.

Poles are the where the function goes to infinity, that thus those values of z are called poles. So obviously poles are not included in the region of convergence. So poles and and and for stable thing, the the the region of convergence is the whole infinite plane, excepting the inside of the circle. So that that is why, we say that all poles must be inside, because the function must converge at all other points, right.

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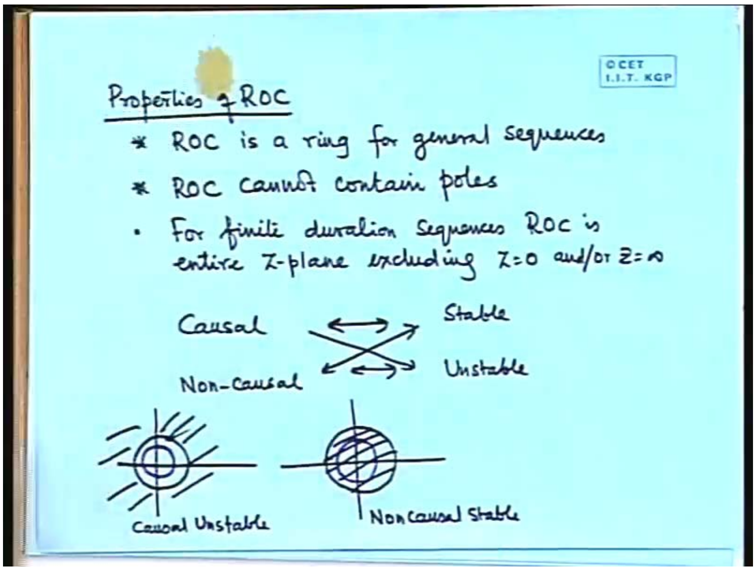

So properties of so in general, what will happen now?

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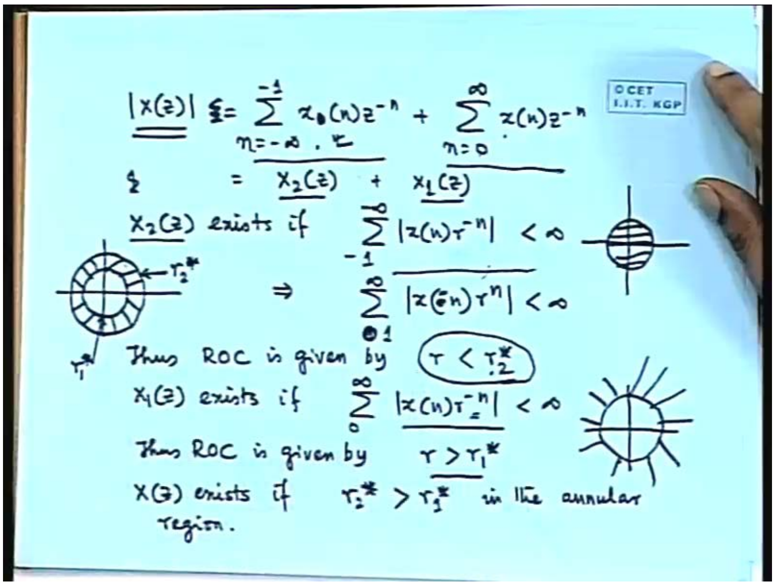

So so you see if if if I if you are take if I take a general sequence, some part of it which is anti-causal, some part of it which is causal. Then if you merge these two, then what will be the condition for x z to exist? That it should exist on the, it will be r less than some r2 star and r greater than some r1 star. So it will be a ring in general. This is r 2 star and this is r 1 star. If that happens, then the Z transform will exist, in this ring by these two conditions. This is r2 star and inside is r1 star, by these two conditions. Then the overall Z transform, which is the sum of these two will exist, in this annular region and if this annular region now, contains the unit circle, then it will be stable and the Fourier transform will exist. If the annular region does not contain the unit circle, then the system will be unstable.

Then then the Z transform does not exist; if r1 is greater than r2, then these Z transform does not exist, because there is no no intersection between the regions of convergence.

