Digital Signal Processing

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Lecture - 6
Z-Transform (Contd.)

Discussing about inverse inverse Z transform, sorry.

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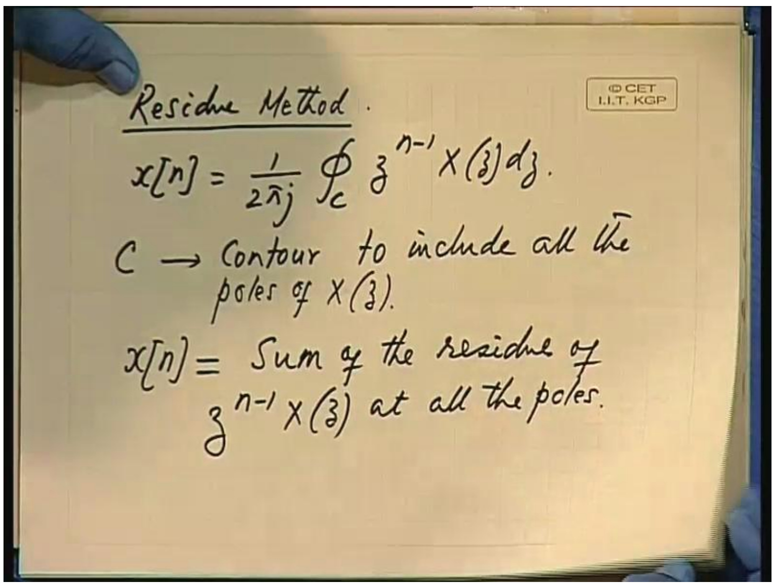

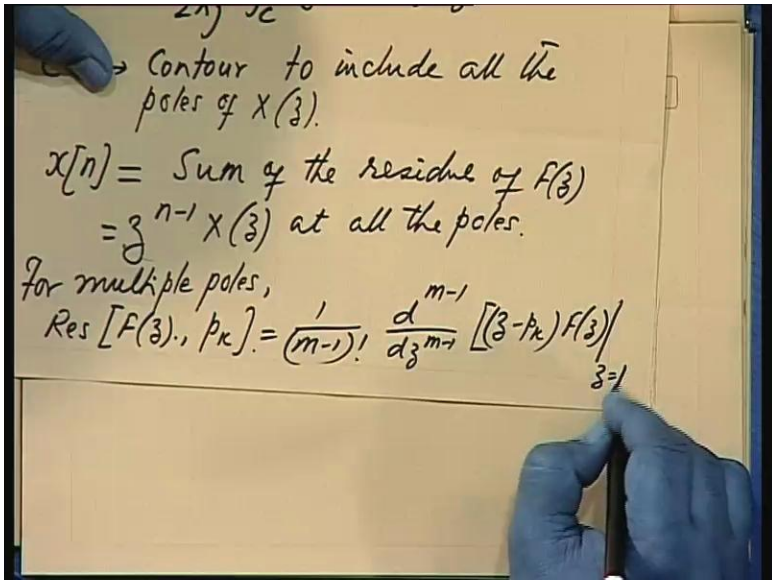

The last method that is the residue method of competition, we will discuss today. x n the sequence is obtained by evaluating, this integral z to the power n minus 1, X z, d z. The contour C is taken to include all the poles, to include all the poles of X z; you choose any contour which can include all the poles.

So, x n can be written as by Cautious theorem; sum of the residues of z to the power n minus 1 into X z at all the poles, okay. And for multiple poles at a particular location, for multiple poles the residue is calculated, residue F z at p K where F z is this product, residue of say F z equal to z to the power n minus 1 into X z.

So, the residue of F z will be 1 by m minus 1 factorial, d m minus 1 by d z m minus 1, z minus p K into F z, evaluated at z is equal to p K, okay. Let us see this with an example, how this can be applied.

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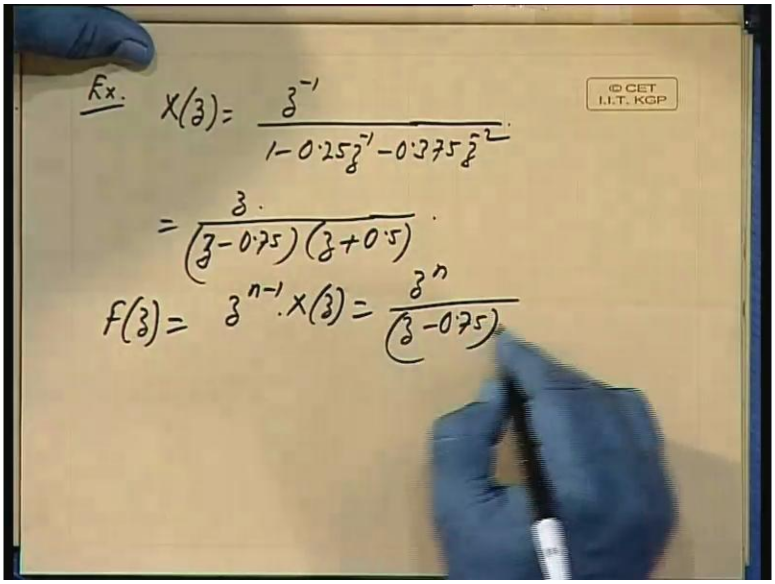

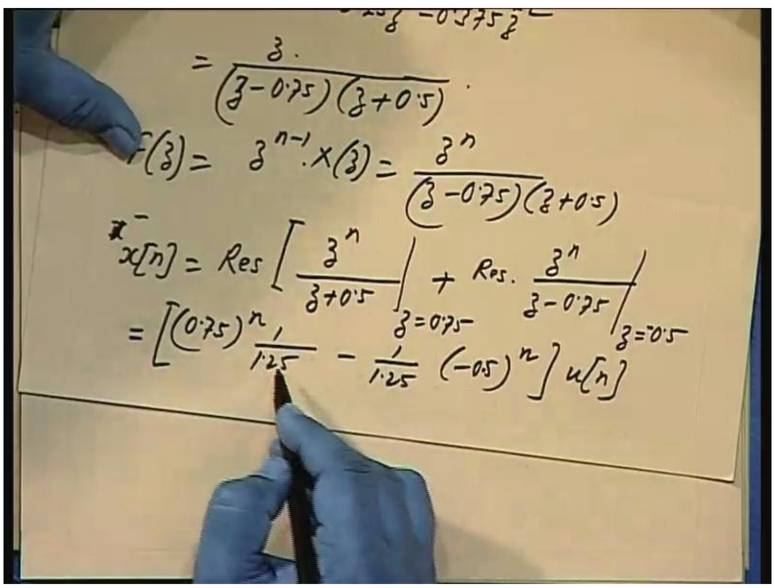

Let us take X z, the same example; z inverse by 1 minus 0.25, z inverse minus 0.375 z to the power minus 2 which we took last time. So, this will be z by I can write; 1 minus okay, z minus 0.75 these factors we got last time, z plus 0.5, okay. So, F z will be z to the power n minus 1 into z which is z to the power n minus 1 into z; that will give me z to the power n by z minus 0.75 into z plus 0.5, okay.

