**Digital Signal Processing**
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**Lecture - 5**
**Z-Transform**

We just give you the definitions of Z transform, we shall continue with that.

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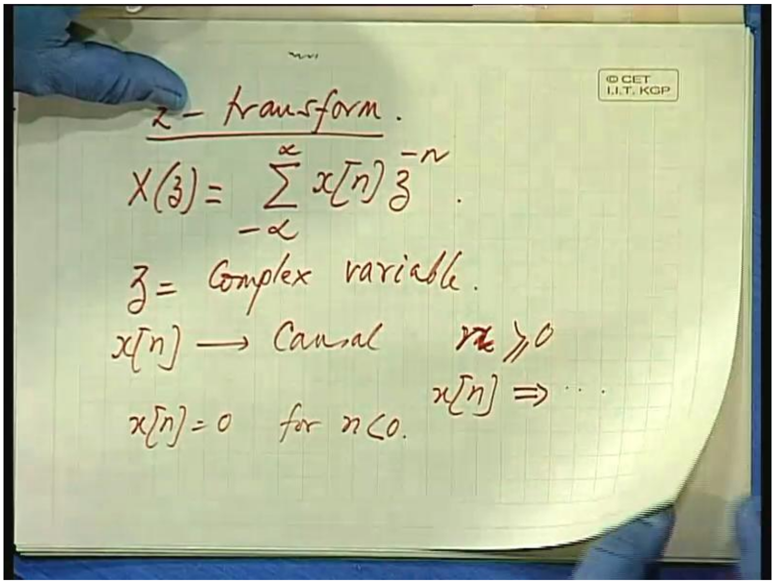


We will see some other important properties of Z transform. We defined, x z as sigma x n z to the power minus n, n varying from minus infinity to plus infinity, z is a complex variable. Now, if x n is causal, if x n is causal; causal means it exists only for n greater than or equal to 0, all right for n greater than or equal to 0, x n exists, okay. This is non-zero, may be zero but x n is 0 for all negative n, this a causal sequence.

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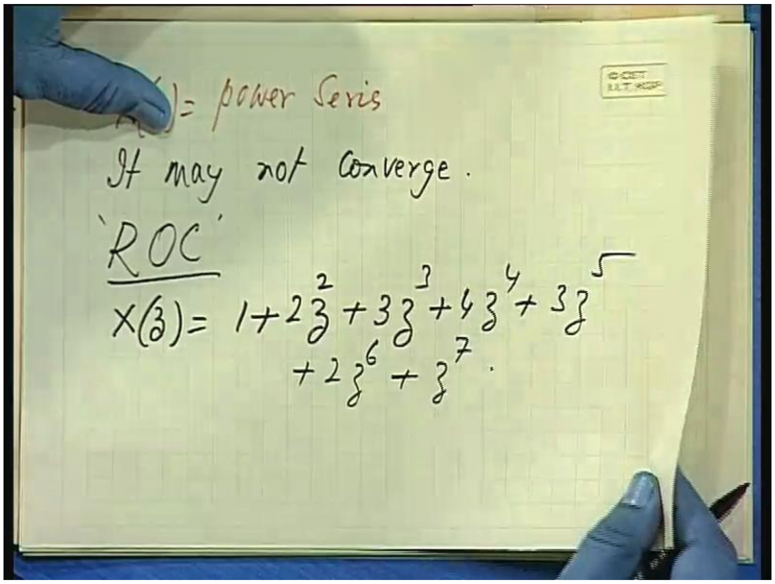


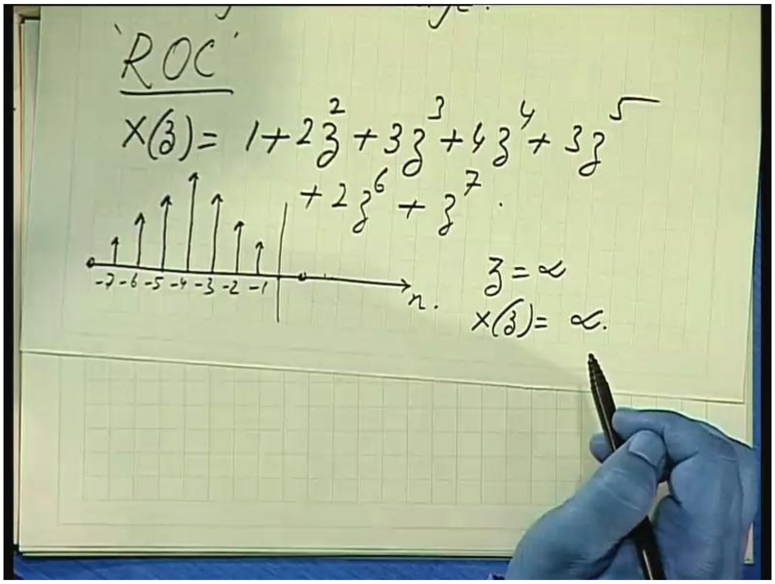


Now, x z is a power series. Let you can see, x z it may not converge at all, it may or may not converge. So, let us investigate under what situation it may not converge, the region in which it converges we call region of convergence or in short R O C, okay. So will put the boundaries where it will be converging or it will not be converging.

Let us take the following series, x z equal to 1 plus 2 z 2 plus 3 z 3 plus 4 z 4 plus 3 z 5 plus 2 z 6 plus z to the power 7. What is this sequence; in terms of z to the power minus one, if we try to see. It is a sequence which exists in the negative domain of time okay, zero, one, two, three, four, three, two, one and then zero. So minus 1, minus 2, minus 3, minus 4, minus 5, minus 6, minus 7 this is n. Now, for such a sequence where for negative values of n only x n exists and this is all zero. The value of x z will be this or does it exist for all values of z, at z is equal to infinity, x z becomes infinity; we would like to see when the function becomes infinity, that means it is not converging.

When z is very very large as it is tending to infinity, the function exist also tends to infinity. So, the function exists except at z is equal to infinity. Let us take, the similar sequence but with a shift.

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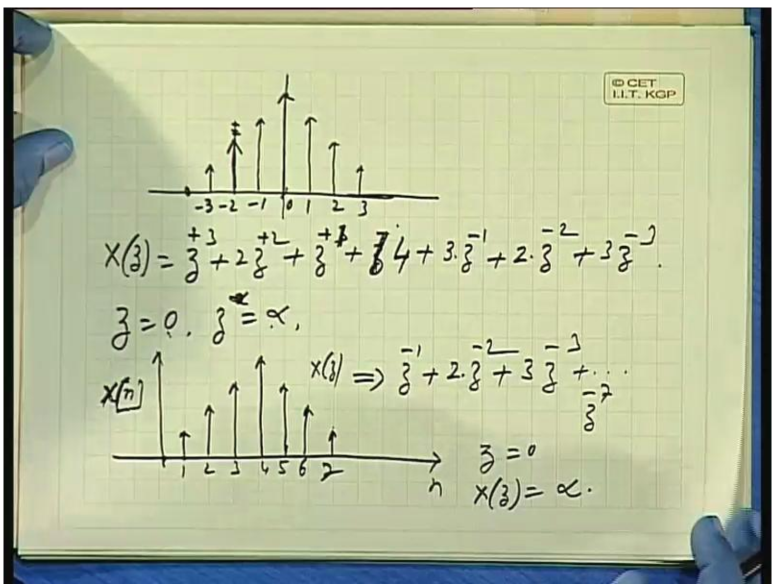


Suppose, we have the same sequence with a shift four, three, two, one, three, sorry sorry this is two, one, zero; so minus 1, minus 2, minus 3 and then this is 1, 2, 3, 0. What will be the value of x z? z to the power minus 3 starts from here plus 2 into z to the power minus 2 plus z to the power minus 3 plus z to the power minus, is it z to the power minus 4? It should be minus 1, then 4 into z to the power 0 plus 3 into z to the power it is sorry, plus 2 plus 1, is it all right, now 2 into z to the power minus 2 plus 3 into z to the power minus 3, okay.