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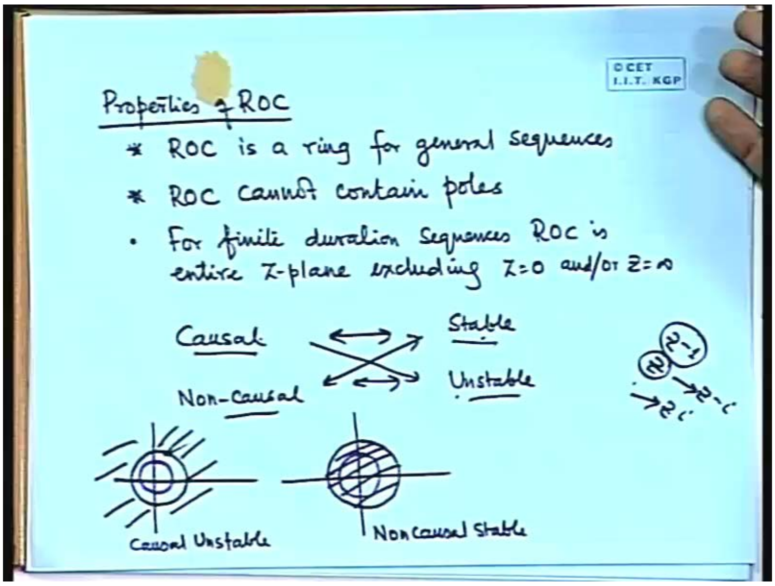

There is no point in z, where both X1 z and X2z will converge, so therefore the sum of x z cannot exist. So the basic properties are that, generally region of convergence is a ring, is an annular region, as we have seen. And and it cannot contain poles of course, because at the region of convergence the value of x z is finite and at poles.. the value of x z is infinite, so it cannot contains poles, region of convergence can never contain poles. Now, interestingly we are we are we in many cases, we deal with finite duration sequences. That is sequences which at zero, then for some between some two values of n, it becomes non zero and again becomes zero. For such sequences, Z transform will always exist, because it is the finite sum, infinite sum will become finite sum. It will always exist, excepting at, may be at zero and infinity, why, because it contains z inverse and z kind of terms. See for positive powers of n, it contains z inverse terms and for negative powers of n, it contains z z square kind of terms.

So if so if these sums... if such terms, that is z to the power i and and z to the power minus i have to be finite, then z cannot be zero here, and z cannot be infinity here. So excepting for those two points; that is z is equal to zero and z is equal to infinity, which you can think of as a infinite radius circle. Excepting those two points at all other points, this sum will be finite simply because, it contains a finite number of terms. So there is no problem, whole plane will be region of convergence, if you have finite duration sequences. Now we must remember that this that, there is no no as such no connection between something between this causal non-causal stable and unstable. That is you can have a causal unstable, you can have a causal stable, non-causal, stable non-causal, unstable, all combinations are possible.

So why we are talking about non-causalities, because as I said that in many signal processing applications; you actually use non-causal filters, so we need to know their properties as well. For example, this this within the, if the region of converge is like this, this is the hatched part; and if this is the unit circle, then it is a causal-unstable filter. Why it is causal, because it is the because the region of convergence is outside of some radius, so it is causal sequence. And it does not contain the unit circle, so it is unstable. So the region of converge, so so this we this region of converge will hold for a causal, unstable filter. Similarly; here you can get an non-causal but stable filter non-causal, because it is a interior of a circle and stable, because it contains a unit circle. So you can have all sorts of combinations, okay.