So, you can evaluate x n, by that method of residue once you have got F z there are two poles, one at minus 0.5 the other one is at plus 0.5. So, evaluate it, it comes as x n. So, okay let me write the residues first; x n will be residue of the first pole that is z to the power n, if I multiplied by the first pole, so z plus not five I get in the denominator this is to be evaluated at z is equal to 0.75 plus residue the second one is z to the power n by z minus 0.75, evaluated at z is equal to 0.5, is it all right, minus 0.5, thank you.

So that gives me, if I put z is equal to 0.75, so it will be 0.75 to the power n, okay; 0.75 will make it 1 by 0.5 plus 0.75, 1.25 and minus 0.5 and the denominator will be minus 1.25 in the second case, and minus 0.5 to the power n, so whole thing multiplied by u n will be x n, it is so simple. This is what; we got by the other method, 1 by 1.25 was 0.8 so I can take that as common.

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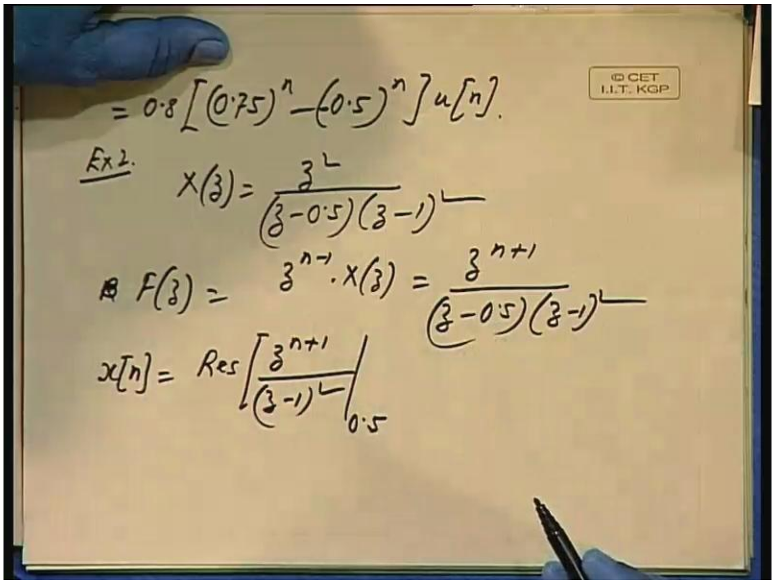

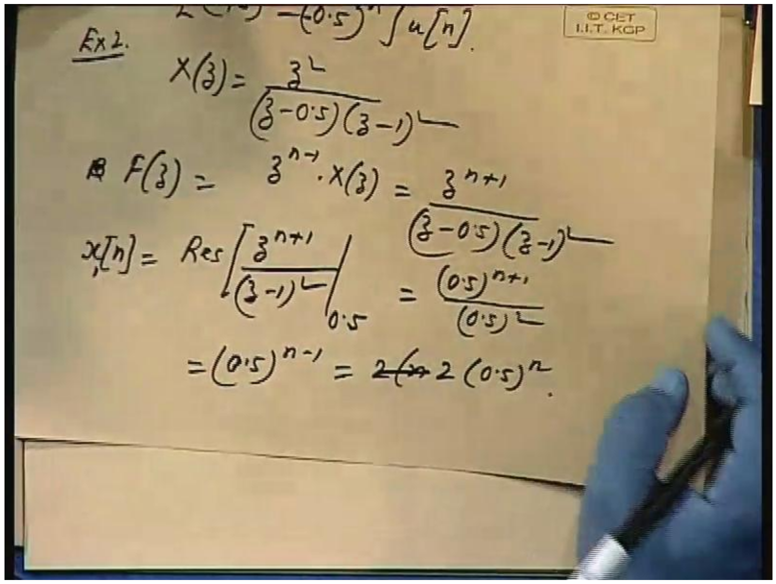

So, that gives you 0.8 into 0.75 to the power n minus 0.5 u n. Let us, take another example with multiple roots. X z equal to z square by z minus 0.5 into z minus 1 whole squared, okay. So, you have to differentiate and then find out the residue. So, residue for the first part, before that let me compute F z; which will be z to the power n minus 1 into X z which will be z to the power n plus 1, z square into z n minus 1, all right.

So, x n will be residue at the first pole, it will be z to the power n plus 1 by z minus 1 whole square, 0.5 evaluated at this pole, is the first one, okay. I will call this as say x 1, this will be 0.5 to the power n plus 1 by this is 0.5 minus 1, so minus 0.5 squared; so 0.52, so that gives me okay 0.5 to the power n minus 1, I may write as 2 into 0.5 to the power n if you permit me, I can write like this. Similarly, the second one I am writing separately x 1, and x 2 and add them together, is it all right.

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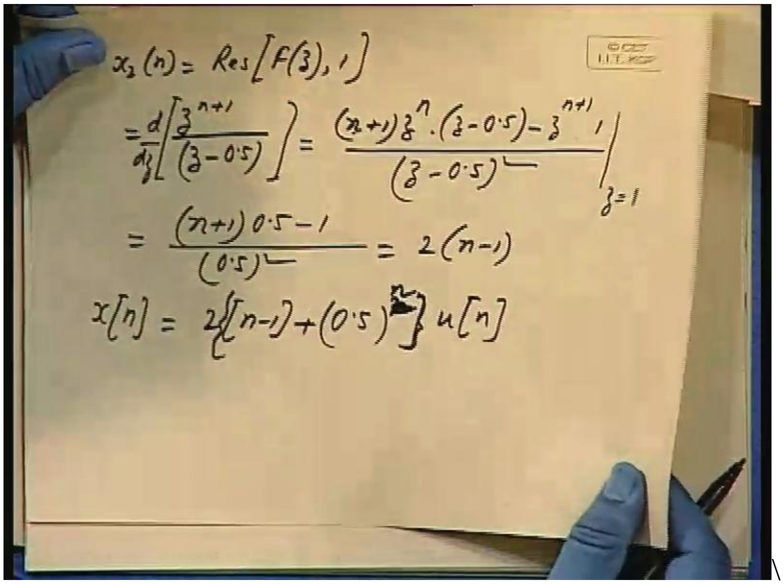

x 2 n for the second one, I will take residue F z at this pole; which will be if you differentiate here, this one will be z to the power n plus 1 by z minus 0.5 d d z of, this one is it not? You have to multiply by z to the power, z minus 1 whole squared and then calculate the function, okay. So that gives me, z to the power; okay n plus 1 into z to the power n into z minus 0.5 minus z to the power n plus 1 into z minus 0.5; the derivative is 1, divided by z minus 0.5 squared and this is to be evaluated at z is equal to 1.

