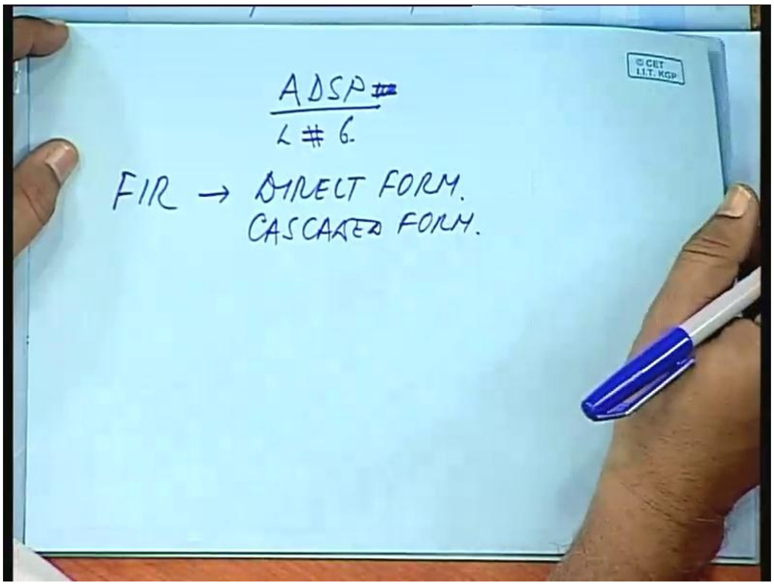
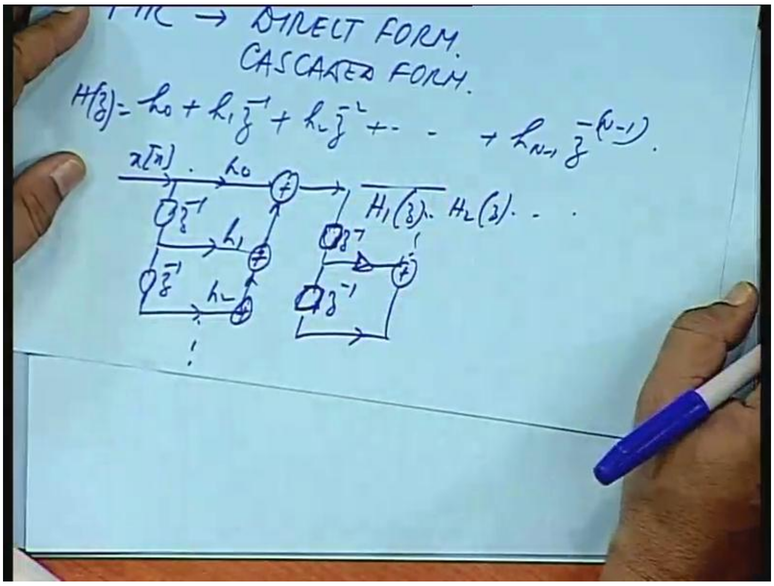
Digital Signal Processing
Prof. T. K. Basu
Department of Electrical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 35
Polyphase Decomposition

Today, we shall be taking up a very important topic Polyphase Decomposition. Now, those who have the electrical engineering background, they know, in electrical engineering, polyphase means or say three phase systems, we have in the phase circuit. Polyphase means, each of the phases will be having lead or lag over the other, lead or lag of angle. So, here it is a little different, but will see the principle is more or less identical, it is the response of a particular channel which will be having a different phase from the other channels. So, there is a kind of a phase shift that we give in the response will see how it is done.