Now, what are the values for which x z becomes infinity? z is equal to 0 it becomes infinity, z is equal to infinity also it becomes infinity. So, its region of convergence will will exclude these two points; at these two points the function becomes infinity, is that all right? If we shifted further by four more steps then we will have a function like this, zero, one, two, three, four, three, two, one okay.

What will be the sequence, corresponding sequence z to the power minus 1? It is x z, z to the power minus 1 plus z to the power minus 2 into 2 plus 3 into z to the power minus 3 and so on, at the end z to the power minus 7, 1, 2, 3, 4, 5, 6, 7 all right. Now, when this function becomes infinity that you have to identify, that you can see from here; when z is equal to 0, anybody? z is equal to zero, okay when z is equal to zero x z is infinity.

So, a sequence that we have considered in the three cases is having the same nature of variation, but it is shifted all right from left to right and the region of convergence also changes; earlier it was not converging for z is equal to minus z is equal to infinity then z is equal to 0 and infinity and now z is equal to 0. So, region of convergence will be depending on the nature of the polynomial, nature of the function x z that you get.

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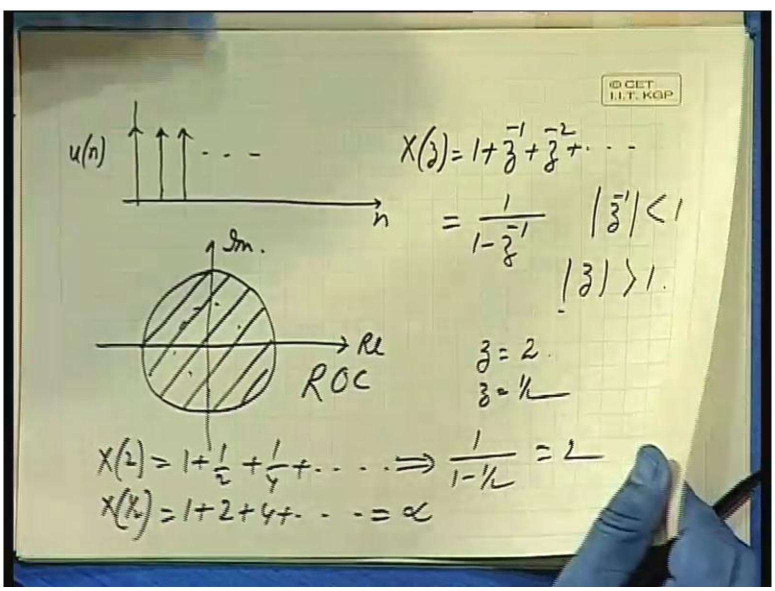

Let us consider this sequence, well known step function which is one, one, one and continues. So, what will be corresponding Z transform? x z, 1 plus z inverse plus z to the power minus 2 and so on; it will be 1 by z inverse, it is g p series provided, z inverse is less than 1 or z is greater than 1, okay.

So, the region of convergence, z is a complex quantity. So, in the complex z plane it will define a unit circle, okay. So, this is a forbidden range and this is R O C, the space outside is a region of convergence; region of convergence is the space outside, this is real and imaginary part, all right. Now, for this sequence you can see for yourself, if I put z is equal to 2 then it is 1 plus half plus x at 2 becomes 1 plus half plus 1 fourth and so on.

And this a g p series with reduced magnitude of these terms, so this will be tending to 1 by 1 minus half which is 2. If I put z is equal to half then it becomes 1 plus 2 plus 4 plus and so on, tends to infinity, okay. So, any point inside if I choose, I have just taken half you can say choose any good, any value, any complex value it will be diverging.

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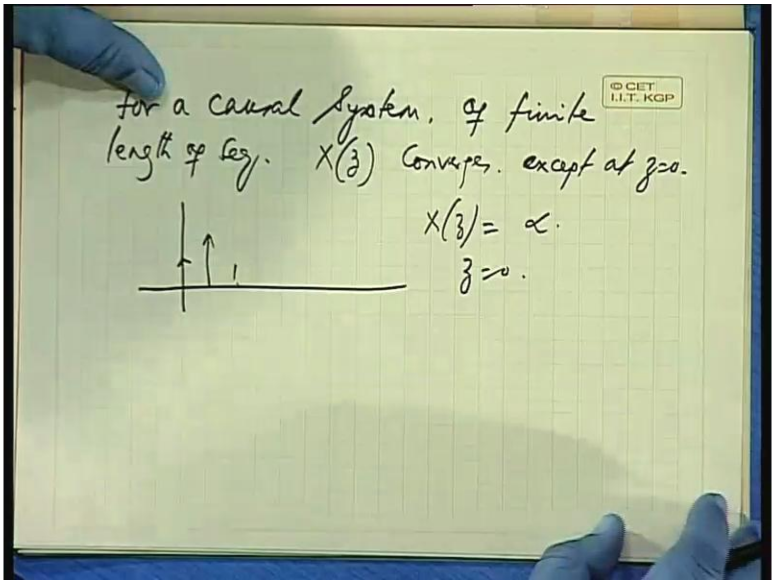

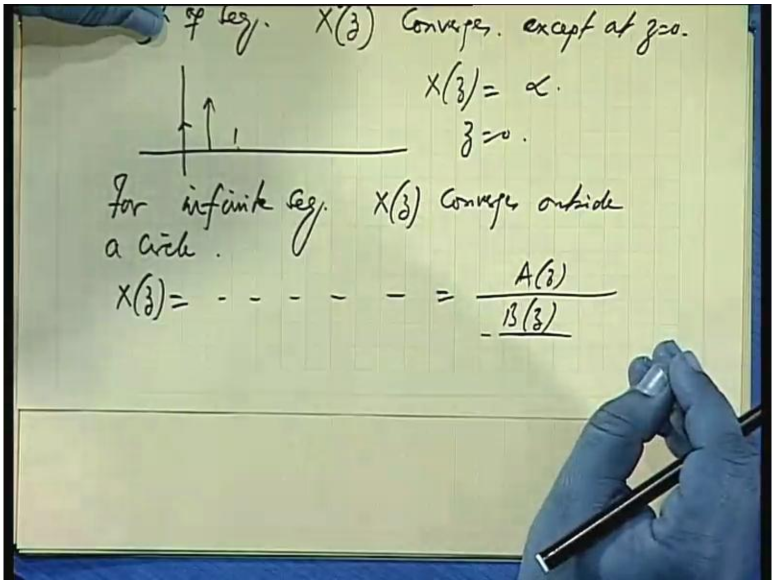


So for a causal system, for a causal system of finite sequence length of finite length of sequence; x z converges because you have got finite number of terms, okay except at zero, except at zero your causal sequence means, the terms will be x zero plus x one into z to the power minus one and so on, all right.

It will be causal sequence means; it will have some value here, some value here and so on. If I put z is equal to zero, z to the power minus one will tend to infinity. So all the terms will be giving you finally, x z will tend to infinity, at z is equal to zero. For infinite sequence, that is sequence of infinite duration x z converges outside a circle. The circle is bounded by the poles okay; poles means if it is an infinite sequence, all right if you can resolve it in terms of some A z by B z, then the roots of B z will be called poles.

