Digital Signal Processing
Prof. T. K. Basu
Department of Electrical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 4
Frequency Domain Representation of Discrete Signals

Today, we shall be discussing about frequency domain representation of discrete signals.

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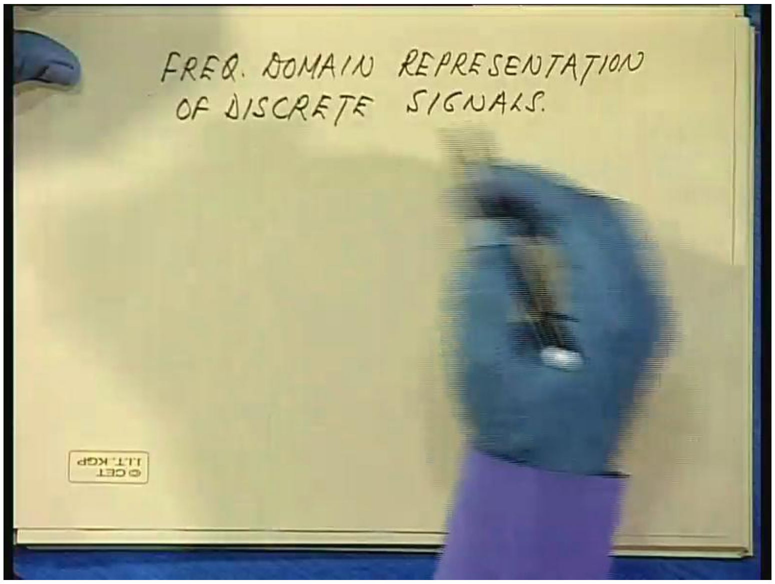

Before we go to the frequency domain representation, we just briefly discuss about how to represent a continuous domain, the dynamic system in a difference equation mode, okay then we will take up frequency domain representation.

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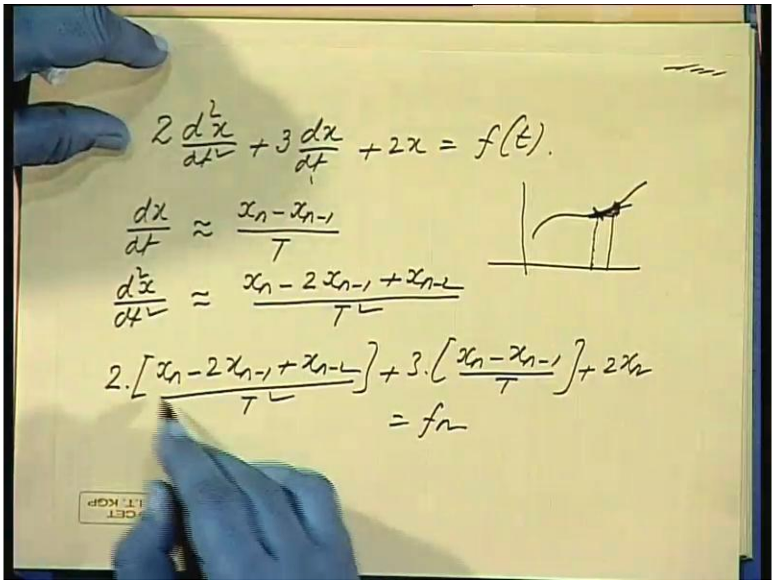

Suppose, you have a differential equation in the continuous domain; 2 into d square x by d t squared plus 3 into d x by d t plus 2 into x, is some forcing function f t. Now, we have approximated d x by d t as, some first order approximation of this kind x n minus x n minus 1 by T, where T is the sampling time, all right.

That means, if there is a continuous function; if you want to evaluate at some interval, at some interval this function is given then the derivative is approximated as the difference between the two successive points and the slope, okay. d square x by d t squared; similarly, will be x n minus twice x n minus 1 plus x n minus 2 by T squared, this we observed last time. So, this differential equation can be written as, 2 into x n minus 2 x n minus 1 plus x n minus 2 by T squared plus 3 into x n minus x n minus 1 by T; 3 into d x by d t plus 2 into x n is equal to f n, okay, all right.

This you can simplify, this you can simplify multiplied by T squared and it will appear finally in this form; some a into x n plus b into x n minus 1, plus c into x n minus 2 is equal to some f n.

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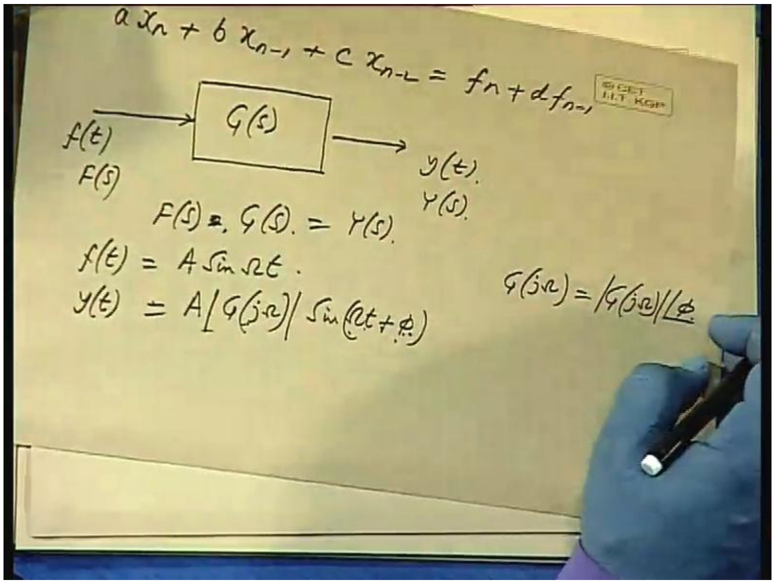

The right hand side could have been also, given in a differential form f t plus some constant time, of dash T and so on. So, that would have generated f n, f n minus one and so on okay. So, this is a general form of differential equation, we can write in the general form, some f n plus a d times f n minus 1 or f n minus two and so on; so this is the difference equation form. That is in the discrete domain, we write a dynamic system equation in this form.

Now, we come to frequency domain representation before we go to the discrete domain, let us concentrate on the frequency response of system in the continuous domain. You are given a transfer function G s, you are having an input f t corresponding output is y t. You convert f t into F s, y t in the transform domain it becomes Y s.

So, F s if you remember is G sorry F s times G s is Y s in the Laplace domain or in the frequency domain, you have also studied the frequency response; if f t is say A sin omega t, a sinusoidal input. If you give a sinusoidal input, what will be the corresponding output? The output can be evaluated very easily and at the steady state condition, it will be A this is multiplied by a gain function which can be obtained by putting j omega, all right.

And then sin omega t, there will be we an additional phase which will be the angle associated with G j omega; if i substitute for s equal to j omega then this will be G j omega magnitude and angle phi. So, that phi gets added with this signal. So, this will be the response of the system y t okay. So in the discrete domain, in the discrete domain we would like to study; if you are giving a discrete signal a discrete signal, discrete sinusoidal function, what will be the corresponding output, let us see, what it means.