(Refer Slide Time: 01:39)

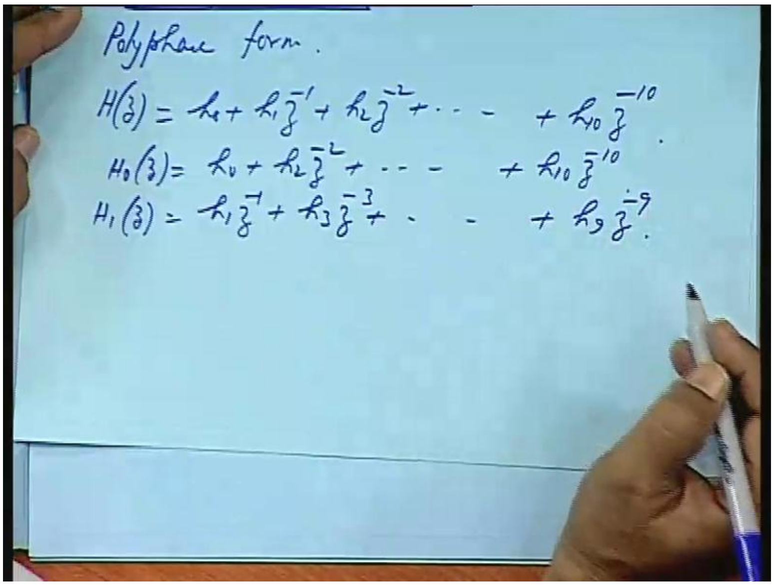


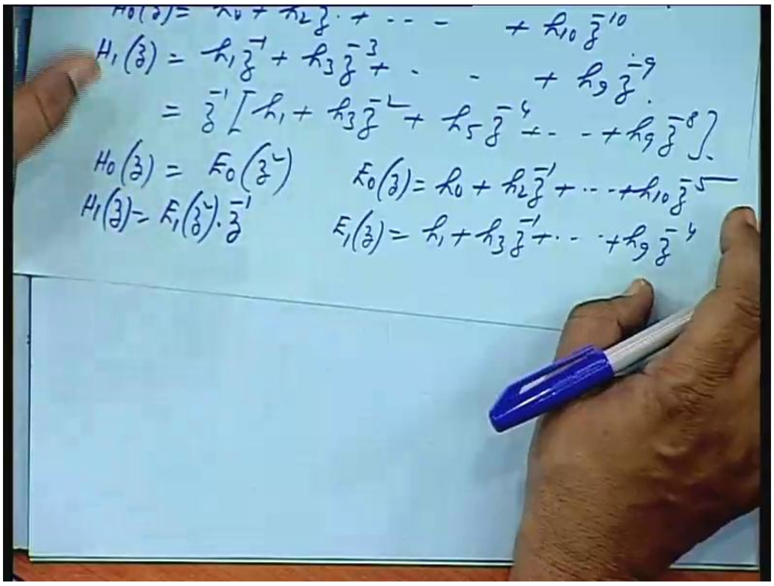


FIR Filters you have studied in the DSP course, FIR filters when you synthesize, we can have a direct form we can have a cascaded form. What is the advantage of cascaded form, why do you cascade filters, direct form, it is like this, h 0 plus h 1 z inverse plus h 2 z to the power minus 2 and so on. Say, h n minus 1 z to the power minus n minus 1. Suppose this is an end point sequence FIR filter. So, you take multipliers, this is x n, then you multiply by h 0 take derivatives multiply by, take the difference operators z inverse multiplied by h 2 and so on. And then add them together. This is the direct form and so on, if I factorize it, in the form of h 1 z into h 2 z and so on. Then, I can have, say quadratic factors, then I will have just structures like this, coming again in this form, this is z inverse, this is again z inverse, added together and so on. So, such blocks will be coming in the h 1, h 2, h 3, the advantage is, whenever there are some drifts in the parameter values, only the roots associated with this block, with this quadratic, only will be affected.

Whereas here, if there is a shift in the value of h 2, it will be affecting all the n minus 1 roots, all the zeros will be shifting, whereas here, that this zero is corresponding to this particular block, is quadratic will be affected, it will not be affecting others. So, you are going to restrict the moment of zeros, due to the drifts in the parameter values, parameter values change because of truncation. You know you have a finite register length, you have an 8 bit or 16 bit register. So, all this coefficients will be truncated, either you round off or you truncate, so there will be some error, so these errors will have some effect on the location of zeros. So, anyway, this is the direct form and the cascaded form, there is also another form.