So, at those points at those values the function will try to blow up. This becomes zero means, the function will be having an infinite values. So at poles, the value of the function will be infinity all right so that will be excluded, the region of convergence will have to exclude those poles, all right. You must have studied in complex algebra, analytic functions; so wherever there are poles, there will be the function will try to go to infinity, so for an analytic function it excludes those poles, is it not?

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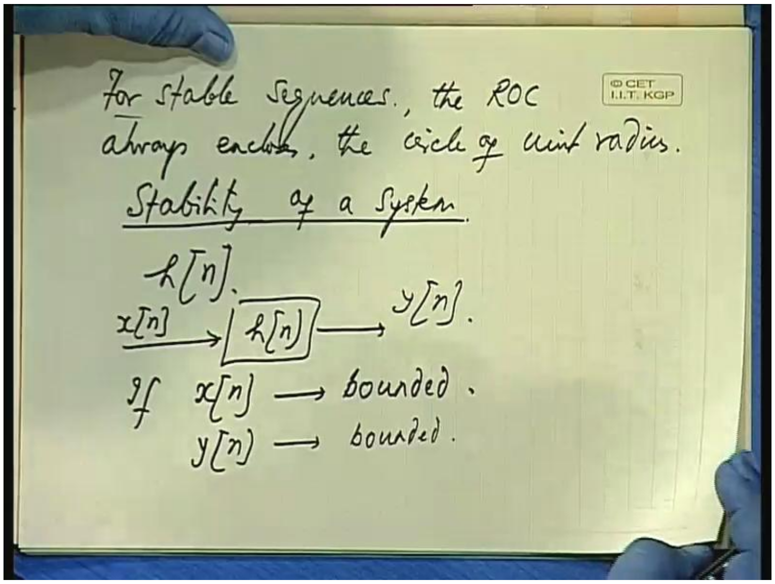


For stable sequences, we have not yet defined stability, what stability? So, next we will take up stability, for stable sequences the R O C, the region of convergence always encloses the circle of unit radius; this point we will discuss after we define what stability is, okay. Stability of a sequence of a system; when we define the impulse response of a system, impulse response of a system h n, this dictates the barrier of the system all right.

Now, what are the conditions on h n, what are the conditions on h n for stability? What do you mean by a stable sequence or a stable system? If this is driven by an input which is bounded then output must be bounded; that means if x n is bounded for a bounded input sequence, if x n is bounded then y n also should be bounded for a stable sequence h n, all right. If that is violated, then the system is unstable all right.

What is it actually telling us, it says that; if I give a input sequence whose magnitude does not grow beyond a certain value, whose maximum value is limited then the output will also have a maximum value which will be limited, which will be less than infinity, output cannot tend to infinity if it does, then it is an obstacle system, all right.

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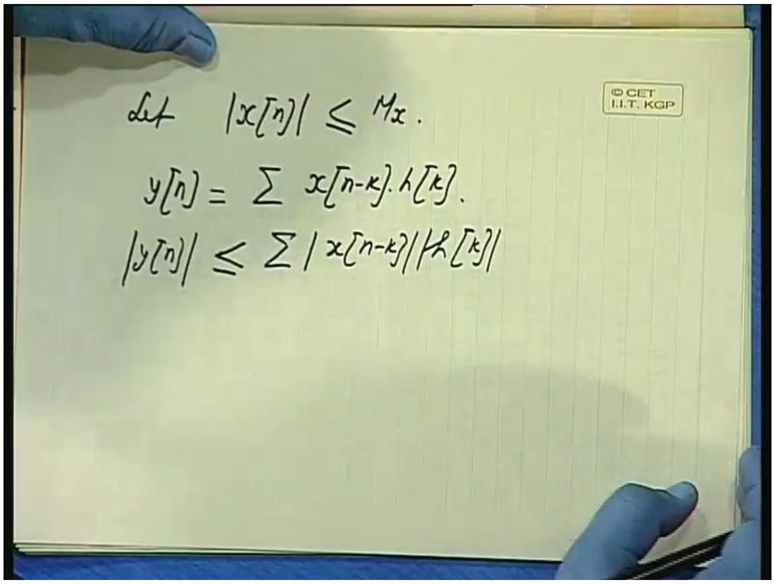


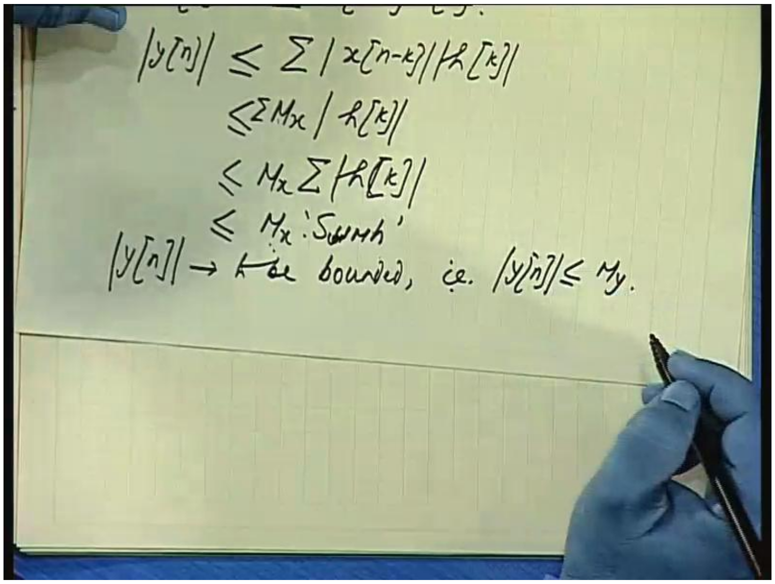


So, let us see the proof of this. Let the magnitude of x n the maximum value, be restricted to M x, so x n can be at the most equal to less than M x. Now y n is sigma x K okay, x n minus K into h K, is it not? So, y n magnitude will be less than or equal to magnitude of x n minus K into magnitude of h K, do you all agree? It has to be less than the product of the two magnitudes and then submitted.

So that means, it must be less than if I take the maximum value of this which is M x into h K; it is definitely less than this product, I taken the maximum value of this summation. I can put the summation, since this is a constant. Summation sign here, so y n if this summation is denoted as sum sum h, I call it sum h to me; it must be less than this product all right.

Therefore, sum h must be finite for magnitude of y n to be finite, to be less than a quantity finite quantity. This to be bounded, that is if this is to be less than or equal to sum M y then M y must be less than M x into sum h therefore sum h must be less than infinity, it must be a bounded quantity, it must be a finite quantity, okay.

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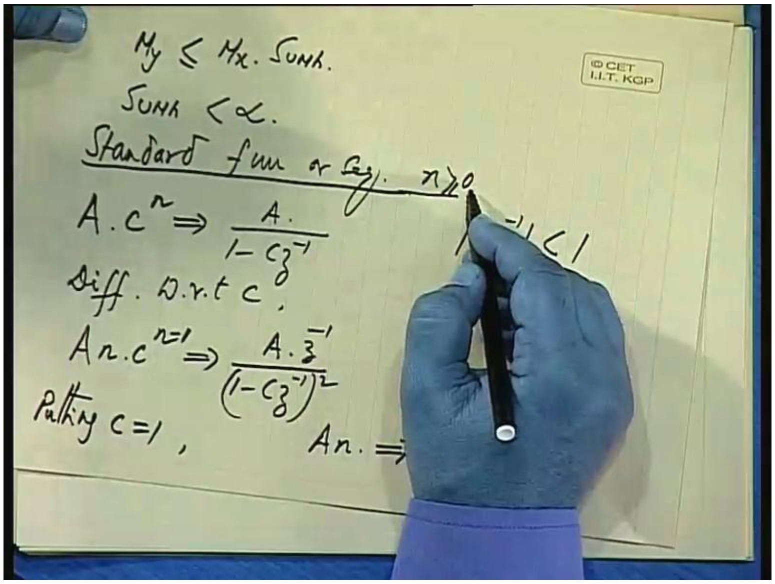


So this is the condition for stability of a system. You add up the magnitude of all the values of h n, it must be converging to some finite value. Now, let us take some standard functions and evaluate the Z transform, standard functions for sequences. We have seen A into C to the power n, if you take the summation; it will be A by 1 minus C z inverse, provided this magnitude is less than 1, is it not? This we saw last time, C to the power n if you take the Z transform the series that you get, this is a common ratio, so that magnitude must be less than 1.