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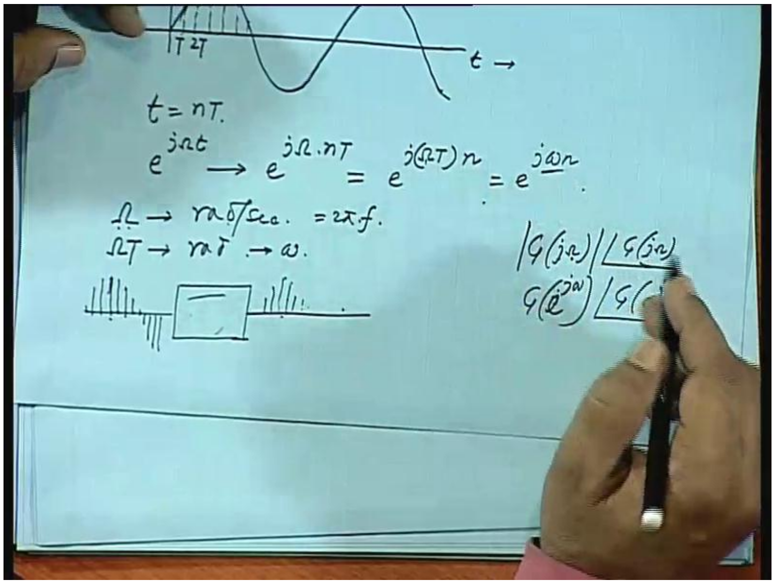

Suppose this is a sinusoidal function, you are sampling it at, regular intervals of time T, okay. So, t is equal to n T, all right. We can say; we are exciting exciting it by a complex sinusoid, I can always write this e to the power j omega t. What will be the corresponding discrete domain representation; e to the power j omega n T, okay. So, that gives me e to the power j n. This capital omega henceforth shall be representing the analogue domain frequency as capital omega that is radian per second.

And omega into T is also a frequency, we call it radian frequency will represent by small omega, okay. Henceforth, this small omega will be used to represent discrete domain frequency which is only radians, all right. And this is in radians per second which is two p i into the normal frequency in hours, okay.

So, this will be e to the power j omega n all right, e to the power j omega n. Now, we are exciting the signal by e to the power j omega n; that means a sinusoid which is discretized. So, what we are trying to see is, if I excited by a signal of this kind; this is the kind of pulse that you are sending all right, that is it is envelope is a sinusoid but is discrete version is taken as a sequence, all right.

You are exciting it by this kind of a signal. And this will give you an output which is also measured at regular intervals. Whatever is output, you are measuring that output also at regular intervals, what will be the ratio of the magnitudes and what will be the changing phase; okay that will be depending on the system's behaviour. So, the ratio of the magnitudes will give you G j omega something corresponding to similar to this function, all right. And the phase associated, that is in find there is a phase shift, so phase will be associated with, again that function G j omega in the continuous domain.

So, we shall be writing something like e to the power j omega; will come to this very soon and angle G e to the power j omega. In the discrete domain, we do not write j omega, we write e to the power j omega because it appears in this form, okay it is a matter of convention.

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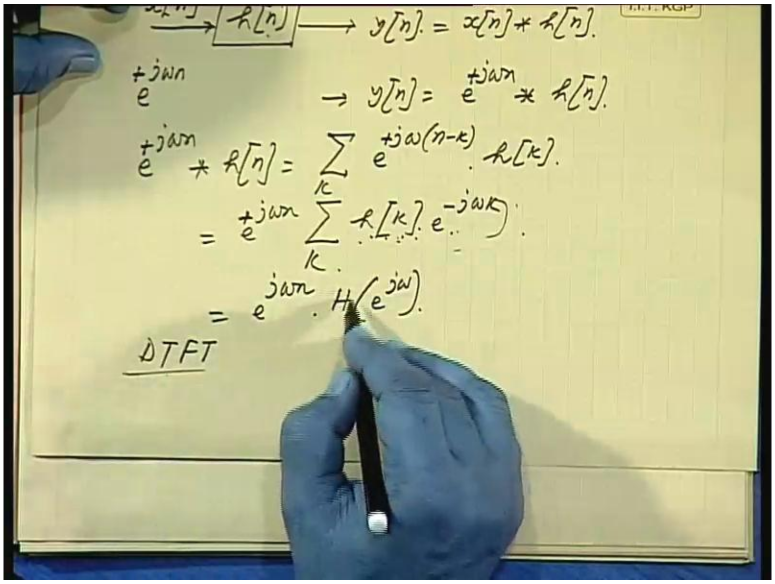

Now, you are observed; if I excite a system whose impulse response is h n, if I excited by x n the corresponding output is x n convolution h n, this we observed last time. Now, suppose x n is e to the power j omega n, so what will be the output? Convolved with h n; okay and what was the convolution expression, if remembered summation e to the power j.

Okay, let us excited by e to the minus j omega n, it will be just e to minus j omega n. So, minus j omega n minus K, h K, do you all agree? I can write e to the power summated over K, e to the power minus j omega n can be taken out; and I can write h K e to the power all right, let us put it as plus itself, okay h K e to the power j minus omega K summated over K, let us write plus j omega.

So now you see, if I excite the system by an exponential complex function like this, the response is; the same function e to the power j omega n, multiplied by this h K e to the power minus j omega K. For a given h K, this is a unique function, a frequency domain function okay. I call this as capital H e to the power j omega, all right.

This represents the quantity here h K e to the power minus j omega K, this is known as frequency response of the system or discrete time Fourier transform of the system, in the discrete domain; in the time domain it is discrete, so discrete time Fourier transform. It is a Fourier transform of the system, h n is converted to H e to the power j omega, all right.

Now, what is the nature of this function, let us see. It is very similar to e to the power j omega H; e to the power j omega is very similar to, G j omega in the continuous domain okay. So, this complex quantity, this is also complex quantity.

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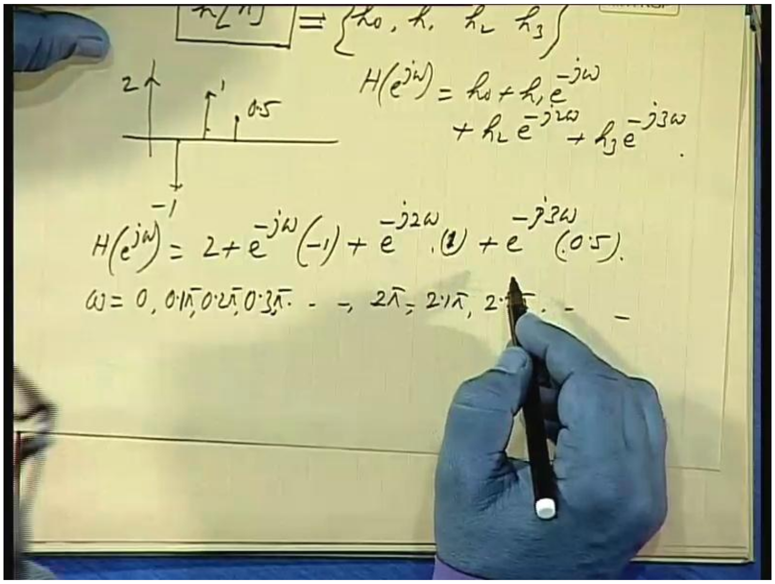

Suppose, I have for a system h n is given as h 0, h 1, h 2, h 3. Like this, 2 minus 1 plus 1 and 0.5, these are the values. What would be the frequency response? H e to the power j omega corresponding to this, it will be 2 plus e to the power minus j omega into minus 1 plus e to the power minus j 2 omega into plus 1 plus e to the power minus j 3 omega into 0.5, okay. It is see; h 0 plus h 1 e to the power minus j omega plus h 2 e to the power minus j 2 omega plus h 3 e to the power minus j 3 omega, is it not? So, I have just use that okay.

You can evaluate this for different values of omega. So, one we take omega equal to 0, 0.1 radian, 0.2 radian, 0.3 radian and so on, you can compute it. You can take instead of such discrete values of 0.1, 0.2; you might as well take 0.1 p I, 0.2 p I, 0.3 p I, that will be very convenient.

Then you can go to 2 p I, then 2.1 p I, 2.2 p i and so on, okay. Substitute these values and then see how it changes. Now, this kind of a function is periodic; can you see this, if I take a step of two p i that is say, omega equal to point one p i, okay what will be the value?