So that gives me, if you put 1 n plus 1 into 0.5, okay n plus 1 into 0.5 minus 1 divided by 0.5 squared. So, that gives me 2 into n minus 1, okay. Therefore, x n is 2 into n minus 1 plus 0.5 squared into u n. It was true; sorry to the power n not square, sorry is that all right. This is to the power n, x 1. Now, there are certain important properties of Z transform, let me just list out some of them, will take up one or two for proof.

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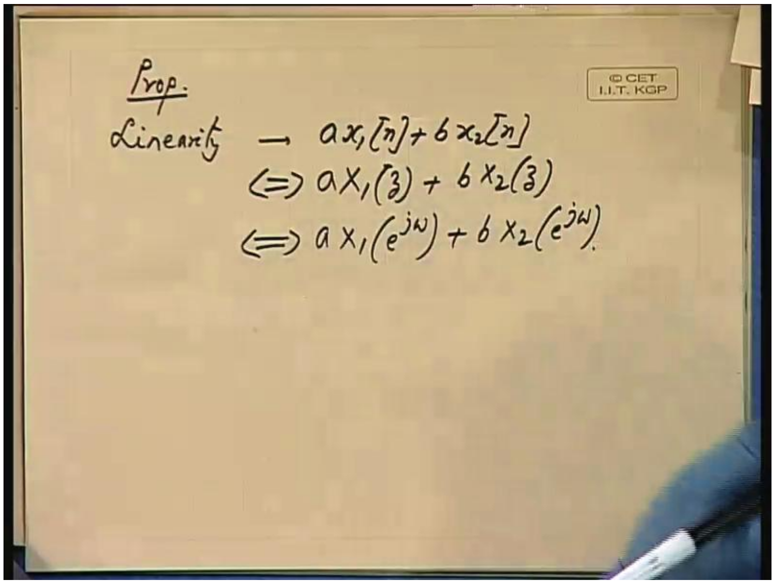

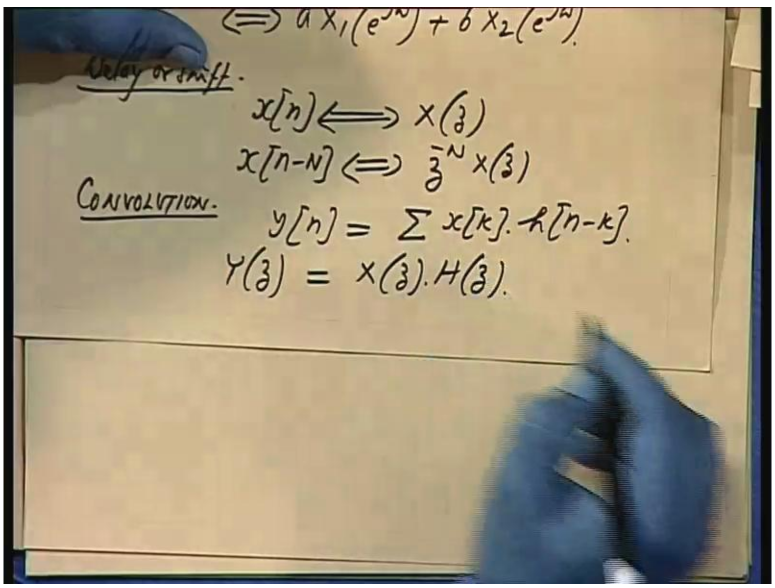

One is very helpful linearity. a into x 1 n plus b into x 2 n gives me, a into corresponding Z transforms X 1 z and X 2 z multiplied by a and b, okay. This we also observed for frequency domain representation, of frequency transform a into X 1 e to the power j omega plus b X 2 e to the power j omega, okay where we discussed about D T F T, discrete time Fourier transform, we will get back to that very soon.

Then delay or shift, this also you observed earlier; if x n is having a transform X z then x n minus say capital N, will be z to the power minus N X z, okay. Convolution, y n if it is convolved in the discrete time domain, x K and h K; if they convolved then in the frequency domain in the Z domain, they will be in a product form. These proofs are very similar to Laplace transforms, where you have seen convolution, a delay, linearity, is a very simple relation.

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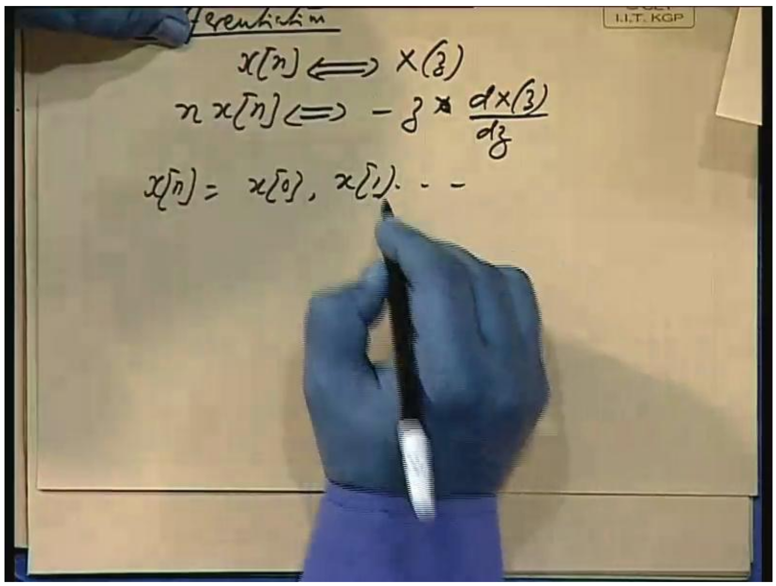

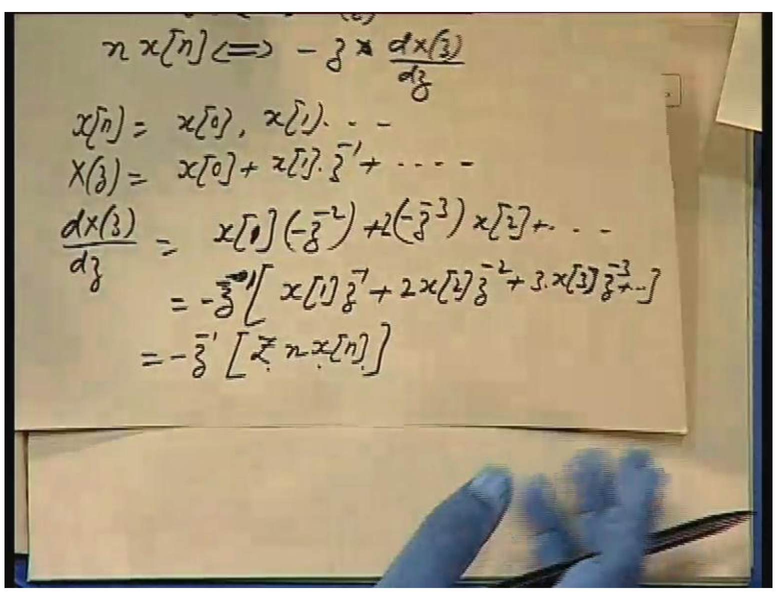

Differentiation, x n is X z, x n is paired with X z then n times x n is minus z, d x z, by d z. Let us take up this one for example; x n by definition is a sequence x 0, x 1 and so on, okay so X z is x 0 plus x 1 z inverse and so on, all right. So, what is d x z by d z will be x 0 derivative is 0 then x 1 and z 1 if I take the, derivative it will be minus z to the power minus 2 plus minus z to the power minus 3 into 2 into x 2 and so on.