If I differentiate with respect to the common quantity C, as you have done in case of Laplace transform, if you differentiate with respect to C differentiate with respect to C then I get, A n C to the power n, mind you; I am considering sequence for which exist for n greater than equal to 0, that is we are taking only causal sequences, okay. And I list out the corresponding region of convergence.

So, A n into C n will give me n minus 1, thank you. If I differentiate, I will get A by 1 minus C z inverse whole squared into z to the power minus 1 okay, minus 1 and minus 1 make final 2 minus 1 will give me plus, what will be the region of convergence? The same C z inverse must be less than 1. If I put C is equal to 1, what do you get; A into n will give me A into z inverse by 1 minus z inverse whole square, what is A into n, if you remember what is A into n?

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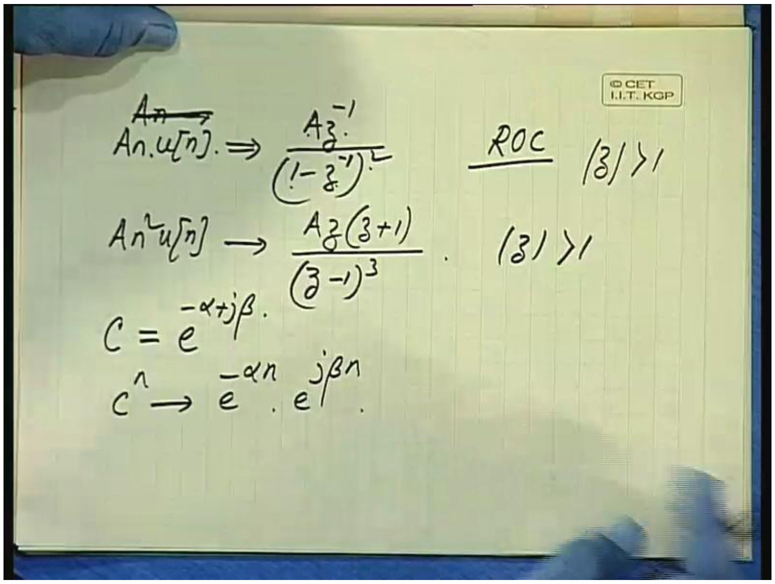


A into n is basically ram function, moment I have defined n greater than equal to 0, so all these functions should have their implied u n as a multiplier, all right so, it is basically a ram function, it is implied though it is not mentioned because I am taking only n greater than equal to 0. So, A z inverse by 1 minus z inverse squared, now what are the regions of, what is the region of convergence, I leave it to you to find out the condition where it becomes infinity, okay.

Similarly, A n squared u n try to check whether you get this; I can always multiply by z to the power minus 2, z to the power plus 2. So, this will become A z divided by z minus 1 whole square, okay. In many books we will find, they write in terms of z all right, Z transform and this becomes z minus 1 whole cube; derivative it yourself and again z greater than 1, z inverse less than 1 means z greater than 1, okay.

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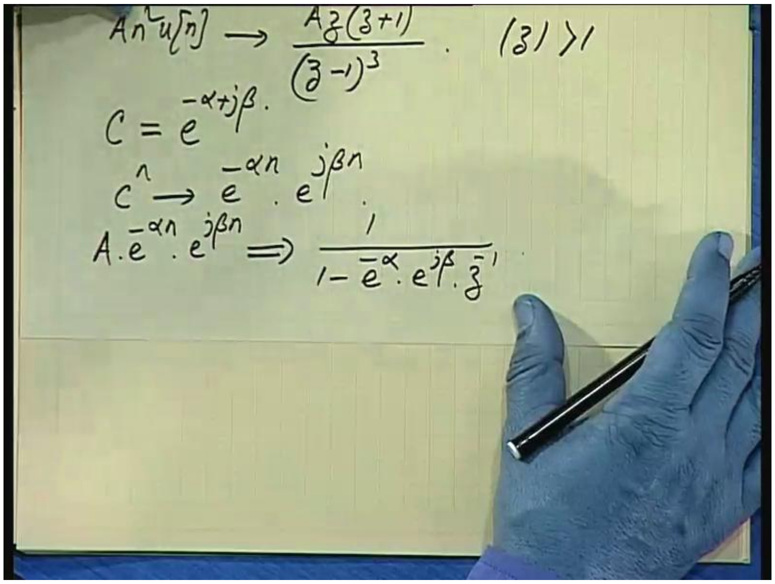

If C is, once again we come back to the first relation involving C. Suppose, C is equal to e to the power minus alpha plus j beta then C to the power n means e to the power minus alpha n e to the power j beta n, do you all agree? Now, what will be the Z transform of this sequence? So, A into e to the power minus alpha n into e to the power j beta n, you just put in place of C this quantity.

So, it will be 1 by 1 minus e to the power minus alpha, e to the power j beta into z to the power minus 1, okay. I just substituted for C e to the power minus alpha plus j beta, if you segregate the real and imaginary part; what do you get? Let us see.

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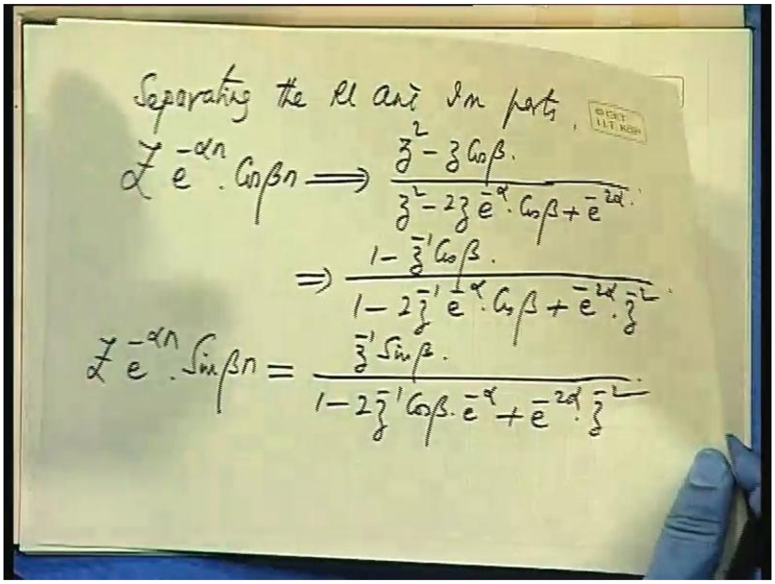


Separating the as you are done in the case of Laplace transform, separating the real and imaginary parts and equating them on both sides; we get Z transform of e to the power minus alpha n cosine beta n, as z squared minus z cosine beta by z squared minus twice zeta twice z e to the power minus alpha cos beta plus e to the power minus twice alpha, okay. Some books write e to the power minus alpha is just r naught, okay.

So you can do that, because e to the power minus alpha appears as it is. One may write also this as, 1 minus z inverse cos beta, this is also written in many books in this form; e to the power minus alpha cos beta plus e to the power minus twice alpha z to the power minus 2. Similarly, Z transform of e to the power minus alpha n sin beta n, will be z to the power minus z to the power minus 1 sin beta divided by the same denominator, okay. We shall be using it later on in partial fractions in certain situations that we will see very soon, I will keep it aside.