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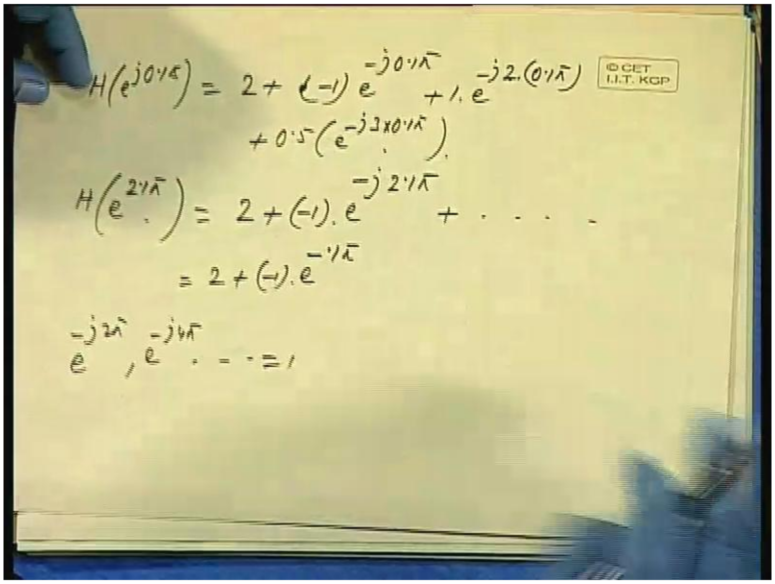

H e to the power j 0.1 p i, it will be 2 plus; okay minus 1 into e to the power minus j 0.1 p i plus 1 into e to the power minus j into 2 into 0.1 p i plus 0.5 into e to the power minus j, 3 into 0.1 p i, okay, whatever be that value. And if I take 2.1 p I, that means after a jump of two p i, what will be the value; 2 plus minus 1 into e to the power minus j 2.1 p i plus all other terms. What is 2.1 p i, e to the power minus j 2 p i and minus 1 p I; so it will be 0.1 p i e to the power minus j 2 p i minus j 4 p I, all will be equal to 1.

So, from the second term and third term will get, it e to the power minus j 4 p i to the minus j 6 p i. So, finally the same terms will be reappearing. So, it is having a periodicity of two p i, okay.

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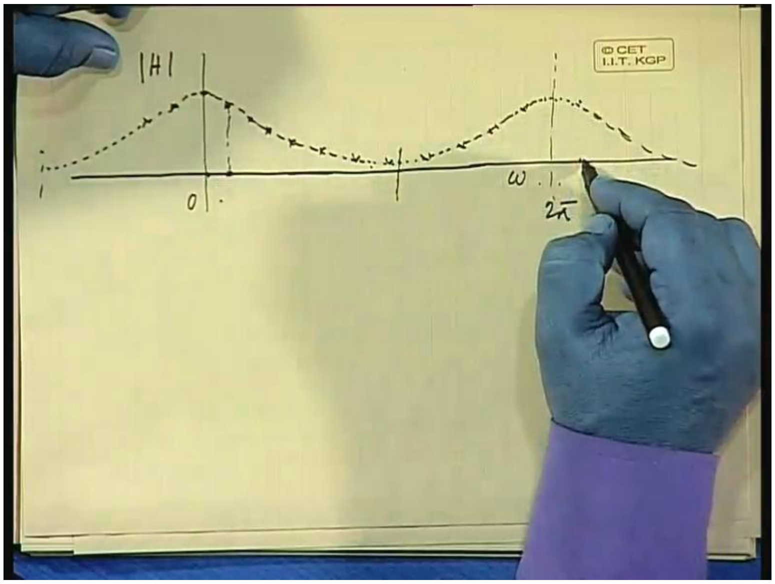

If we make a plot; therefore suppose at 0, what will be the value of this function at omega equal to zero?

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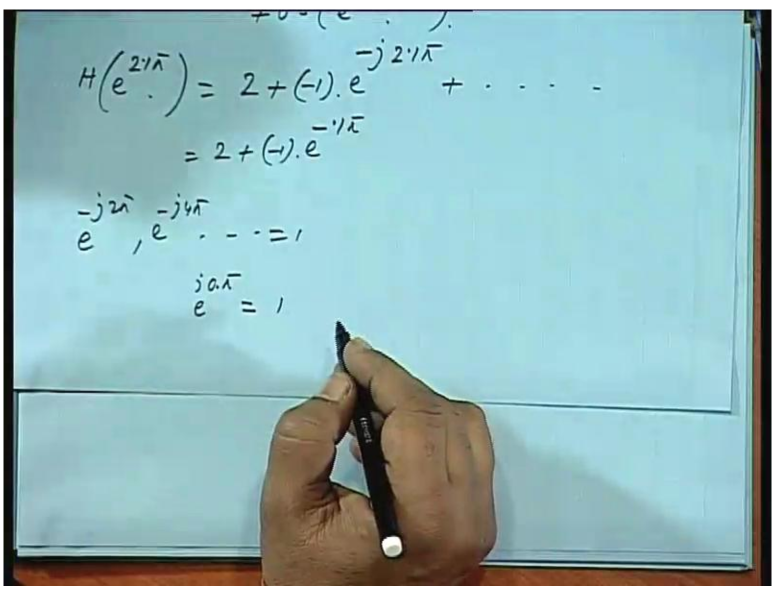

It will be just two; these all be reducing to one, sorry e to the power 0 p i. So, e to the power zero p i, how much is it?... All it will be reducing to 1, so it will be, is it all right?

So, this this will be 2 plus minus 1 plus 1 plus 0.5; whatever be that value it will be coming here. At 0.1, we have measured the magnitude and phase, so if I take the magnitude; now if you take minus omega, if I take minus omega these signs will be just opposite, okay.

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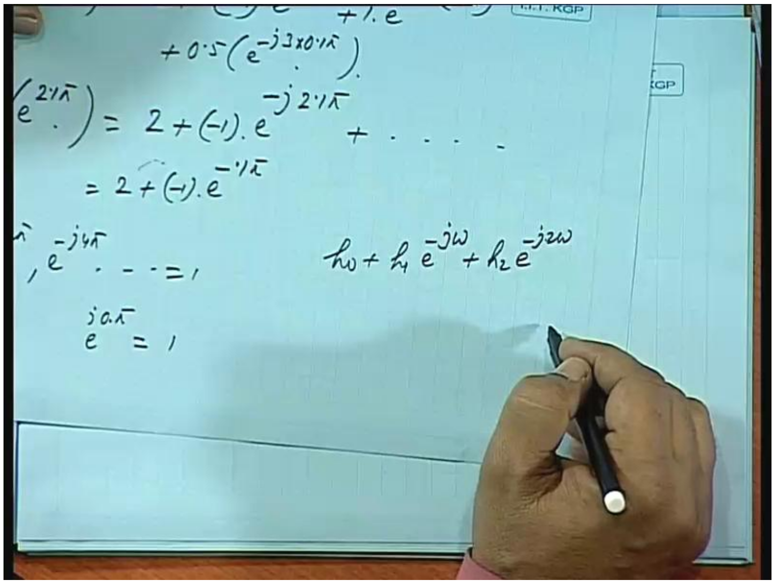

Your series was h 0 plus h 1 e to the power minus j omega plus h 2 e to the power minus j 2 omega and so on. So for any value of omega, whatever imaginary part you get for minus omega, you get just the conjugate, okay. So, the magnitude remains same, it is the phase which will be opposite, all right.

So, magnitude will be for both positive and negative direction of omega, you will get the same magnitudes. So, it may be function like this, okay but then it is symmetric; it will have a repetition after every two p i. On this side also, if you produce it backward then it will be after minus two p i, minus four p i and so on, okay. So, there is a period of 2 p i, the values will be lying here. So, 0.1 p i and 2.1 p i will be identical, is it all right? The response of the system corresponding to 0.1 p i and 2.1 p i will be same.

What is 2 p I, what is it in terms of absolute frequency?