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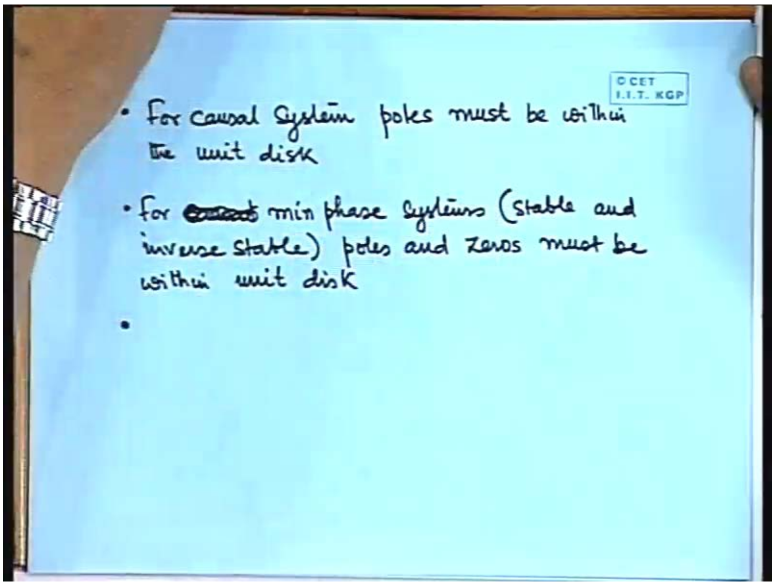

And obviously for causal system, poles must be within the unit disk. That is if the system has to be stable, then the region of convergence must include the unit circle. So this should be extended.

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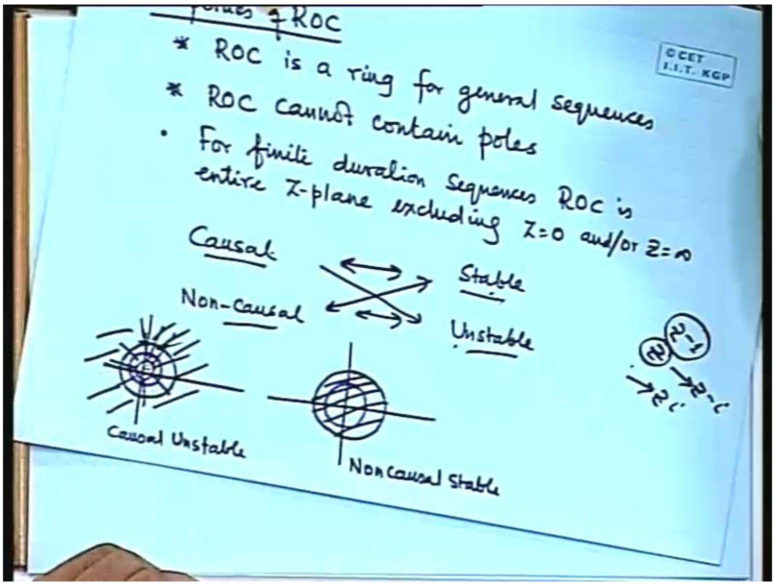

Then it will be causal-unstable and and this whole region cannot contain, the poles. So therefore where are the poles? Then poles are here, inside. So that means, they must be inside the unit circle. If the... if the system is causal-unstable, that is that kind of systems we are mainly interested in.

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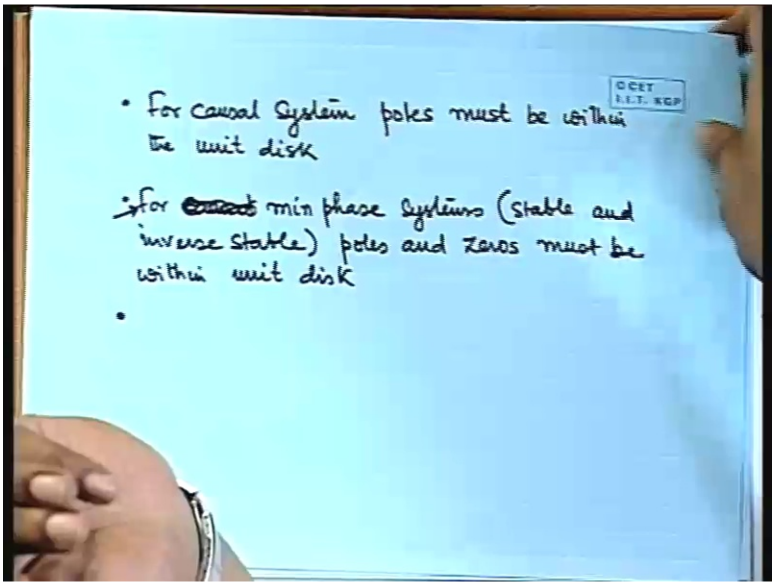