If, I take z to the power minus 1 common it will be minus z to the power minus 1, minus z common, okay. Sorry, z to minus 1 common, sorry z to the power minus 1 common will be x 1 z to the power minus 1 plus 2 into x 2 z to the power minus 2, 3 into x 3 z to the power minus 3 and so on; which is nothing but z inverse, this is a Z transform of what, n x n is it not. As, if 0 into x 0 plus 1 into x 1, 2 into x 2, 3 into x 3 these are appearing, the 0 term is missing because it is multiplied with 0, okay.

So, it is Z transform of n times x n series. So, if I transfer this to this side, we get the result minus z into d x z by d z is equal to z transform of n x n, okay.

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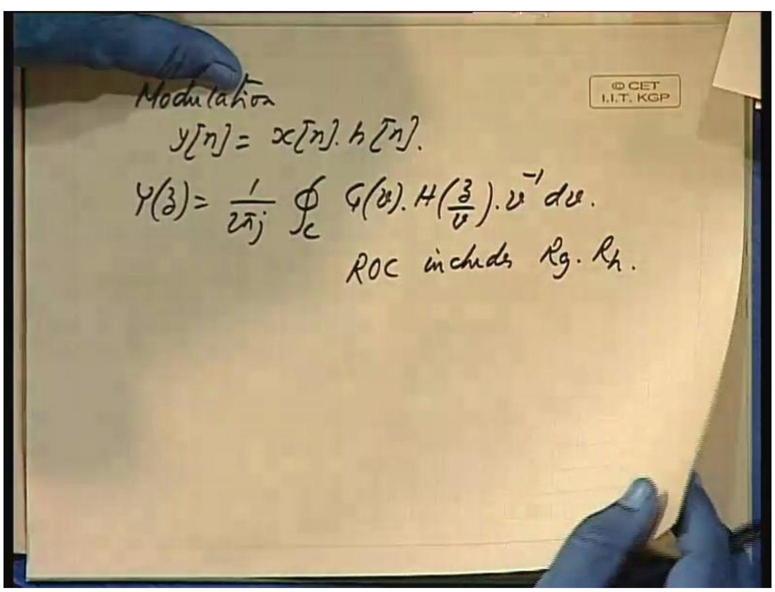

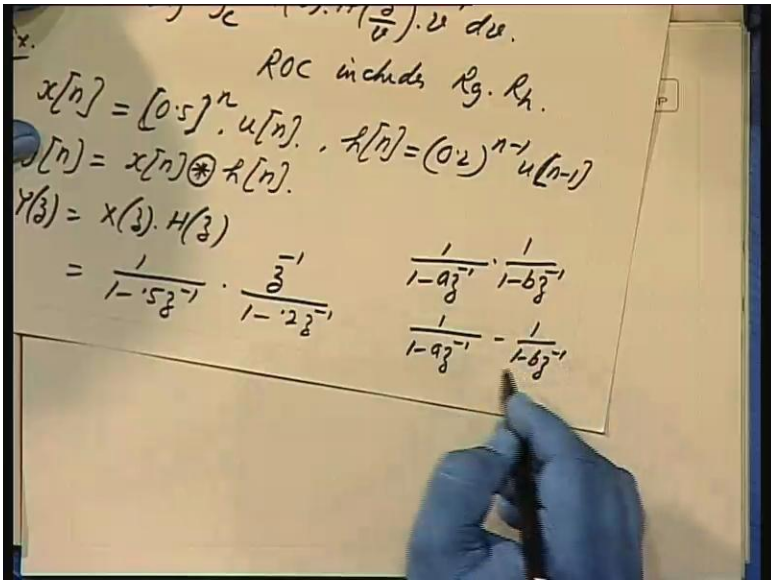

Then modulation, that is y n is x n into h n, all right. Then Y z will be 1 by 2 p i j, G v, H z by v v to the power minus 1 d v. The ROC, region of convergence includes R g at R h, okay. We will be taking up, some simple application of Z transform in a real systems, okay. Then we will go back to discrete time domain and frequency domain presentation once again and try to find out, the common link between Z transform and frequency domain transform, okay.

Say, we are given x n, x n as 0.5, 0.5 to the power n, u n. h n, 0.2 to the power n minus 1 u, n minus 1. What will be y n which is convolution of x n and h n? Sometimes we use a circle around the star for denoting convolution, because star is quite often used for complex sequences also complex conjugates, so to avoid confusions, okay we use this.

So, let us use that relationship for convolution. Y z will be therefore; X z into H z and what is X z for this one? 1 by 1 minus 0.5 z inverse, is it not? For 0.5 to the power n and H z, this is delayed by one step, so it will be z to the power minus 1 and 0.2 to the power n; Z transform is 0.2 z inverse, is that all right? So, you know the product make partial fractions, okay.

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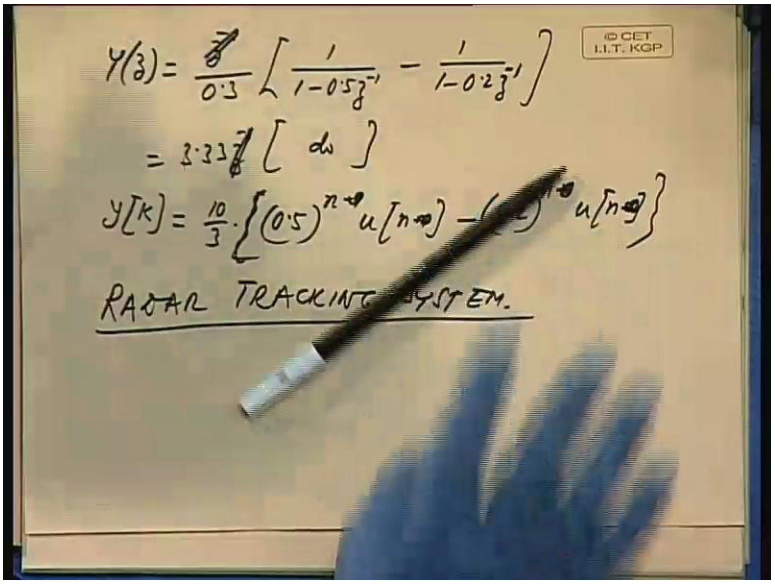

Let me use this paper okay. So, Y z is z to the power minus 1 by 0.3, 1 by 1 minus 0.5 z inverse minus 1 by 1 minus 0.2 z inverse, seen such cases it is very simple. If you have the product like, 1 by 1 minus a z inverse into 1 by 1 minus b z inverse then you straight away take the partial fraction as 1 minus a z inverse minus 1 by 1 minus b z inverse, okay. Once you will get cancelled, so what you will be left with; a minus b in the numerator, so divide by a minus b. That is what I have done; the difference between 0.5 and 0.2 is 0.3, so divided by 0.3.