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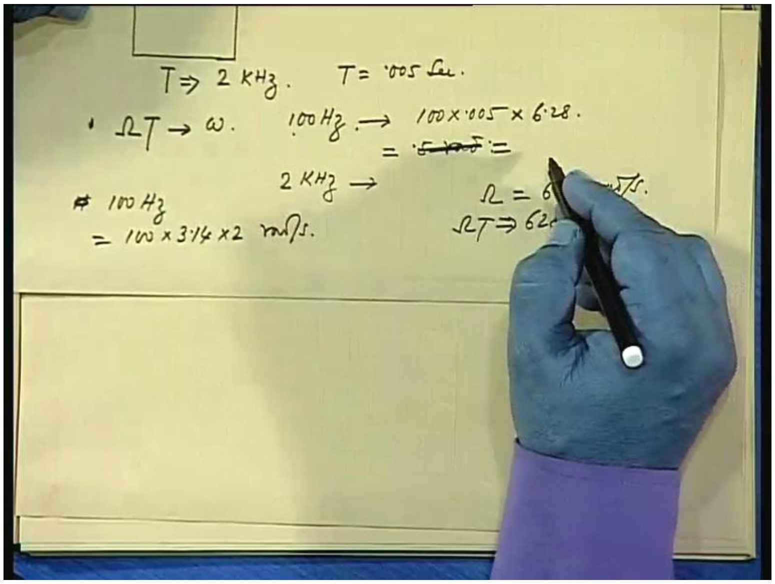

Suppose, you are having a system where you are having a sampling time corresponding to a frequency of 2 kilohertz, is 0.5 second or 0.005, 0.005, T is equal to 0.005 second, okay that means 2 kilohertz.

So, one kilohertz will correspond to half that sampling frequency, all right. So, omega any omega into T which will give you the corresponding frequency; say hundred hertz, if I consider 100Hz, what would be its a presentation in the discrete frequency domain? It will be 100 multiplied by 0.005 radian, all right so, 0.5 radian all right.

2 KHz will correspond to…. is it all right? Point, so hundred hertz, actually 100 Hz is 100 into 2 p I, okay so many radians per second, all right? So, its omega is corresponding to 628 radians per second. So, omega into T will correspond to 628 into 0.005, so it is not 0.5 radian into 2 p I, so 6.28 so that will be the radian, okay.

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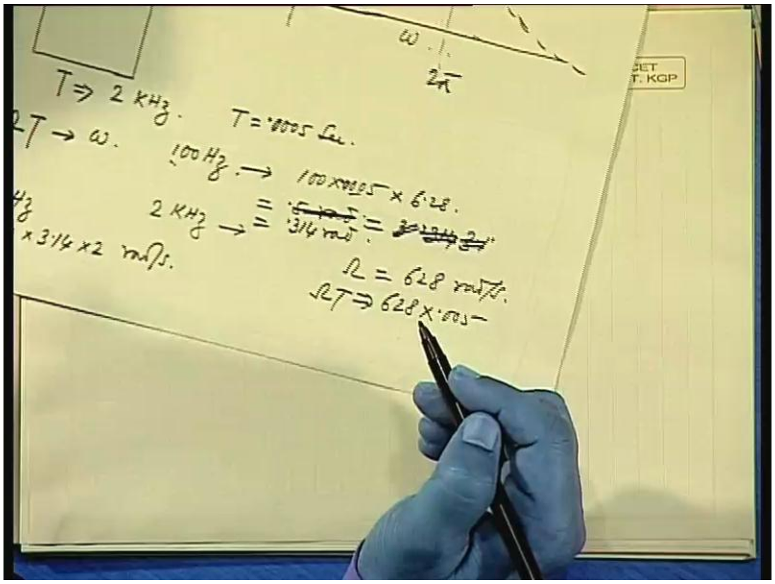

Similarly here, so this is 3. Or 0.314, 0.314 radian, is it all right? No, 3.14 radian okay, hundred hertz will correspond to three point one four; it is hundred, one tenth it will be, you see there is a 0 here… One by two thousand is three 000, is it all right, okay there was a slip. So, this will be equal to 0.314 radian, all right.

So what I meant is, in terms of the sampling frequency, sampling frequency corresponds to 2 p i.

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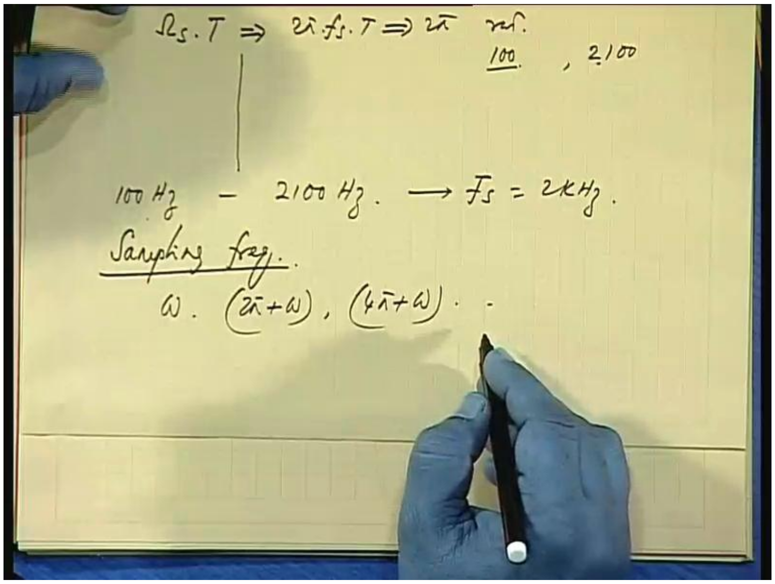

In the radian domain, sampling frequency will correspond to 2 p i, is it not, because; omega s into T is 2 p i into f s into T and f s into T is 1, so that gives me 2 p i radians, okay. So, any fraction of that will be, that fraction of two p i. So, hundred hertz was one twentieth of two kilo hertz, so one twentieth of two p i.

So, 100 hertz therefore and two kilo hertz plus hundred hertz, that is 2100 hertz will appear identical; the response of the system correspond to corresponding to hundred hertz and twenty one hundred hertz will be identical, is it not, that is what you have observed.

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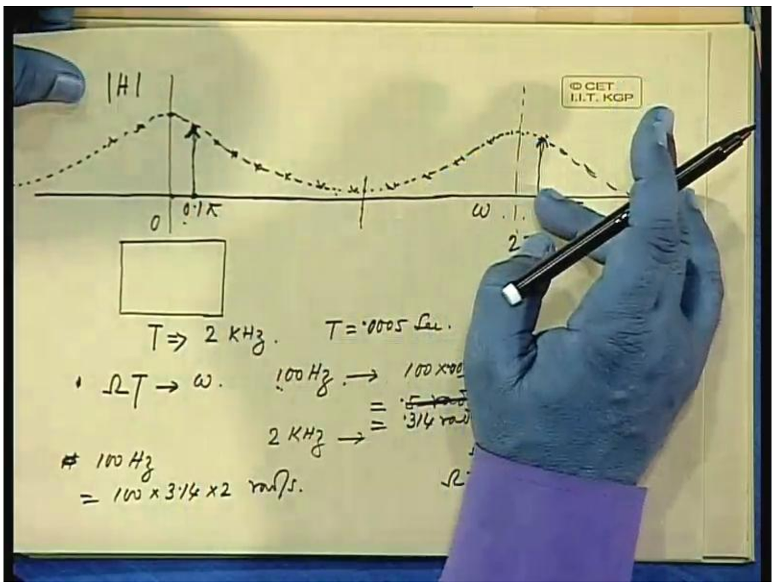

1, 0.1 p i and 2.1 p i, 01 p i corresponds to say; one tenth sorry, 0.1 p i may be corresponding to some frequency like hundred hertz say, and this would be 2.1. That means at a gap of that sampling frequency, two p i correspond to sampling frequency you get the identical response.