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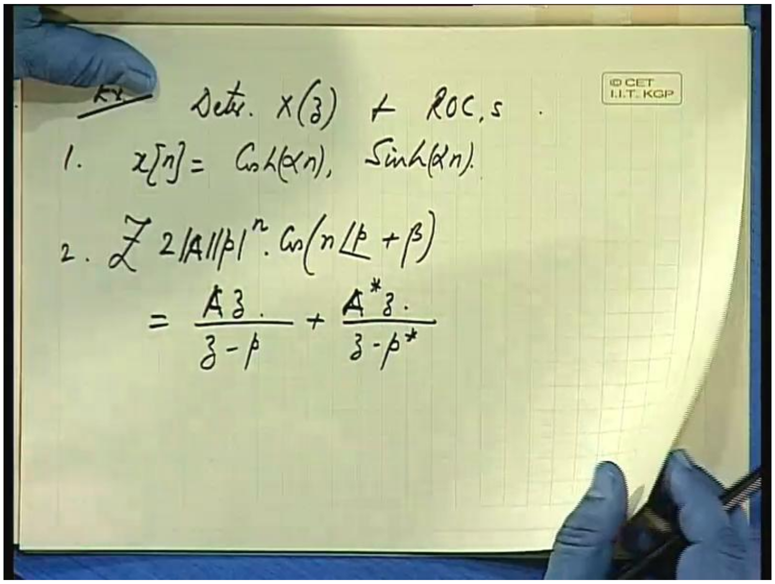


I would like you to try, try to find out the R O C’s determined X z and R O C’ s for the two functions; x n equal to cos n and sin n, okay, alpha n, cos alpha n, sin alpha n. And second, prove that Z transform you try this yourself e to the power n, I am writing this C is basically A, okay; A cos n angle of p plus beta is C z by z minus p plus C star z by z minus p star, A sorry I have change it to A, okay to avoid confusion with a earlier C, okay.

So, A is a complex quantity all right. Now let us see, how to in you can evaluate the Z transform of many such functions with the same simple principle of addition.

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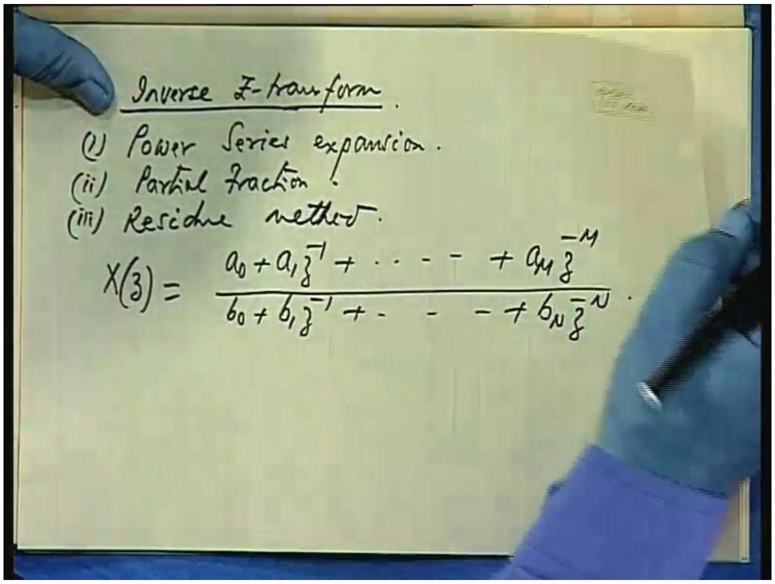


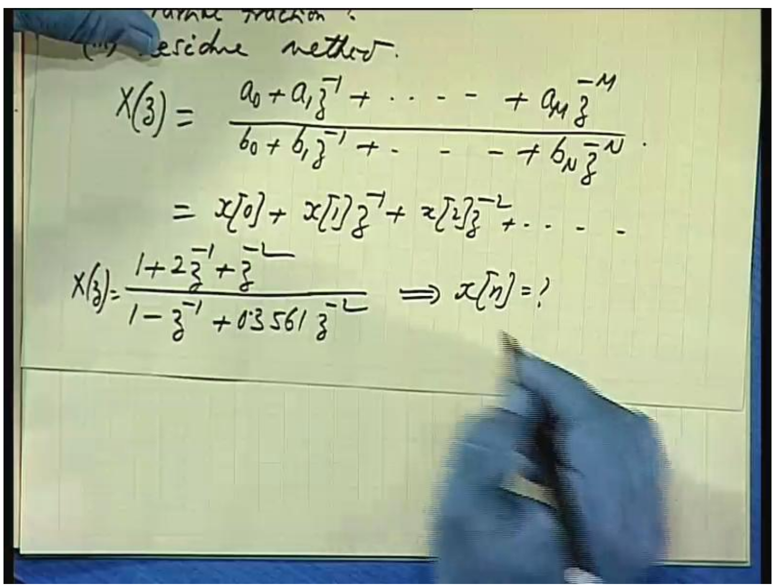


Now, let us see how you get the inverse of these functions? Given a function X z, how do you find out the sequence x n that will be our next task, inverse Z transform okay. So, there are three standard methods very simple, some of you must have studied in control systems also. Power series expansion, second one is partial fraction and third one is residue method, residue calculation, okay.

So the first one, X z suppose; we give you X z as a 0 plus a 1 z inverse like this a polynomial a m z to the power minus m divided by b 0 plus b 1 z inverse plus b N z to the power minus N. By long division, you generate this as x 0 plus x 1 z to the power minus 1 plus x 2 z to the power minus 2 and so on; all right you can divide it and then get these.

For example, if I have 1 plus 2 z inverse plus z to the power minus 2 divided by 1 minus z inverse; say I have taken a simple quantity, z to the power minus 2, 0.3561, z to the power minus 2, what will be the sequence if X z is given by this ratio of two polynomials, what will be x n? It will be an infinity sequence.

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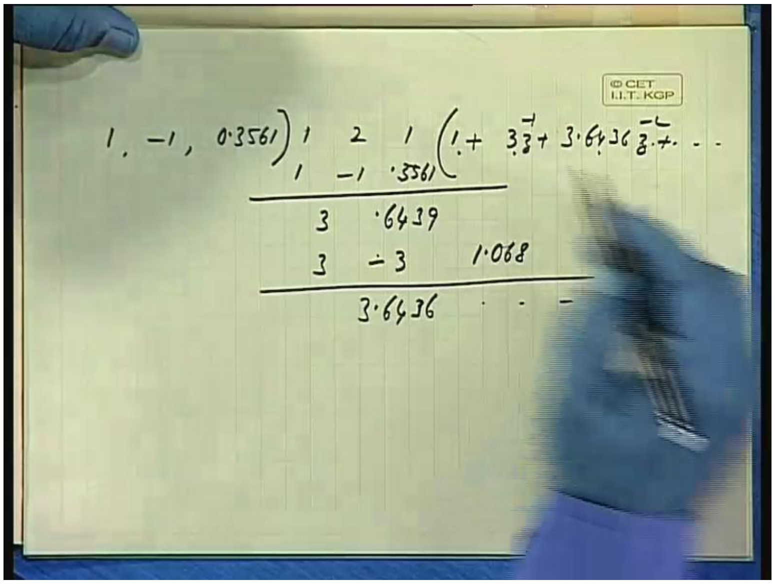