And there was z inverse in the numerator, so I have taken it out. So that is approximately 3.33 z inverse into this. So, what will be Y K? It will be 3.33 or you can write 10 by 3 either way, solution for this to be delayed by one, inverse of this to be delayed by one step. So this is 0.5 to the power n minus 1 u, n minus 1 minus 0.2 to the power n minus 1 u, n minus 1, okay. So, this is a solution. Yes, please?

z inverse will come as n minus 1, partial fractions; 1 will get cancelled, there is a z inverse already coming in so this will not be there, so thank you very much. So, I need not have z inverse, this itself will give me the z inverse, thank you. So, it will be just n no then 0.2 u n, yes it will be sorry it will be this; z inverse will automatically come from the subtraction.

Now, let us take an interesting application, radar tracking system. In a radar tracking system in a radar tracking system, what do you do, you have a source of signals, you are sending pulses all right. Suppose, you are tracking a moving object all right, so the pulses will be going at the velocity of light of course and then that will reflected back, you are sensing the reflected signal, you are finding out the time delay that divided by 2 will be giving you the distance of the object, all right. Now you keep on measuring it, all right.

So, if you are sending signals at regular intervals and receiving signals at regular intervals then you try to compute, you try to estimate its position and also its velocity. If you know its position at a certain instant of time and its velocity; you can predict after say one second, what will be its possible location. You know its direction velocity and position, so you know its direction; so along that direction, you know after one second it will be in this position.

So you fire the next shot, all right you fire the shots to the object, to the enemy object, you know the velocity of the missile and then you can destroy that, okay that is the idea. So, let us see very simple mathematical model, for a radar technique system.

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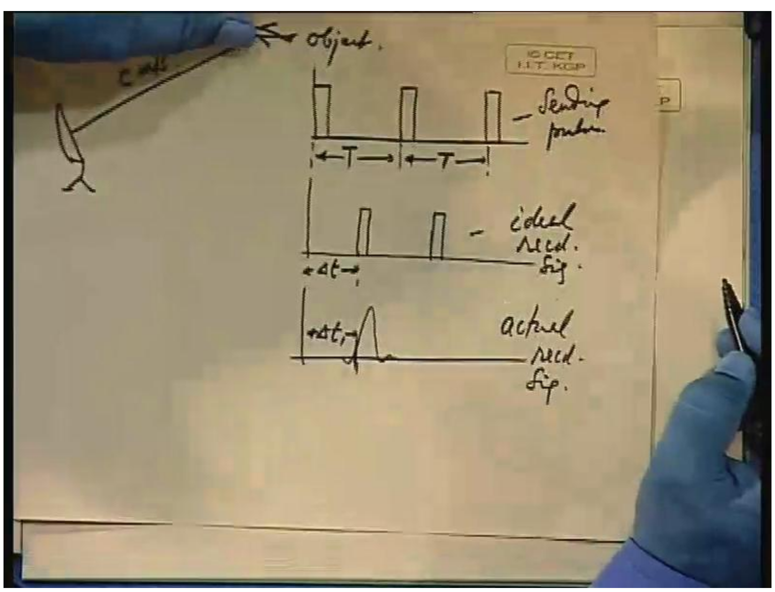

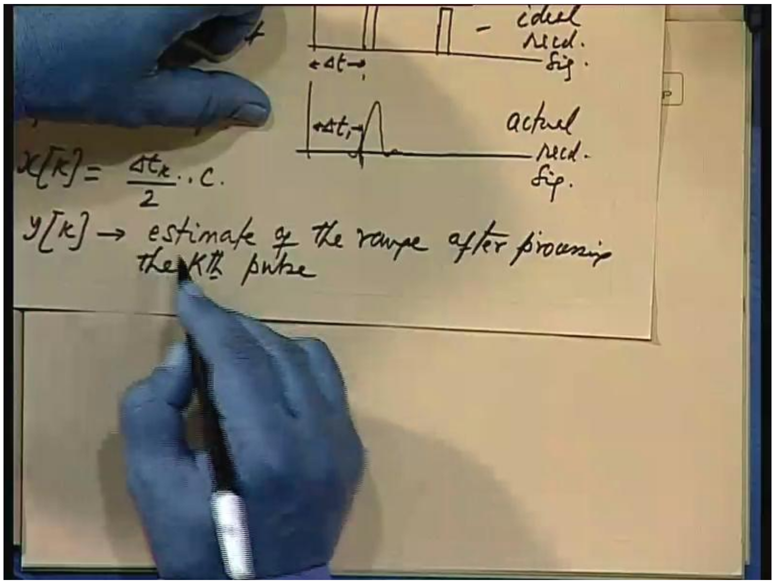

So, I will show you a very simple, schematic, diagram. So, this is an enemy plane and this is radar. So, you are sending a signal, this is object, c meters per second; it is a velocity of light, the signal is sent and you are sending the pulses like this.

So, this is T at regular intervals, all right. The received signal ideally that any instant, this is delta t, this is the received signal, of course this delta t will keep on changing because it is changing its position. So, the received signal, the time occupy a time taken by the pulse to return back will not be the same throughout, that is changing so this gap is changing, okay.

But the actual signal is contaminated. The received signal, actual received signal will be somewhat like this. So, there is threshold that we take and this is our actual time we are accounting; okay, it will be contaminated by noise and there is a filtering effect of the space. So, this is the actual received signal and this is the ideal received signal, this is a sending pulses.

Let us take X K, as a measurement of the range, measurement of the range from the K th pulse okay, so what will be X K? It will be delta t k by 2; delta t k is a time for the K th pulse to return, all right. So, delta t k by 2 into the velocity of light, so many meters, okay. Y K is an estimate of the range after processing K th pulse, okay that means; when I ask you to make a measurement,

measurement does not give you any measurement for that matter, measurement does not give the exact value.

For example, in your high school physics experiment or even here; you take a measurement ten times and then take the flat average sometime all right. In schools, your you learn to take only the arithmetic average, later on you must have observed, you take the most probable value that means the value which is occurring more often is taken as the estimate, estimate. You can always make an estimate, now there will be always some measurement error all right, personal error and so on, so instrumental error.

So, depending on the complexity of the experiment, you make an estimate of any item any variable from the set of measurement, okay.