So, 100 Hz and 2100Hz will give me the same response when the sampling frequency is 2 KHz. So, if you change the sampling frequency then only you can distinguish the response corresponding to these these two different frequencies, not otherwise, number one.

Number two; if I change the sampling frequency then the response due to these two will not be same, okay. So, sampling frequency is very important, important. So any omega and 2 p i plus omega, 4 p i plus omega will have identical response; physically what it means, let us see.

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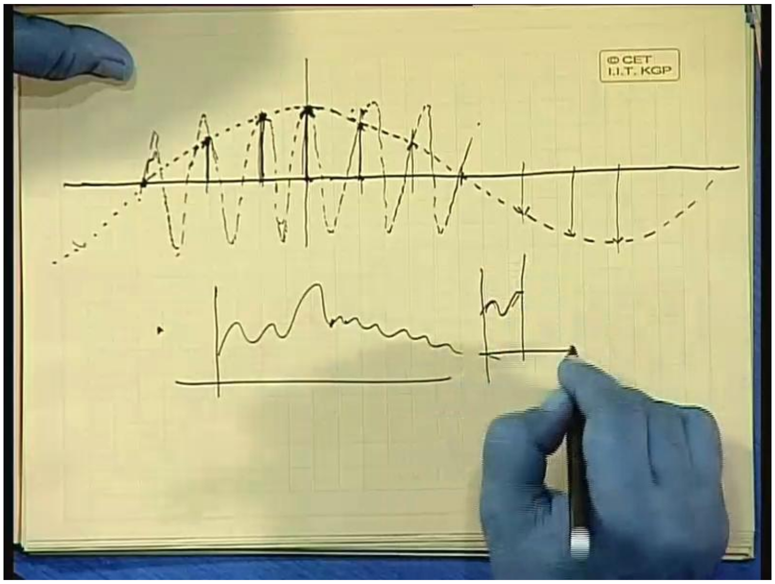

Let us have a sinusoid, I am taking just a cosine function; see varying slowly……. these are the sampling points, okay. I am also having….. another sinusoid like this, okay. Now, I am sampling at this rate; see the sampling rate is relatively slow, means the time gap is sufficiently high sampling time is quite high. You have scanned these values from this signal, somebody else has got another signal he has scanned at the same rate but they also fall on this line. Now which particular signal from these values, these discrete values which particular signals can you identify, there is an ambiguity.

I could have had many more such signals, many more such signals which will have these as the sample points. I am not drawn it for many other frequencies, it is a little difficult to draw it here. Anyway I could have shown you, many more signals can be represented whether same discrete values, okay.

So, there is an ambiguity. So what should be the; suppose the signal is having is like this, is having lot of kings here and there, that means number of frequencies are involved. And what should be proper sampling rate; so that we can trap, all possible frequencies. So, the highest frequency that is present here will have, like this say you take the maximum number of variations in that signal; that means where that high frequency component appears and then sample it there, okay let us see what it means.

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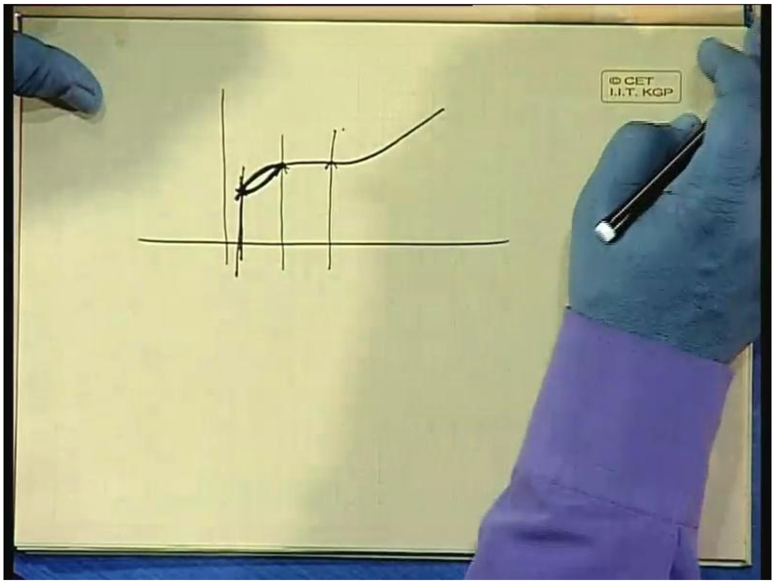

Any function, I sample two points, these are the two values; if I just all okay, let us take these two values from this point to this point the function has gone this way, it could have gone this way also. So, I could have represented with the help of these two sampled points, I could have represented an analogue function; the continuous domain function which has a part like this, that is also possible. So, there is a path like this either a convex or a concave form.

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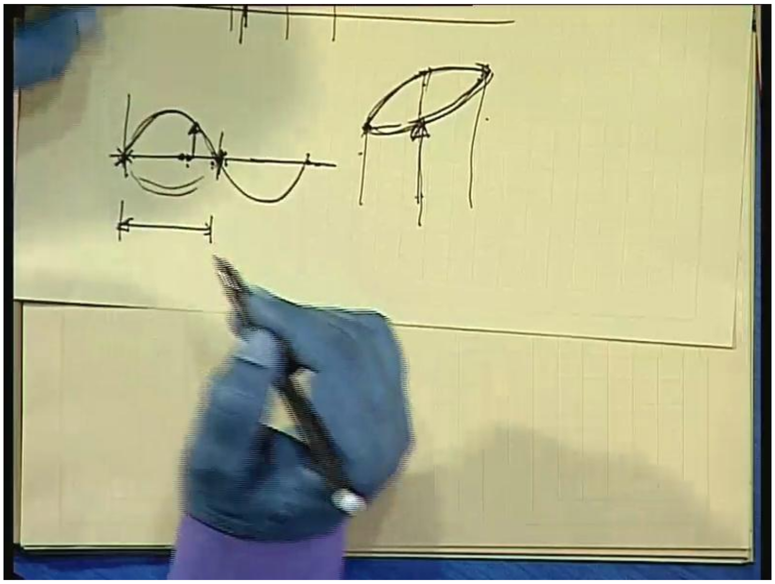

If you want to determine, if you want to determine if between the two successive points; there should not be any ambiguity then it should not go through a maxima, without having another sampling. If I would have had another sampling here then I could have determined, it has gone this way and not this way, another sample point should have been here; that means between say on the two sides of a maxima or a minima, I should have sampling okay. That means, if between two points there is a maxima or a minima then I must have another sampling there, okay.

Let us consider this, if I sample it here and here both may may mean zero; it could have been this way or this way, this is half the frequency present. So, I must have at least a sampler which will sample values in-between. So, if I can sample somewhere here then I know it is going this way, it cannot be this way, okay. This is half the frequency, so I must have a rate faster than half that frequency, sorry this is corresponding to this is twice the sampling frequency.

See, if I sample at this rate, sampling frequencies twice this frequency, okay. So, it must be more than twice that frequency, okay, physical it means this. If I can have a sampler whose sampling rate, your sampling at this rate which is twice the frequency of the signal. This a highest possible frequency presents here, so if I want to trap that, then I must have a sampling frequency which is higher than twice that frequency; that is sampling time should be half of this less than half of this.

This will derive sometime later also, mathematically…. Let us consider for the time being, let us consider Laplace transform of a continuous domain.