And remember that, what did we define as a as a minimum phase system? It is something which is stable and inverse stable that is Hz is stable and Hz inverse and an H inverse z is also stable. So if you want to have that; now now what is a what are the poles of H inverse z? They are the zeros of Hz. Just imagine z by bz, so simply speaking. So H inverse z will be Bz by z. So the zeros of Hz are the poles of H inverse z and vice versa, so if both have to be stable, then both poles and zeros of Hz, must be within the unit circle because zeros will become poles. So if you if if Hz has a zero outside the unit circle, it will become a pole of, it will become a pole of H inverse z. So then H inverse z will be unstable, so that cannot happen.

So for minimum phase systems all poles zeros are inside the unit circle poles and zeros both, okay. This is just and a I am just trying to emphasize; the fact because at this level, you know normally at a lower level when we study, say for under graduates, we do not bother about these things too much, we use stables and things like that. But here I wanted to stress, that point that that that that region of convergence things and see and and also see, see one thing we also assume is that, all Z transforms will be will be expressible as some Bz by z, that is not correct. There are some Z transform, which can which can appear as a said as a ration of two two polynomials. All the Z transform may not be that, there are there may be any any arbitrary form of z, right. So, not necessary that Z transforms are are are always expressible, as you know z minus A by z plus A, kind of forms. So if we we we… in our earlier classes, we we we generally assumed that form and then then talk about other things. So here we have not assumed anything like that, right.

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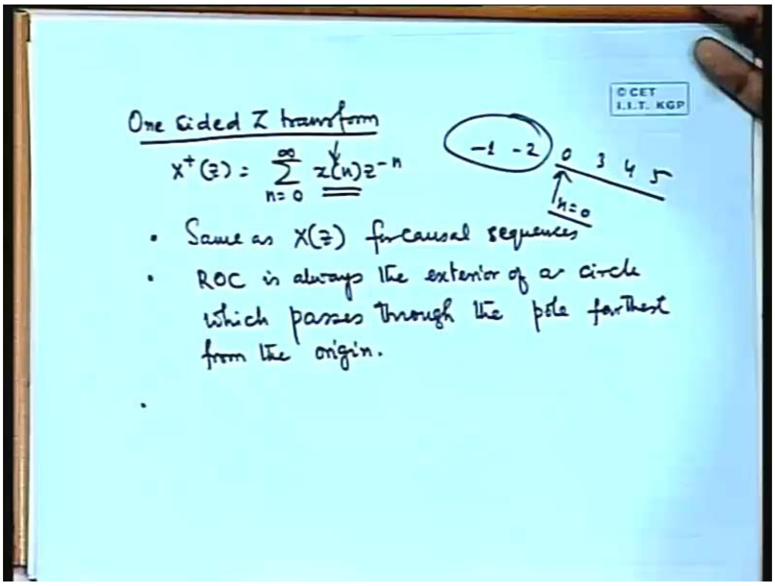

And we sometimes talk about, you know so far we always talked about, what are known as bi, I mean two sided Z transform. That is this summation is from minus infinity to plus infinity. That is very necessary when this x n can contain anti causal terms, otherwise it will not be properly defined. See, I you can take an one sided transform, then you are not at all talking about, what happen to x n when when n was negative. So for all sequences, so suppose; you have a sequence say minus one, minus two, zero, three, four, five and this is your zero point.

This is the n is equal to zero value. Now you can have anything here and just, if you match these you will have the same one sided Z transform. So then the inverse will not be unique, right. So whenever if we have only, if we know that this x n is the causal sequence, then the one sided and two sided Z transform will be same. Then also that is problem, because if you are just given a differential equation and if you have and if it has non-zero initial conditions and if you are just given the input from zero, how do you get the output; because output was non-zero, before n is equal to zero. So you must be given something about it. So then if you are given initial conditions, then only you can use the one sided Z transform. If you are not given the initial transition**** if you are not given the initial condition, you cannot use the one sided Z transform, right. So there are some such cases, so okay. So this is I I wanted to discuss this point, so because sometimes this will arise.