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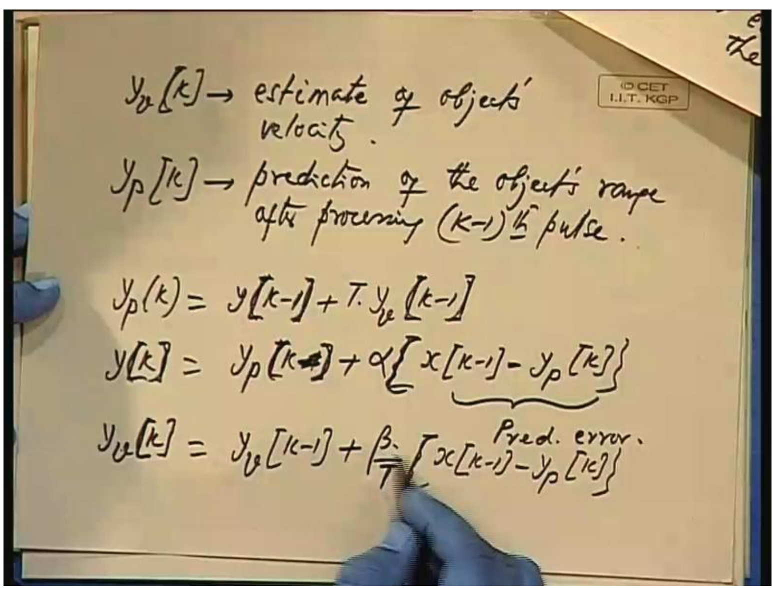

Then y v K is an estimate of the objects velocity, and y p K is a prediction of the objects range. When you whenever you are making a prediction, prediction is based on the previous measurements all right; so after processing K minus 1 th pulse, so on the basis of K minus 1 readings you make, a forecast, you make a prediction what will be its position at the K the instant. And you have made an actual measurement.

So, that gives you an error, prediction error. So, you take that error as a feedback and let us see; how these estimates are modified with the help of that feedback. So, y p K the predicted value, how much is a predicted value? Say, up to this instant of time I have got, y K minus 1; y K minus 1 as the estimate, what will be my prediction in the next instant? If I know its velocity, if I have an estimate of its velocity then velocity multiplied by that time gap capital T that we are considering, so multiplied by that time gap will be the additional increase in the range, so T into Y v K minus 1.

Suppose, this is K minus 1th instant, so at K minus 1th instant; if I have estimated the value of the velocity of the object, if I multiplied by the time interval, I will get the increase in the value delta y, okay. So, that multiplied by the present estimate, that will be giving me the prediction. Similarly, y K whatever the estimate at the next instant; that is after possessing K number of measurements, what will be its value? y K, how will you estimate?

What is y K? It is the predicted value, whatever was predicted to be the value at this moment from the previous instant, y p K minus 1 plus we take a correction factor into the error, all right. And what is the error measurement, and using sorry third bracket; so let me be consistent, minus, is that all right? This was the prediction error.

So, prediction error multiplied by a constant, normally less than 1 plus the prediction met in the previous instant, it should be; prediction is made at this instant, remind you prediction is made at these instant from the previous data. This is prediction error, okay and y v K, what will be the estimate of the velocity? It will be whatever velocity was estimated in the previous instant plus the prediction error from the prediction error, I will estimate the prediction, prediction error in velocity.

So, beta times 1 by T times the same prediction error, is that all right. Whatever is the error in the position that divided by that time T, gives me the error in the estimate of the velocity. So, that multiplied by a factor beta, if I add with the previous estimate of the velocity; that will give me the present estimate of the velocity, oaky. This is known as, an alpha beta tracker. Now, let us take Z transform of these.

So, from here if you permit me, these three equations I will write, for say the for example; the second equation what will be the Z transform of this?

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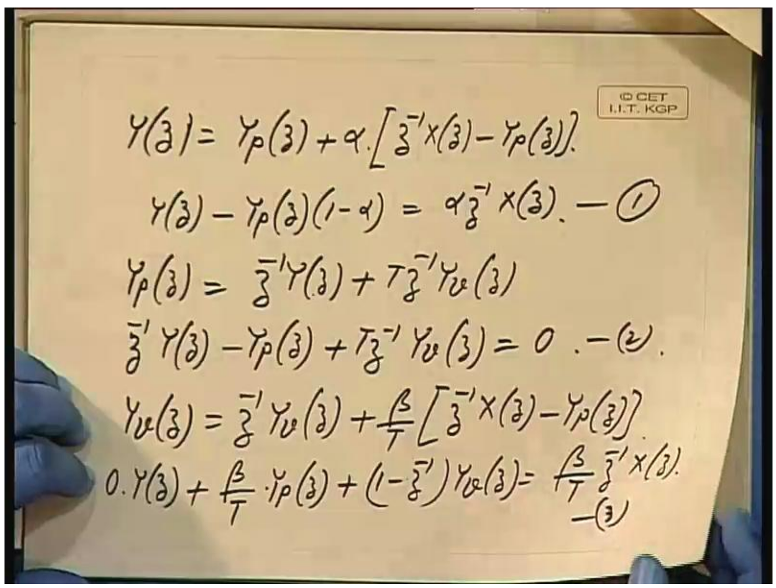

Y z is equal to Y p z, Y p z plus alpha times x K minus 1. So, x z into z inverse minus y p z, is that all right? Similarly, first equation here will give me y p z, okay let me arrange these it will be Y z minus Y p z into 1 minus alpha, okay, is it all right, Y p z this one, okay. On this side alpha z inverse X z, is it so, is this all right?

Next one, y p K is equal to y K minus 1 plus T y v K minus 1. So, if I take the Z transform, Y p z equal to z inverse y z plus T z inverse y v z or in other words z inverse into y z minus z inverse into y z minus y p z plus T z inverse y v z equal to 0. And the third, third equation this one gives me, y v z okay, equal z inverse y v z plus beta by T, z inverse X z minus y p z or in other words; I can write it 0 into y z, there is no y z term here, okay. Plus beta by T y p z plus 1 minus z inverse into y v z is equal beta by T z inverse X z, this is equation three.

I have tried to arrange the three equations in such a way; that you can write a very simple form, a matrix equation.

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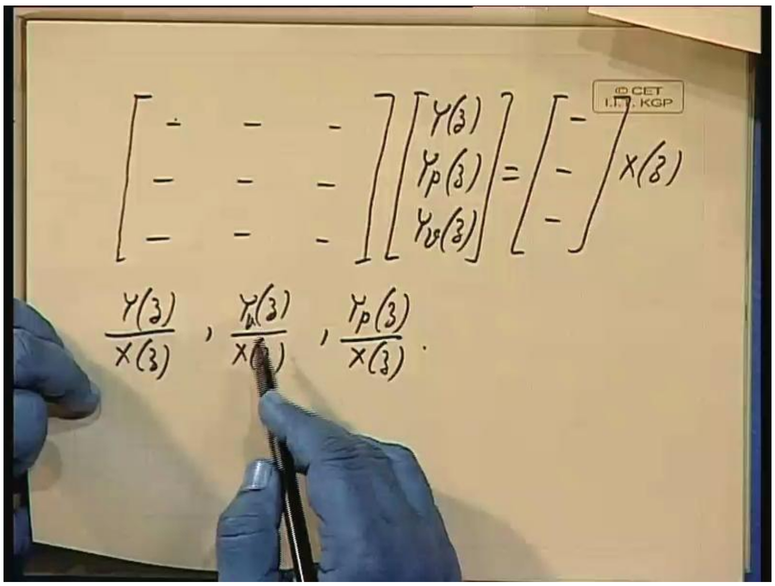

I would request you to fill up the blanks Y z, Y v z, Y p z, and Y v z will give you something into X z, these elements you fill up from these equations. So, it is very simple, matrix equation. You can find out, Y z by X z, you can find out Y v z by X z, you can find out Y p z by X z; try at home, what will be the transfer functions?