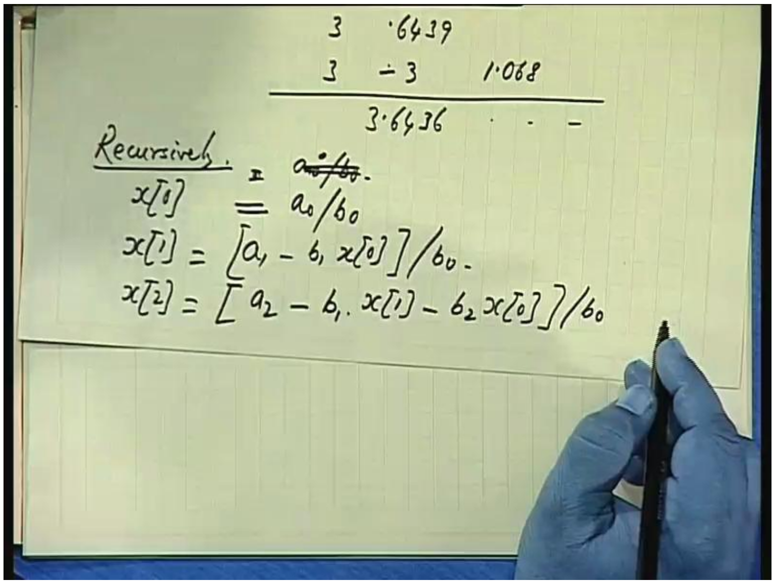

So, you divide it, take the denominator of coefficients 1, minus 1m, 0.3561; I am not writing z to the power zero, z to the power minus one, z to the power minus two, just put little gap in between the coefficients; similarly one two one, 1, 2, 1, okay. So, the division gives me 1, 1, minus 1, 0.3561. So, after subtraction I get 3.6439 then next division gives me, 3, 3 then minus 3, 1.068l; three into this much so that gives me, 3.6436 and so on, okay.

So, next value will be 3.6436. Now this is x zero, this is x one, this is x two and so on because this is coming as a constant. What you are getting is 1 plus 3 z inverse plus into z to the power minus 2 and so on, so this will be endless, so will be the sequence. So, the coefficients of z inverse, z to the power minus 2 and so on, they will represent x zero, x one, x two, okay it is as simple as that.

Recursively, we can write this as the same result we can write x 0, x zero is basically a 0 by b 0, okay sorry it should be here, okay. x 1 will be a 1, sorry b 1 into x 0 whole thing, divided by b 0. x 2, I write one or two more terms and then a 2 minus b 1 into x 1 minus b 2 into x 0, divided by b 0.

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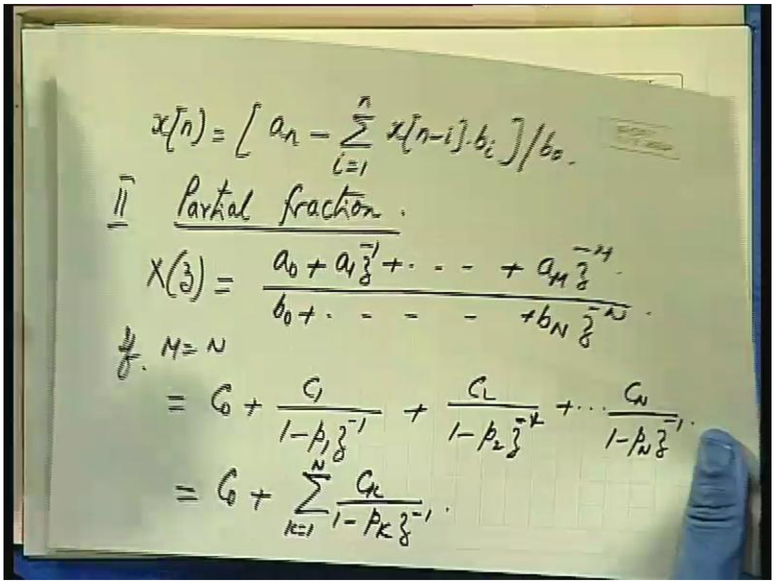


So generalizing this, it will be X n equal to a n minus i equal to 1 to n, x n minus i b i by b 0. So, you can use, I would request you to write as short program. Take any polynomial a and b, a z and b z and then get the values of x n from this recursive relation, okay may be up to n is equal to ten or twelve. Write a short program and then see for yourself, give the coefficients and then find out the values, okay.

The next method is partial fraction, X z equal to a 0 plus a 1 z inverse etcetera plus a n z to the power minus m divided by b 0 up to b N, z to the power minus N. If M is equal to N then we can write; if M is equal to N then we can write this as, some C 0 plus C 1 by 1 minus p 1 z inverse plus C 2 by 1 minus p 2 z to the power minus 1 and so on.

C N by 1 minus p N z to the power minus 1 equal to, I can write this as a very general term C K by 1 minus p K, z to the power minus 1 okay; K varying from 1 to N. Now, from here see from the polynomial in the denominator, we have evaluated the routes so the routes have to be evaluated here and then you can write this one, for each one of them you know the solution.

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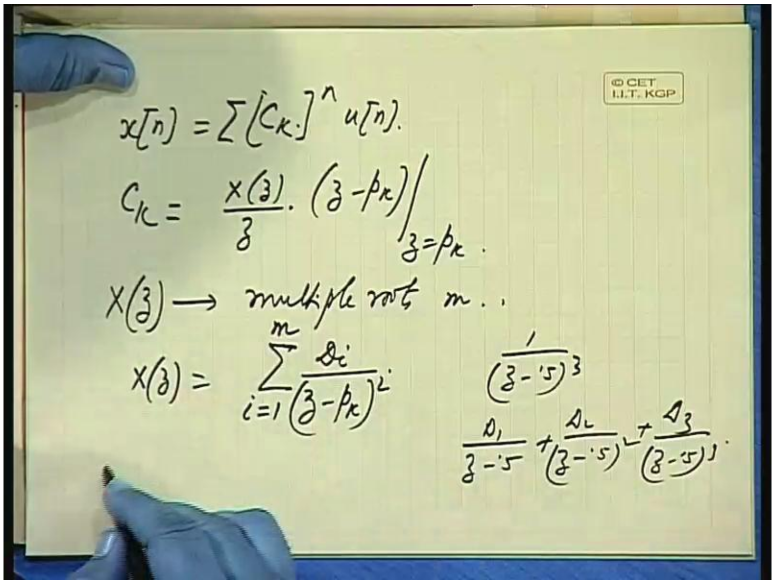


So, X n will consist of this sequence C K to the power n summation C K, okay. C K, C 1, C 2 etcetera all to the power n, u n all right. And all we evaluate C K, it will be X z by z into z minus p K evaluated at z is equal to p K, as you have calculated residuals in case of Laplace transform. If, X z has multiple roots, multiple roots say; m number of multiple roots m number of multiple roots then that is say, x z contains a term like this, D i by z minus p K to the power i, i is varying to m varying from 1 to m, okay.

There is m number of routes are there, at the same place p K then the corresponding residues; so if it is 1 by z minus say 0.5 to the power 3, then I will have a term say D 1 by z minus 0.5 plus D 2 by z minus, sorry z minus 0.5 squared plus D 3 by z minus 0.5 cube. So, these D i's, the general term D i can be evaluated m minus i factorial, sorry excuse me m minus i d z, m minus i, z minus p K, exactly identical method as you have done in case of Laplace transform, okay.