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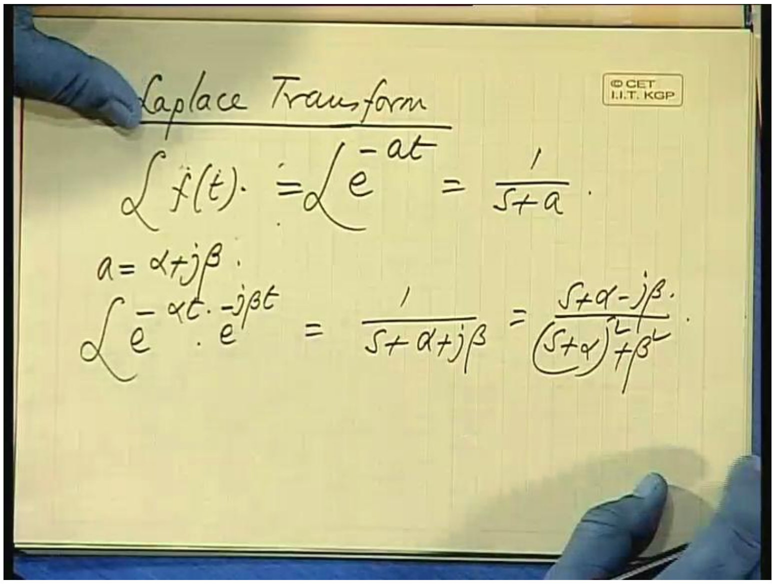

Hurriedly, we shall go through the standard Laplace transforms then we will go for disc domain transform. What is a Laplace transform of function f t; which is say e to the power minus a t? So, we know this is 1 by S plus a, okay. So, if I put a equal to alpha plus j beta then Laplace transform of e to the power minus alpha t, e to the power j beta t; would be 1 by s plus alpha plus j beta, okay which is S plus alpha minus j beta by S plus alpha whole squared plus beta squared.

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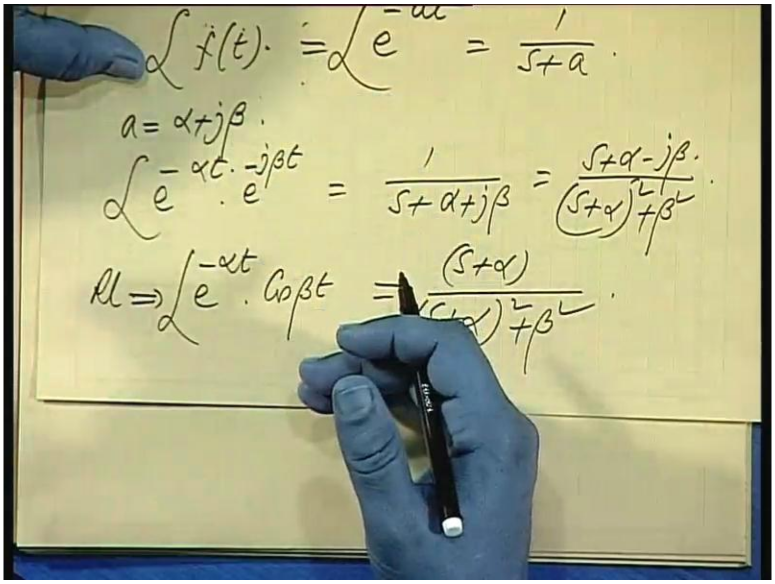

If you segregate the real and imaginary part then the real part of this will be e to the power minus alpha t cosine beta t; so Laplace transform of this will be real part of this which is S plus alpha by S plus alpha whole squared plus beta squared. Why, I am doing this, is this that by the same logic; we should be deriving the Z transform of a different time domain function, discrete time domain functions which need not go through the entire derivation.

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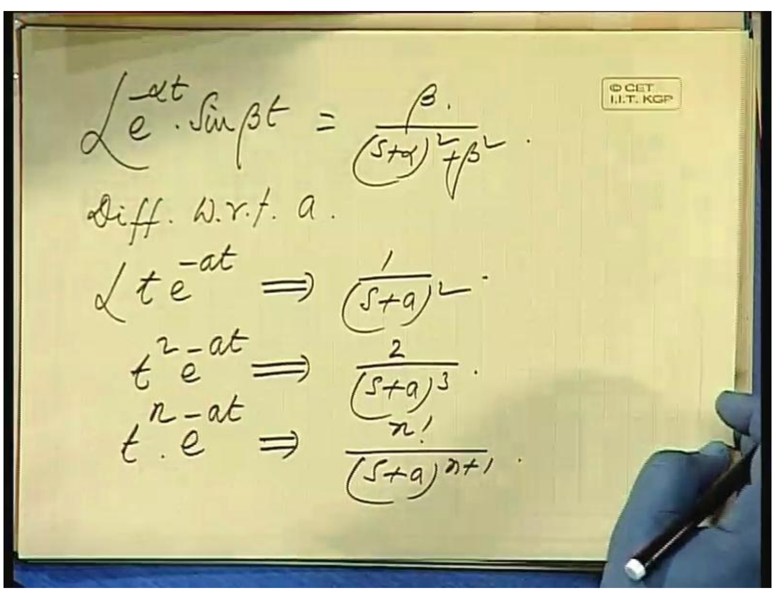

Similarly if I take the imaginary part, e to the power minus alpha t sin beta t will be beta divided by from both sides minus beta t minus j will get cancelled, so it will be beta by S plus alpha whole squared plus beta squared.

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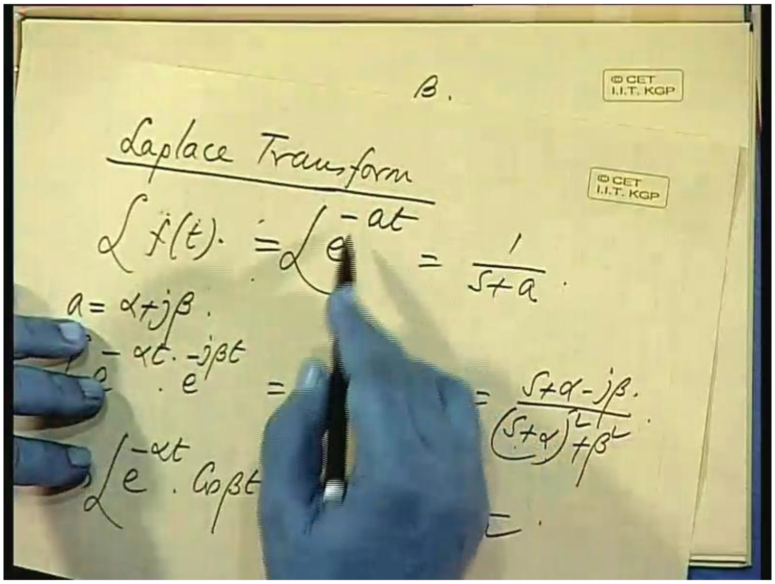

If you differentiate, once again e to the power minus a t, if it differentiate with respect to a, a is the common quantity, common variable. So, if I differentiate with respect to a, I will get t into e to the power minus a t; and that gives me on this side, 1 by S plus a whole squared, we can the negative sign will get cancelled.

You differentiate once again, t square into e to the power minus a t, will give you 2 by S plus a to the power 3. And successively, if you keep on differentiating with respect to a; we will get factorial n by S plus a to the power n plus 1, okay. If you put a equal to alpha equal to zero, you get the Laplace transform for a pure sinusoid sin beta and for sin beta 2, will be makings of this relation very soon.

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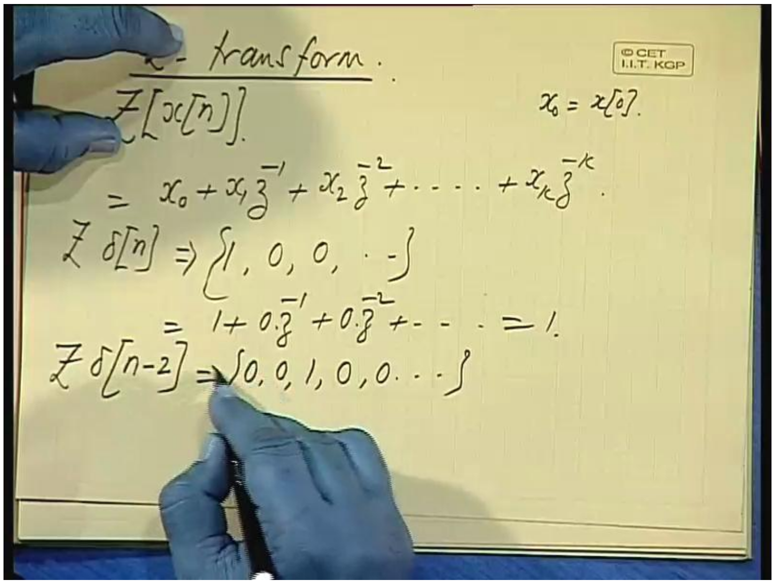

Now, you come to discrete domain transform, known as Z transform. We defined Z transform of a sequence x n, as say transform will be shown like this; capital Z of the function x n or the sequence x n, is equal to x 0 where I write x 0 as nothing but x at 0, in many books they write x at 0 also it is one and the same thing…wherever retains okay. K may tend to infinity; if it is an infinite sequence then it may tend to infinity, it might be finite sequence also, okay.