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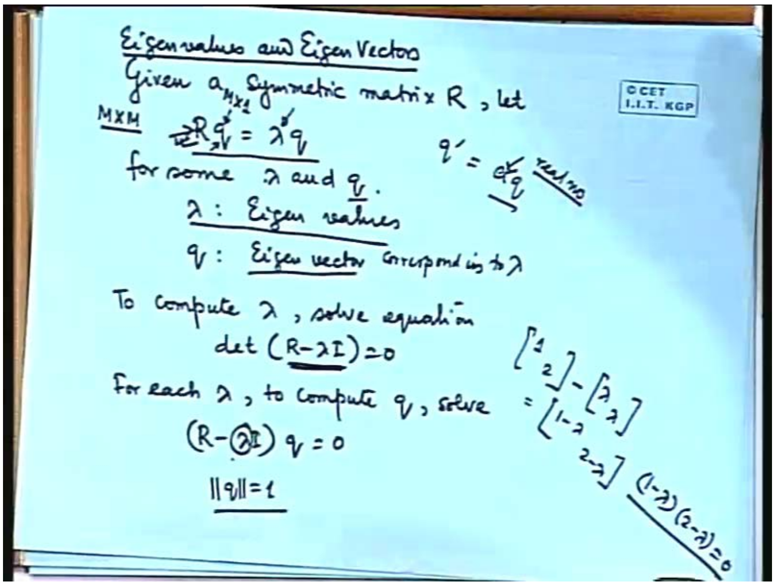

Now we will talk about another concept. This is a totally different concept, which talks about matrices. We will consider real matrices for the time being, that is whose elements are real numbers and we are we will here, we are talking of square matrices, okay. So which is may be, is this familiar to you Eigen values? and Eigen vectors? Do you know what are Eigen values? I think it was it is it is covered in all under graduate programs, but may be so we will just go through it, okay. So because we will in many cases, actually what happens; is that you know matrix is a complex quantity, I mean it is a collection of m square numbers in general. So it is very difficult to understand, what is happening. So sometimes you know, if you transform the matrix in certain ways then, you can understand nice properties of the matrix, okay this is this is happening.

That is very interesting to I mean important to understand, so therefore we will be transforming the matrices in various ways. When we will also try to know in some cases we will we will try to know, I mean things like what is the strength of a matrix? What do you mean by strength of a matrix, after all what is the matrix? A matrix can be seen as a as a transformation of vectors. See, if you multiply a a matrix by a vector, you get another vector.

So when many times we would like to know that, what is the relationship between these two vectors.

See if I take a vector along this direction and then multiply this vector by the matrix, two things will happen; first with the length of the vector will become larger, may become larger or smaller and its direction may change. These two things will happen, if you if you generally multiplied by a matrix. Now we we will be interested to know whether it goes larger or whether its goes smaller. For example, what is stability, when we have x x k plus one equal to A x k, our famous linear system equation. So what are we doing, we are taking a state vector and then multiplying it by the matrix A, to get the next state vector. So if it converges we say that, the length of this state vectors must continuously decrease, otherwise why should it come to zero? So so it must be related to this, the to this property of the matrix; that whether it stretches the vector or it contracts the vector, whether it through which angle does it turn and things like that.

So to characterise that, Eigen values are important. That is why we say that, the Eigen value should be less than unity for discrete systems, okay. Pole should be within the unit circle. So Eigen values of the matrix should be less than unity, so that every time it will get multiplied by its length will be less than the length of the previous vector, right. So to to understand these concepts, we need to understand this concepts are typically, analysed through Eigen values and and Eigen vectors, okay.