That means, the input signals the input measurements from the measurements not from the signals, X z is a measurement so with the measurement, how are you making estimates of the velocities, position and the prediction, okay. Now, if you have a general equation of this type; say, a 0, y K plus a 1 y K minus 1 plus a 2 y K minus 2 and so on, up to a N y K minus N.

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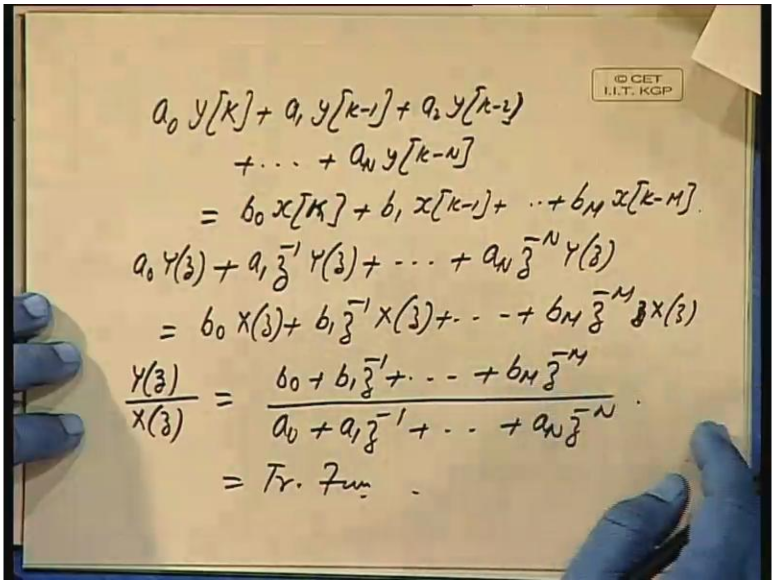

On this side, you have b 0, X K plus b 1 x K minus 1 and so on, up to b M x K minus M. This is obtained is a generalized form of a difference equation, obtained from differential equation. Once earlier we have discussed, how from a differential equation you can approximate and you can make certain approximations about the derivatives and you can get a difference equation.

Now, difference equations will not be identical, for the same differential equation; if you change the step length, if you because the factor T squared T, T Q, would depending on the order of the derivatives. So, it all depends on the step length that is a sampling time that you choose, okay. If you change the sampling time, the difference equation coefficients will also be different, okay. This is something very important.

Now, such difference equations can be converted into, in the cab be converted into Z domain. So, you get a transformed equation. So, you get a 0, Y z if you assume zero initial conditions to start with; that will be z inverse for Y z, a N Z to the power minus N Y z, it will be b 0, X z plus b 1 z inverse X z and so on, up to b M z to the power minus M X z.

I can take Y z common, similarly on this side; I can take X z common and Y by X will be b 0 plus b 1 z inverse b M z to the power minus M divided by a 0 plus a 1 z inverse and so on, a N z to the power minus N. We call this as transfer function in the discrete domain; as you have done in the continuous domain Laplace transform, you take and then then take the ratio of the output by input. So, this is the transfer function the discrete domain.

Given a transfer function we can find out its impulse response; just take the Laplace the Z inverse of this and you have seen for calculating Z inverse there are three methods; you can use any one of them, so you make factors, find out the factors factorizing this and then making partial fractions either calculate the residue or by the partial fraction method, you can find out the solution. Otherwise, by long division you can calculate H K. So, this is normally written as H z or G z.

Now, let us come back to discrete domain functions once again; before we go further because we will be requiring the discrete domain descriptions, especially in the frequency domain and. Like the the properties that we just now discussed in the Z domain; you have also in the frequency domain similar properties that is some of them probably discussed earlier.

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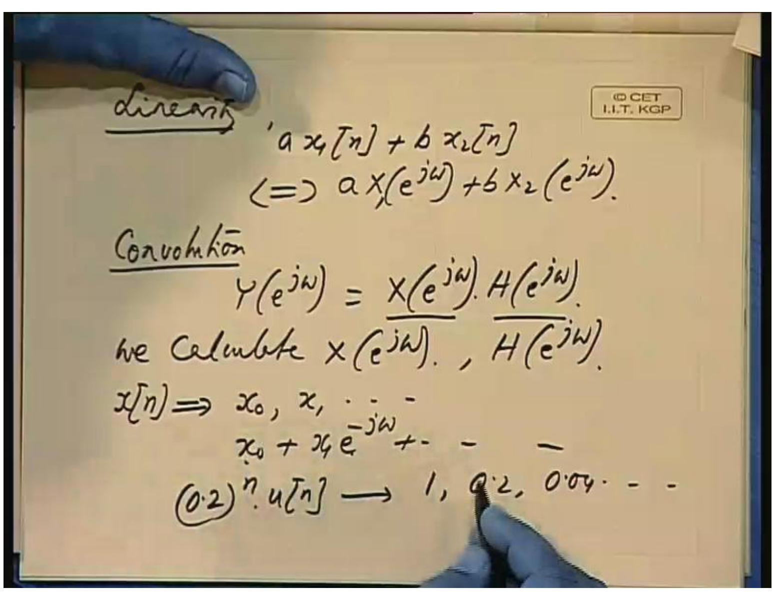

Linearity, the same principle a times X 1 n plus b times X 2 n; will give you a times X 1 e to the power j omega plus b times X 2 e to the power j omega, probably we mentioned it earlier, delay or shift as very similar to Z transforms. Then your convolution that will also be similar; now for convolution for convolution, suppose we have output Y is convolve convolved with X and H, so it will be in the frequency domain, X e to the power j omega, H e to the power j omega. If you can find out the values, it is okay, if you can get an analytical analytical expression and if you can take the inverse.

So, we calculate x e to the power j omega and H e to the power j omega, all right. So, if you are given a sequence x n as x 0, x 1 etcetera; so calculate x 0 plus x 1 e to the power minus j omega and so on. If it is a mathematical description; that is given and if you can find the convergence, if you can reduce it to a close form, say like if the sequence is 2 to the power, say 0.2 to the power n u n then you know the sequence is 1, 0.2, 0.04 and so on. So, you can write the expression since it is a convergence series; we will be getting the solution very identical to Z transform, X e to the power j omega will be 1 by 1 minus 0.2 e to the power minus j omega, all right.