So, let us see what would be the Z transform of a delta function; it is 1 then 0, 0 the sequence is like this, is it not? A delta function is existing only at n is equal to zero, after that it is all zero. So, what will be the Z transform of this sequence? It will be 1 plus 0 into z to the power minus 1 plus 0 into z to the power minus 2 and so on, which is 1.

What is the Laplace transform of a delta function? That is also one. For a delta function, both in Laplace domain and Z domain the transform function is one. What is the Z transform of a delayed impulse, say delayed by 2 steps, 2 intervals? So it will be 0, 0, 1, 0, 0 this kind of a sequence, okay.

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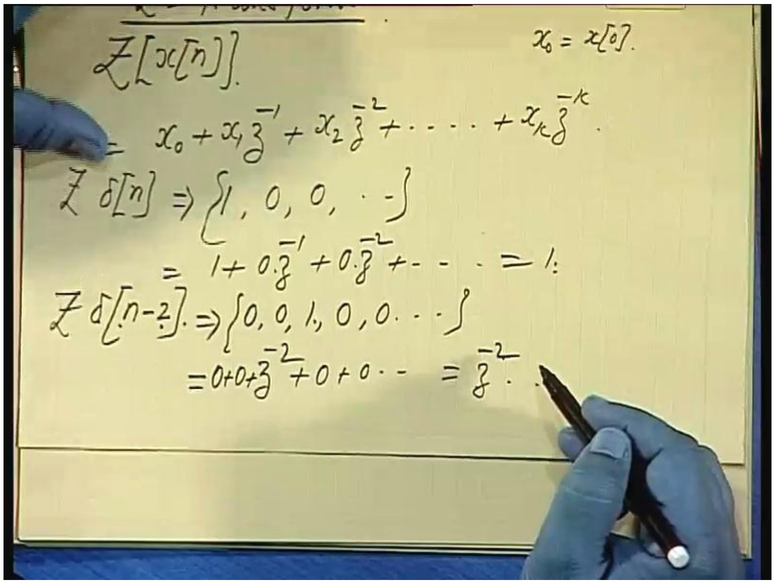

So, what will be the Z transform of this? 0 plus 0 plus 1 into z to the power, minus 2, is it not? So, it is z to the power minus 2. So, if there is a delay of two steps; one gets multiplied by z to the power minus two, okay any x n okay, any x n.

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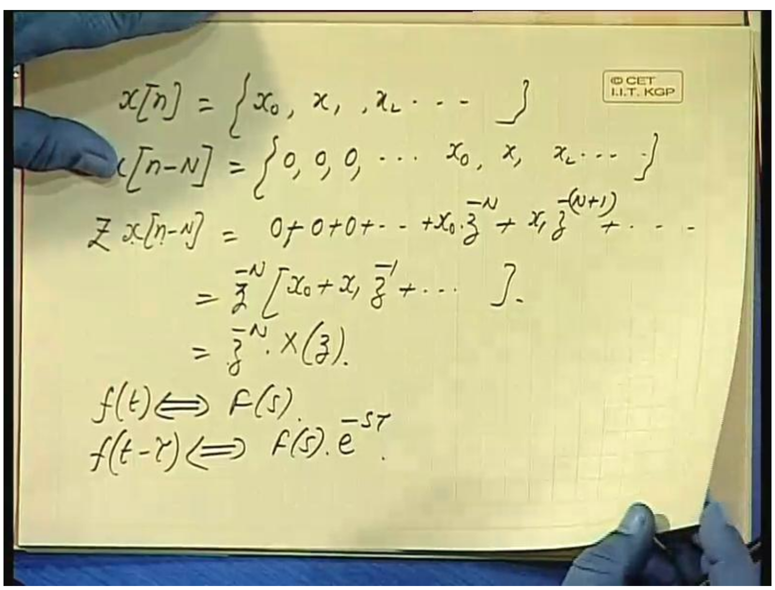

If, I shift by some n steps it will look like; 0, 0, 0 after n steps only, it will appear, is it not? So, what will be the corresponding Z transform? So, 0 plus 0 plus 0 then starts with x 0 into z to the power minus N plus x 1 into z to the power minus N plus 1 bracket and so on. So, z to the power minus n can be taken common, it will be x 0 plus x 1 z to the power minus 1 and so on, which is nothing but x z, okay.

So, x z gets multiplied by z to the power minus n, if there is a shift of n steps. In the Laplace domain, if we remember; if f t gives you F s then f t minus tau was represented by F s, it was getting multiplied by S tau, e to the power minus s tau. So similar to this, here it gets multiplied by z to the power minus N, if the function gets shifted by n steps.

Let us now consider a unit step, what be the Z transform of a unit step?

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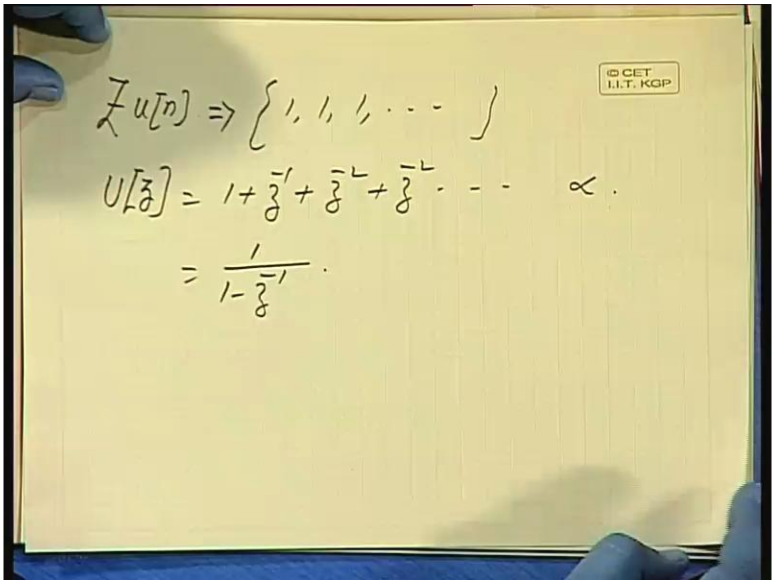

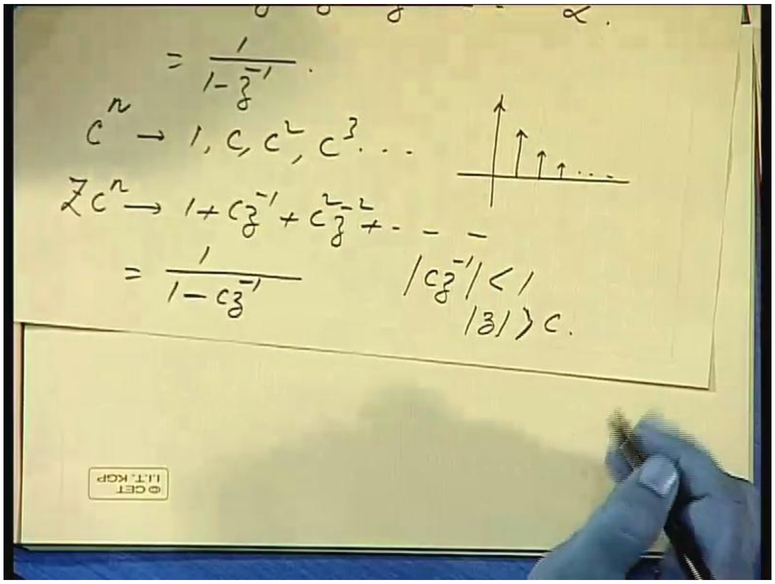

Unit step is given by 1, 1, 1 okay, so what will be Z transform? So, U z, I write U z is 1 plus z to the power minus 1, z to the power minus 2 and so on up to infinity and this is a G p series;1 by 1 minus z to the power minus 1, okay. What will be an exponential sequence, c to the power n? c to the power n is basically, c to the power zero is 1, c, c squared, c cube and so on, all right.