So what is the, this is the definition of an Eigen value; that if you take the R is an, let us say M into M matrix, q is an m into one vector. So if you can find out a vector q and a number lambda, such that this equation is satisfied, then the number lambda is called the Eigen value and the vector q is called the Eigen vector, corresponding to lambda. So you have for a matrix, you have a number of Eigen values and for each Eigen value you may you will have Eigen vectors, this is of case, right. Now put the point is how do you know the Eigen values? If you solve, if you just take it this side; you can get R minus lambda i equal to zero, rather R minus lambda i into q is equal to zero, you take it this side. So it turns out that; to get lambda you have to solve this equation, determinant of R minus lambda i should be equal to zero. So this will be a in general this will be a, this equation will be an mth order polynomial equation in lambda, see lambda is an unknown, all other the numbers, right.

So if you have one two as R minus lambda lambda lambda I, so so you have one minus lambda, two minus lambda, as this matrix. So what will be its determinant? It will be one minus lambda into two minus lambda equal to zero, this is the equation. So it becomes a second order polynomial in lambda. Now if you solve this polynomial equation, you you can always solve it. So it will give you n numbers, because an nth order polynomial will again roots, so those numbers are the Eigen values, once you so so now after solving this you have got the Eigen values.

Now we put the Eigen value in this equation; that is substitute the value that value of lambda, then you get an equation like this. It it becomes like an equation Ax equal to zero. So you solve it. It is nothing but a set of linear simultaneous equations; whose unknowns are the coefficients of q one, q two, q three, that is the elements of this vector. Just simply solve the equation R minus lambda into q equal to zero, you you will get the Eigen vector. It turns out that, Eigen vectors are length independent; that is if q is an Eigen vector then, if you multiply q by any number two, three, four, five any real number and that is if you make stretch all the numbers by the same amount, it is still remains an Eigen vector.

So we generally talk about, Eigen vectors of unit length. Actually Eigen vector only indicates a direction; there is no concept of length, I mean along that direction any vector of any length will be an Eigen vector, that is that is obvious for here from here.If you if you take a q dash to equal to some aq, still this equation is satisfied. If if q satisfy this equation and and a is a number, then q dash also satisfy this equation. So it is also an Eigen vector. So we generally talk about Eigen vectors whose length is equal to one, we fix the length generally.

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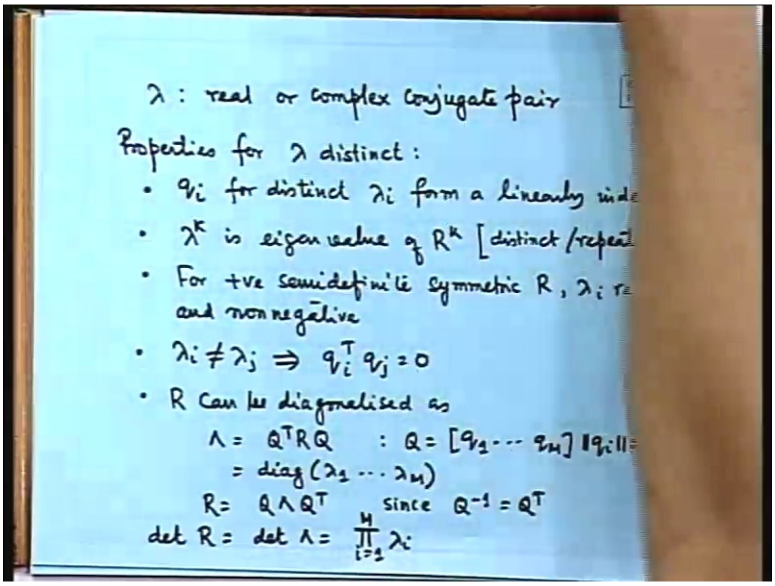

But lambda can be a real or complex conjugate pair remember, because this equation is a, this equation oh sorry