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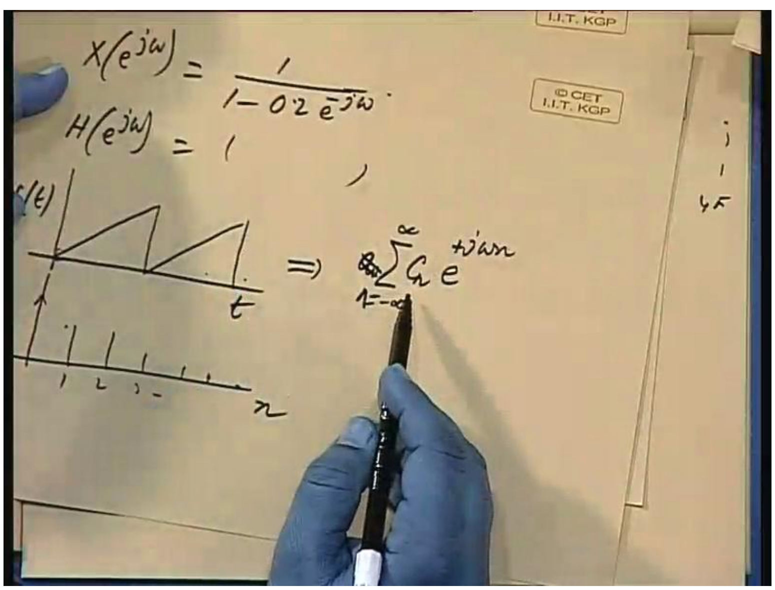

This kind of a series, if you take the discrete discrete time Fourier transform all right, so the frequency domain description of this sequence will be this, is a g p series I have reduced it.

Since e to the power j omega is always having a magnitude 1, so 0.2 times this will not blow up, it will be a convergence series okay because the magnitude is reducing, all right. So, from here if you similarly get e to the power H, e to the power j omega, in such a form close form and take the product and then take the inverse; how would you compute the inverse, how would you compute the inverse?

Many of classical Fourier series you have seen; a periodic function, any periodic function say this a periodic function in the time domain, can be expressed in the frequency domain, can be expressed in the frequency domain as a 0, I will write in a complex form C n e to the power j omega summation, all right, okay plus, in varying from minus infinite to plus infinite. If you combine together and then write in the complex form, it will be like this, okay.

So, C n e to the power j omega and we get lines, these are called harmonics, okay. I can just number them n harmonic number 1, 2, 3, 4 and so on, these are the frequencies present. Now what is the frequency transform of the sequence x 0, x 1, x 2? What is the frequency transform? x 0 plus x 1 and this we saw earlier; this is a periodic function all right of period two p i, is it not? We discussed earlier, this is having a periodicity of two p i.

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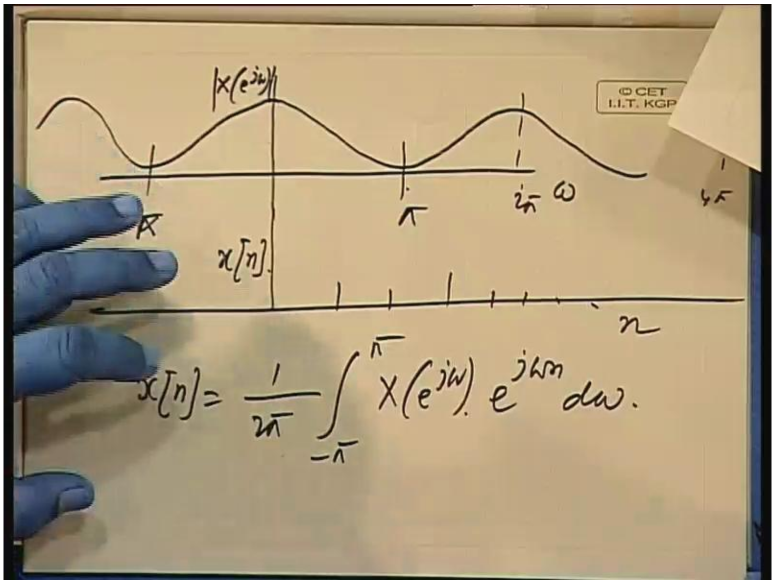

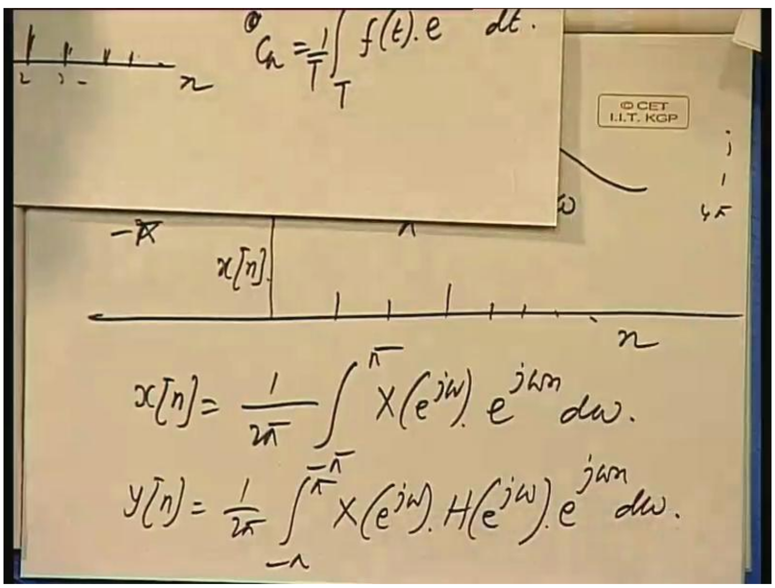

So, if I now take omega and X e to the power j omega, u take it magnitude as well as phase; either way it is periodic function, may be like this and this is 2 p i minus to plus p i, 2 p i and 4 p i and so on, so it is a periodic function. In the time domain, if it is a periodic function then in the frequency domain; it is a discrete discrete line, is it not.

So, you just stretch your imagination in the frequency domain, if it is periodic then what is it in the time domain? It will be lines and precisely that is what we started with, it was x n, is it not, all right. How did you compute these magnitudes in the normal Fourier series? You integrate say a 0's and sorry a i's and b i's, how did you compute or complex C n?

Integration over one period then the function f t then e to the power plus j omega n t, d t was it plus or minus, minus okay. So, in the time domain if it is periodic; the corresponding frequency domain constants are evaluated from here, divided by T, okay. In this case by the same argument, I can evaluate x n, what is the periodicity? 2 p i 1 by 2 p i minus p i to plus p i, okay; X e to the power j omega, e to the power I will take plus j omega and d omega okay, that will give X n.

So, once you have got X, sorry you have got X and H and if they are convolved in the time domain; so in the frequency domain you have to just take their products to calculate output and then take inverse of that product, and for taking the inverse will referred to these expression. So, our Y n will be 1 by 2 p i minus p i to plus p i product of X into H, all right.

So given two sequences, take the frequency domain transform, take the product then take the inverse; that is integration over the period 2 p i, after multiplying by e to the power j omega, okay. So, we will stop here for today and we will continue further in the next class. Thank you very much.