Suppose, c is less than one and so on, then what will be the Z transform? 1 plus c z inverse plus c square z to the power minus 2 and so on. And this is again, c z inverse is a common ratio; so this is again a G p series which can be written, as where this quantity must be less than 1, c z inverse must be less than 1 then only you can represent this, this means z should be greater than, okay.

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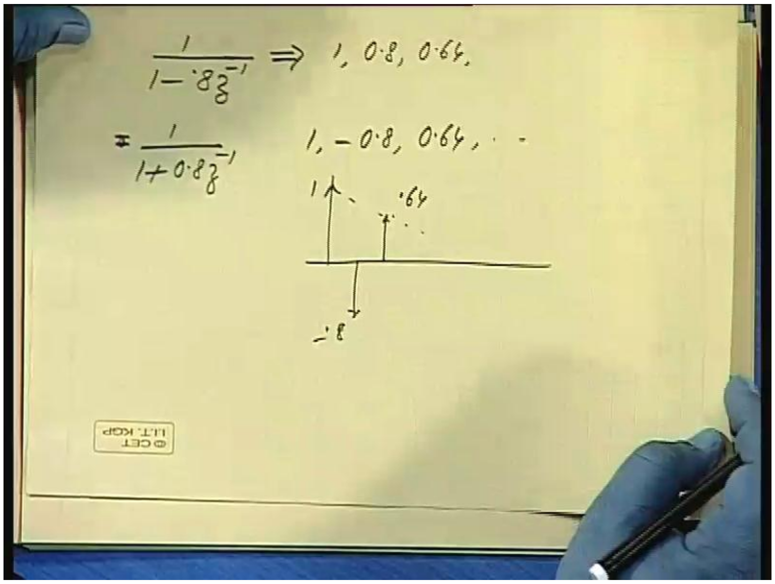

Suppose; you are having a sequence, a function 1 minus 0.8 z to the power minus 1, so c is corresponding to 0.8, okay. What will be the corresponding inverse transform? Inverse transform, inverse transform like Laplace transform; we just take the table and then see the values of the corresponding inverses.

So, corresponding to 1 minus c z inverse, 1 by 1 minus c z inverse I know, the sequence is c to the power n. So, c is 0.8, so it will be 1, 0.8, 0.64 and so on, okay, this is the sequence. If, I have 1 by 1 plus 0.8 z inverse then c is minus 0.8, so it will be 1 minus 0.8 plus 0.64 and so on. So, it will be like this, 1 minus 0.8 plus 0.64 is progressively decreasing, but it is alternative in sign, okay.

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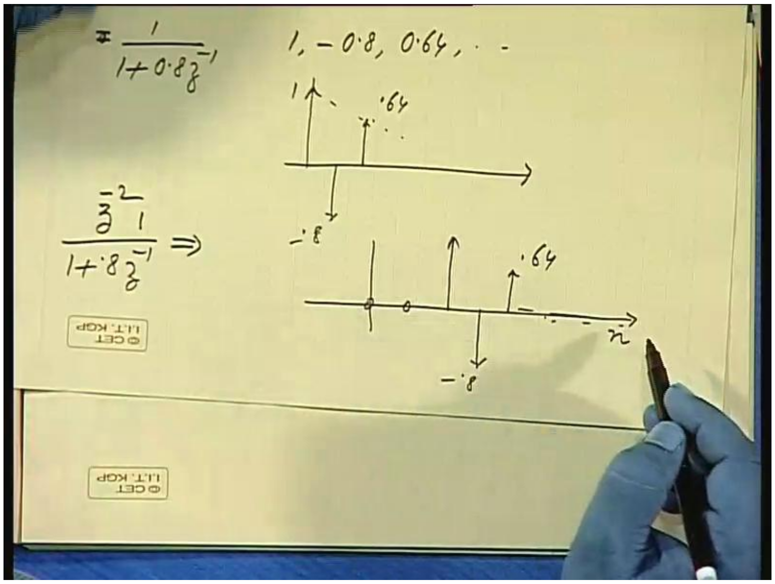

What would be z to the power minus 2 by 1 plus 0.8 z to the power minus 1; what would be the corresponding time domain representation? So, z to the power minus 2, we have observed if a function is delayed by n steps then the original function gets multiplied by z to the power minus n. So, we know the inverse corresponding to 1 by 1 plus 0.8 centi- inverses, this one, so the whole thing gets shifted by two steps.

So, it will be zero, zero then one then minus 0.8 then 0.64 and so on, it is a sequence of this type, okay. See, if I multiplied by z to the power minus 2, the sequence of this type.

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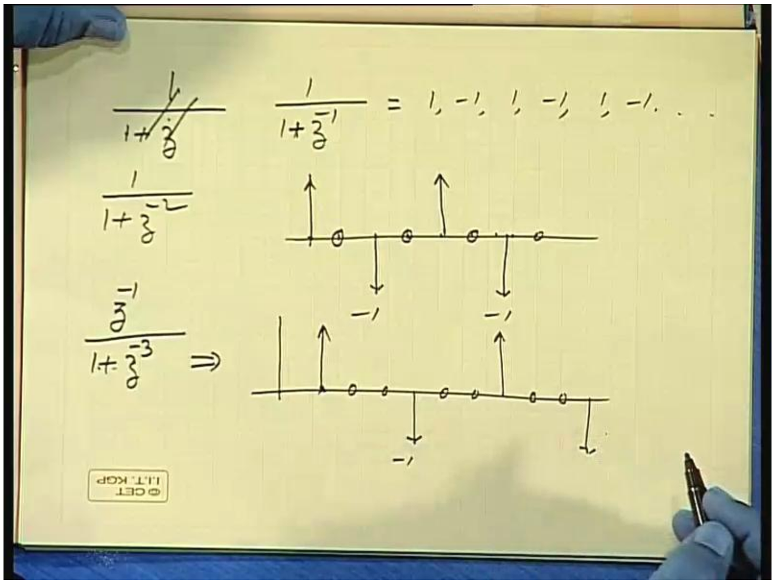

Now, let us come to another interesting function; what will be okay, what will be the inverse of 1 plus z to the power minus 1? It is minus 1 to the power n, so it will be 1, minus 1, 1, minus 1, 1, minus 1 and so on, okay. What will be 1 plus z to the power minus 2, 1 plus z to the minus 2? Now, look at it, look at it this way; z to the power minus 1 corresponds to one delay, if I had two delays then that correspond to z to the minus two, all right.

So, I will get the same sequence with two delays. So, this is 1, minus 1, 1, minus 1 and so on, in-between there will be only zeros, all right. I do not have anything like z to the power minus one, see in-between I will just put some zeros, okay. Now, you tell me; z to the power minus 1 by 1 plus z to the power 3 minus three, what will be the z inverse of this? Shifted sequence, so its starts from here; so it will be one then it will be two zeros minus 1, two zeros plus 1, two zeros minus 1 and so on, okay. So in the next class, we will stop here today.