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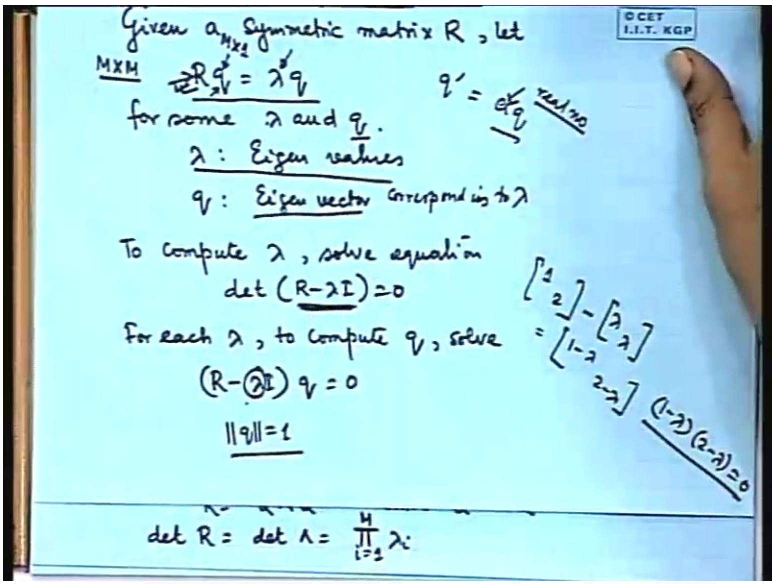

this equation is a nth order equation, so it will have real or complex roots but since since all the coefficients of this equation are real; therefore if we have complex roots they will they will occur in conjugate pairs, always otherwise you cannot get real coefficients, correct. So therefore lambda is a will occur real or complex conjugate pair.

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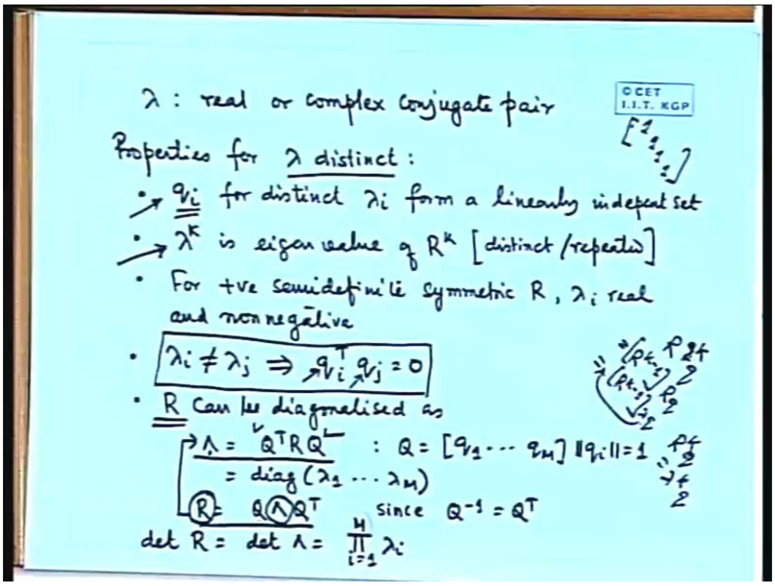

Now we will look at properties for lambda being distinct. Actually what happens is that, if lambda is not distinct, then the direction of the Eigen vector also is not always guaranteed. For example, I mean a best example is take the identity matrix, that is an nth order matrix; which is one one one one one all ones, okay. Incidentally.. it is the diagonal matrix, but but but all its Eigen vectors are I mean, what are its Eigen values, all ones. So there are all Eigen values are repeated, because R into q is equal to lambda into q, for any q.

And for lambda equal to one, because it is an identity matrix, so so therefore R into q is equal to q for any q, remember that for any q. So basically, so if if all the Eigen values are repeated then, these space of then then any vector is an Eigen vector. So so you cannot say that even the even the direction is fixed, you cannot say that. So in that case, if you I mean you have to choose your direction, but so let us talk about the case; because if you want to know properties, then then you should understand these properties. That is if the lambda distinct, then what happens? So then we get very interesting properties of the Eigen values and the Eigen vectors, which we are going to extensively use in this course.