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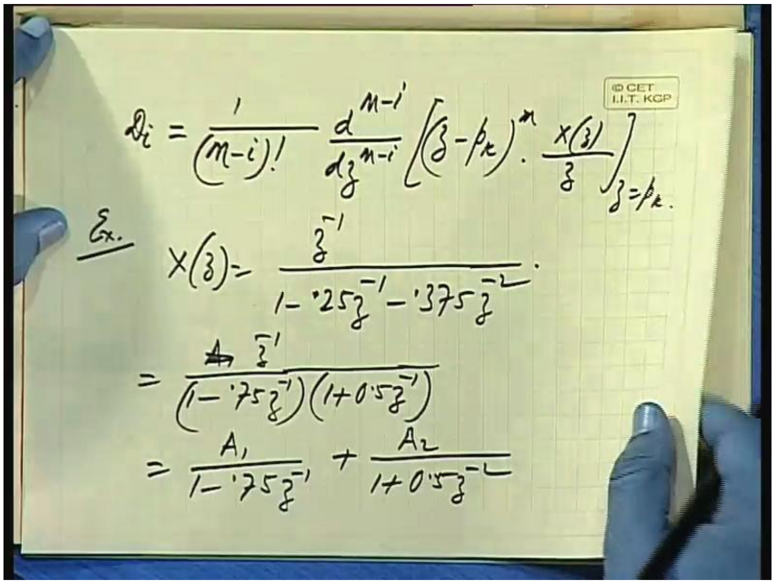


Let us take one or two small examples. Suppose, X z is given whereas z inverse by 1 minus 0.25, z inverse minus 0.375 z to the power minus 2, I can evaluate the residues; this is factorable you can this is quadratic, you can find out the factors, okay.

So, 1 minus 0.75 z inverse; okay let me write in the factor factored form first, 1 plus zero 0.5 z to the power minus 1, okay, the factors are actually like this. So, this I write as A 1 by 1 minus 0.75 z inverse plus A 2 by 1 plus 0.5 z to the power minus 2, okay. So A 1 and A 2, one may try very simple combinations of 0.75 and 0.5.

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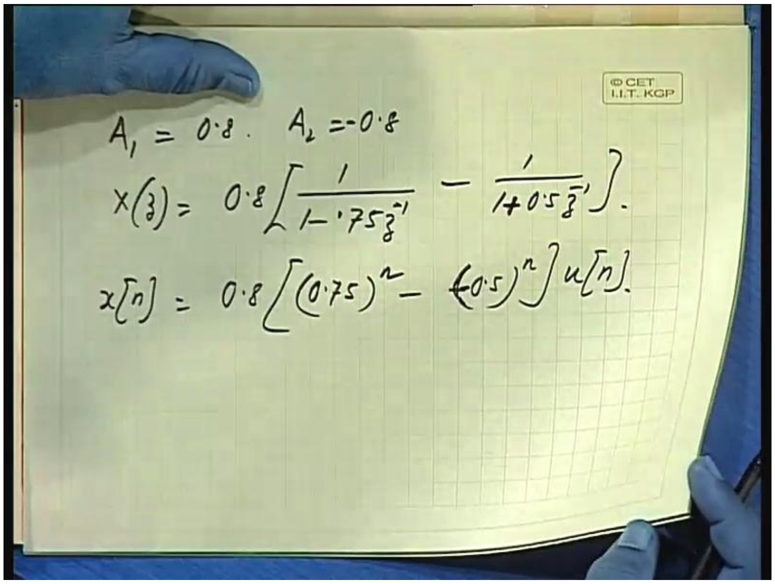


You can see A 1 is equal to 0.8; A 2 is minus 0.8. So, X z becomes 0.8 into 1 by 1 minus 0.75 z inverse minus 1 by 1 minus 0.5 z inverse. So, what will be x n, very simple 0.8 corresponding to this it is 0.75 to the power n minus, minus 0.5 to the power n, whole thing multiplied by u n.

So, these are closed. This is minus 0.5 this was 1 plus 0.5 thank you, thank you 1 plus 0.5 z inverse. Now, if B z has the quadratic with complex routes what will you do; if you remember in your early classes, in Laplace transform we studied the complex routes. If the denominator contains complex routes when you try to resolve in say, s plus alpha Whole Square plus beta square form, remember?

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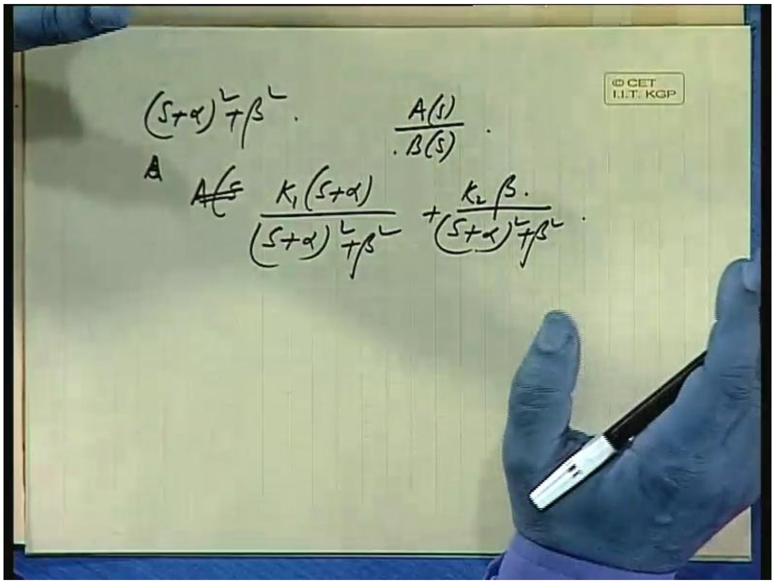

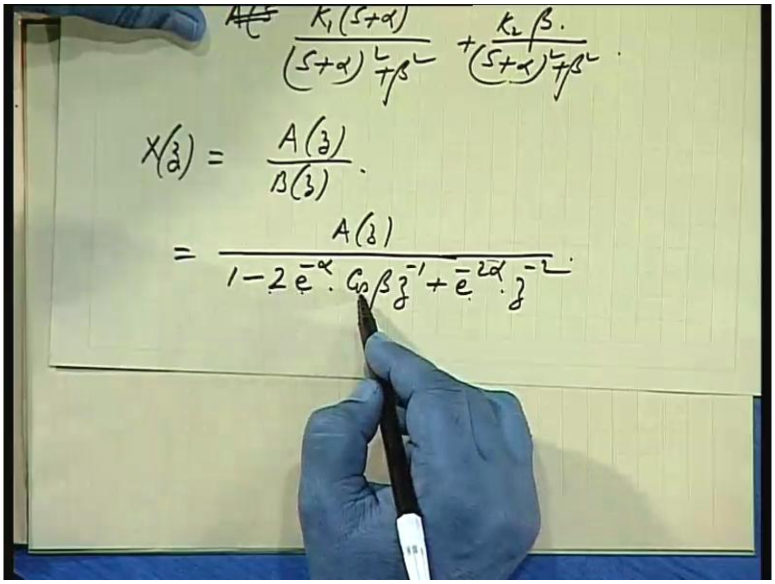

And we tried to say, if you are given some function A s by B s, all right and B s contains a factor like this then you try to dissolve it into; some A into some K 1 into s plus alpha by s plus alpha whole squared plus beta square plus K 2 times beta by s plus alpha whole plus beta square, all right. Corresponding to this, you get a cosine function as inverse and for this you get a sin function, exactly similar procedure we shall follow here.

So, X z, X z if it is given as A z by B z and B z is having a quadratic then we can write B z as 1 minus 2 into some e to the power minus alpha, cos beta z inverse plus e to the power minus twice alpha z to the power minus 2, okay.

So, the quadratic that will be given to you, quadratic that will be given to you will tell you the last term; z to the power minus 2 whatever is a coefficient that will be corresponding to e to the power minus 2 alphas. Take that square root e to the power of minus 2 alpha then the middle term should be equally 2 into e to the power minus alpha into cos beta, so you know cos beta, okay you can calculate sin beta also.

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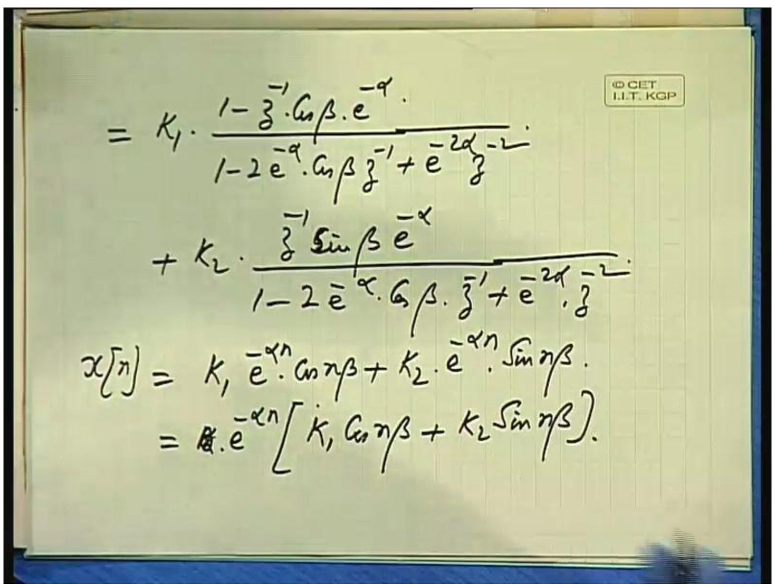


So, this finally you reduce into the same form, some K 1 times 1 minus z inverse cos beta e to the power minus alpha divided by listed denominator plus K 2 times, z inverse sin beta e to the power minus alpha by 1 minus 2 e to the power minus alpha cos beta z to the power minus 1 plus e to the power minus 2 alpha z to the power minus 2.

So, corresponding x n will be after evaluating K 1 and K 2, it is just a matching that you have to do for the numerator function. So, you get K 1 times, what is the inverse of this? e to the power minus alpha, cos beta, okay. So cos n beta and this is minus alpha n plus K 2 times e to the

power minus alpha n, sin n beta, okay. So that, gives me K 1 into e to the power minus alpha, sorry e to the power minus alpha n I can take out; K 1 into cos n beta plus K 2 into sin n beta.
It is better to write in separate forms, sin n, beta n, cosine n beta. Somebody may write cosine n beta plus theta, whatever you want okay.

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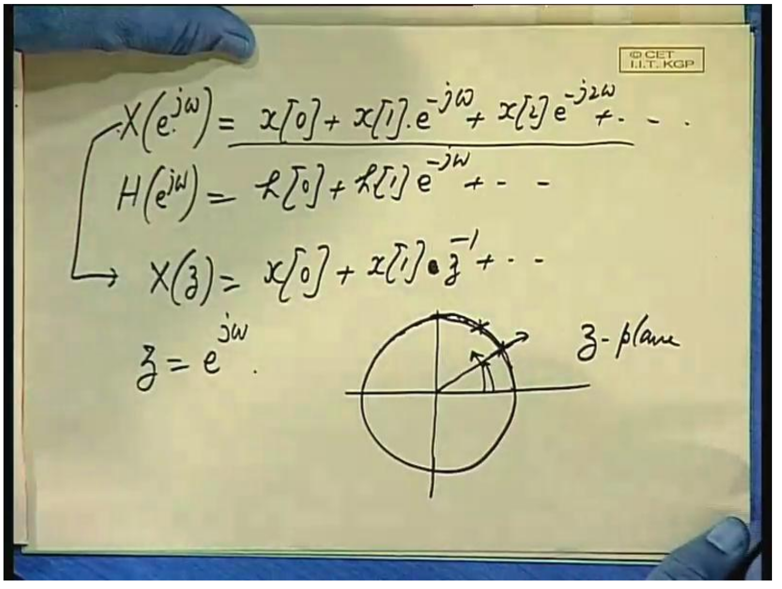


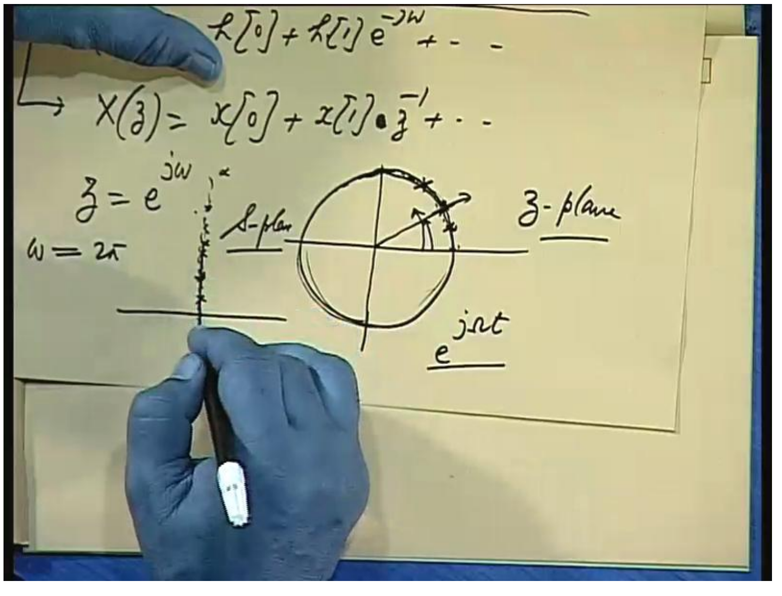


Now, sometimes we try to link up the discrete time Fourier transforms; that is if you remember earlier we are talking about, frequency response of the system or frequency domain representation of a signal, that was x 0 plus x 1 e to the power minus j omega plus x 2 e to the power minus j 2 omega and so on, is it not?

If you remember, we are writing H e to the power j omega frequency response of the system whose time domain response, impulse response was H n. So, this was h 0 plus h 1 e to the power minus j omega and so on, okay. You see, in X z the expression is x 0 plus x 1 z to the power minus 1 and so on; so from the z transform you can directly get the frequency domain representation of the sequence, just put z is equal to e to the power j omega, you get the same same function.

So, in the z domain what is e to the power of j omega, it is a unit circle as omega increases sorry, omega increases along this; e to the power j omega, omega is equal to say two radians, means I will go here by two radians.

So these are the points on the units circle in the z plane; this is a z plane, z plane is a complex plane okay. That means some specific values of z which will be defined on this unit circle, if I take then X z represents nothing but the discrete time Fourier transform of the sequence, we collect DTFT, this is known as DTFT; discrete time Fourier transform of a sequence X n, is it all right?

So just take the unit circle, take the points on this so that presents x e to the power j omega. Now, when I comeback that is when is complete two pi, omega 2 p i, again I comeback to the same point, same valuation, all right. So, I keep on making circles as omega tends to infinity from zero; if i keep on increasing the value of omega on the in the z plane on this circle, on this perimeter, I am just describing the same points, all right which is one another same thing. This is the point that you shall take up later on, when I talk about stability of systems and well in the filter design.

As in the case of continuous domain, you have varied along the imaginary axis of s plane; along the imaginary axis of s plane you have varied the frequency from minus infinity to plus infinity or from 0 to infinity, if you take 0 to infinity. Basically, points on this represent different sinusoids or exponential periodic functions of this nature, is it not?

You consider the frequencies of the imaginary axis. We will try to establish a relation between s plane and z plane parameters, that is if I shift a point in the continuous domain along this, what does it mean in the z plane? Now that mapping of a particular zone in the x plane onto z plane that means from the continuous domain, how to translate it into discrete domain, is that all right. Thank you very much, we will continue with this in the next class.