So first of all let us see first, this property, this property is of course true for any, in mean repeated or distinct Eigen values. That is if lambda is an Eigen value for R, then lambda to the power k is an Eigen value of R to the power k. That is very simple to prove, because R to the power q; say R to the power k q is equal to what? is equal to R to the power k minus one into R q. R is the matrix, so it is equal to R to the power k minus one into lambda q. So you go on replacing that, now lambda is a scalar, so it will come here. S ay again take another R out in this case, every R will generate a lambda. So you will get that R to the power q kq is equal to lambda to the k q; which means that if lambda is an Eigen value for R, then lambda to the power k is an Eigen value of R to the power k. Very simple property to prove and this happens, for whether it is repeated or distinct.

Now interesting thing is that, this Eigen vectors have very nice properties; which are which is what is going to be exploited, in the in this course. They have very nice properties and the one of the most important properties is these, that is for distinct Eigen values. The Eigen vectors are orthogonal. You know we always we always prefer, whenever we choose a coordinate system; we always prefer that our coordinate systems are perpendicular, because there in we can go along one without changing the others, without affecting the others. So we always look for perpendicular coordinate systems, and it so happens, that for any matrix these Eigen vectors are perpendicular to each other, right. When a when a when a vector is perpendicular another, its inner product is zero, true.

So this is a first property, this can be again even easily proved, how you will prove this? Think about proving this properties, why, I am telling you, if you if you if you if you we can discuss this in a in a future class. They are generally easy properties to prove; you have to all you have to use is R q equal to lambda q, no high great knowledge is required, but if you do this then you will then the properties of the Eigen.. vector and Eigen values with you will will, I mean we will get embedded more deeply in your mind. So so try to prove this; we can discuss these in a tutorial class, they are they are they are very.. I mean quite simple to prove actually.

So and so obviously if qi transpose, q j equal to zero, this is the next property which is important. This is actually actually this is a more general property than this. This says that, for distinct lambda i all the qi, they form a linearly independent set. Means what? What is a set of, these are vectors, right? This is our vectors. So what what do we mean by a set of vectors are linearly independent? When any one of them cannot be generated by a by by a linear combination of the others, which means that all of them are need. You cannot really remove one and then say that, okay okay using the other I can generate this one. That is not possible, so all these sets are needed.

So whenever you require, so so they are the minimum range of vectors needed, to actually characterise a full end dimensional space. See if you have only have vectors in this plane and if you and if you try to characterise them in terms of three vectors; say you say that I all vector, I will express in terms of x1 x2 x3. Then somebody is going to say that, it is not needed. You do not need x three, you just try it try to work with x1 and x2, any in any description that you can get with x1, x2 x3. You can also get with x1, x2 because, this is a two dimensional space.

So in general when you have our vectors are all M dimensional q's in the because, I mean the the dimension of this space M, so and and this qi is our linearly independent vectors, M linearly independent vectors, because there are M Eigen values, capital M. So which means that using these, so that these vectors can actually; there are two things which are being said that, these vectors can serve as a coordinate system, because they are linearly independent so they will characterise the whole space. So any vector can be expressed, as a as a sum of these, that is the first thing which is being said. And second thing which is being said is that, these coordinates, they are they are they are they are not only coordinates, they are perpendicular set of coordinates. So these vectors really serve as as very useful coordinate basis for for representing general n dimensional vectors. So in many cases we will represent our vector processes in on a coordinate system, which will be based on the Eigen vectors of their auto co-relation sequences and then we will get we will get very good understanding as to what is happening to the process.

So so basically these are used for coordinate transformation, and that is used so, if you a if you have R as a matrix, this is the last result and we will stop after this. That if you have a matrix R then, it can always be expressed like these; that is you can always and a and vice versa, these two results are there where this becomes a diagonal matrix. So you can diagonalise any symmetric matrix R by using the Eigen vectors, this q's are the Eigen vectors. Yeah and similarly if you have if you have a so similarly, R can be expressed as this as the sum like this with these Eigen, with a with a set of diagonal matrices, all right. So just remember these and we will we will start using it in the next class, thank you very much.