(Refer Slide Time: 04:57)

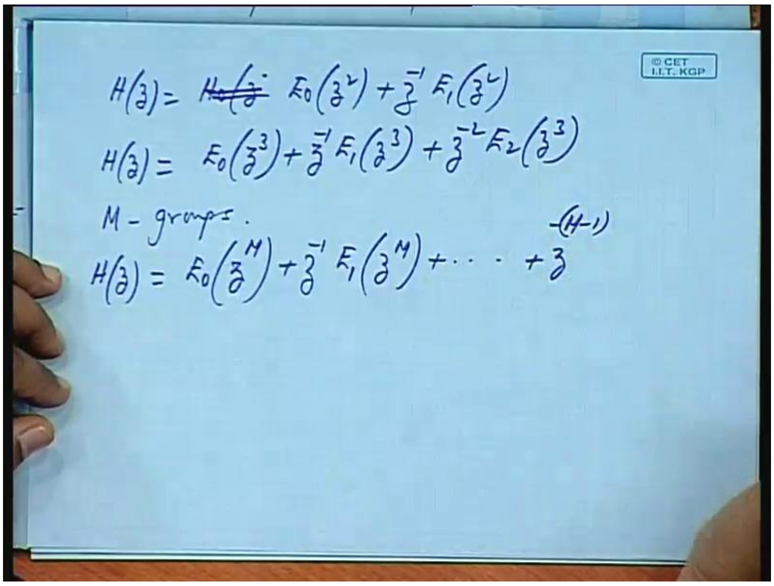


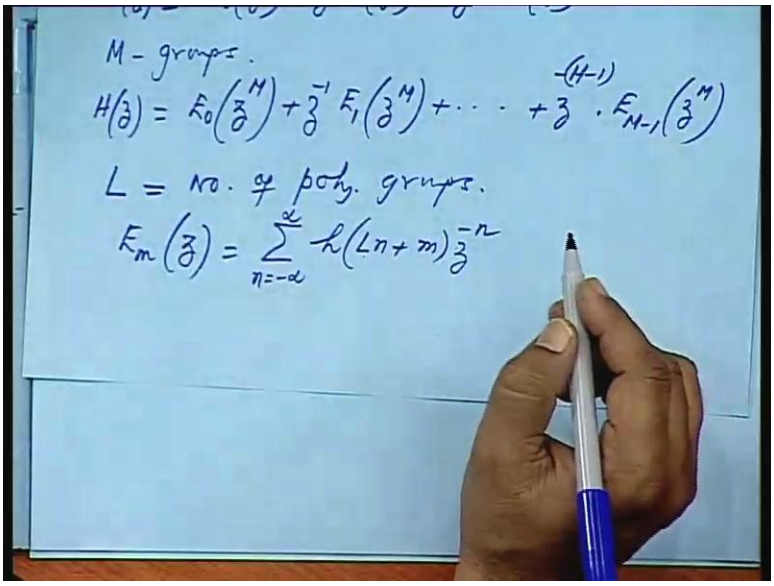


You must have studied it, that is the lattice structure, we are not going to discuss about the lattice structures right now. And there is a fourth form, that is the one, that we are going to discuss today, the polyphase structure, a polyphase form. It is like this, H z we can write as h 0 plus h 1 z inverse. Say, I have up to h 10 z to the power minus 10, I can pick up the alternate terms; that means, even terms and odd terms, I call it H 0 z, H 1 z , I may write like this, h 9 z to the power minus 8.

Now, this is a function of z to the power minus 2, so H 0 z, I can write this as, some E 0 z square, is a function of z square, where e 0 z will be h 0 plus h 2 z inverse plus, Finally, h 10 z to the power minus 5. Similarly, H 1 z can be written as, e 1 z square, where e 1 z is h 1 plus h 3 z inverse and so on, h 9 z to the power minus 4. Do you agree, it is a polynomial in z to the power minus 1, both of them e 0 and e 1. So, that polynomial, z if you replace by z squared will get h 0 and h 1 z, if I take out z inverse, this is H 1, this is e 1, e 1 z square, so e 1 z means this.

(Refer Slide Time: 08:18)



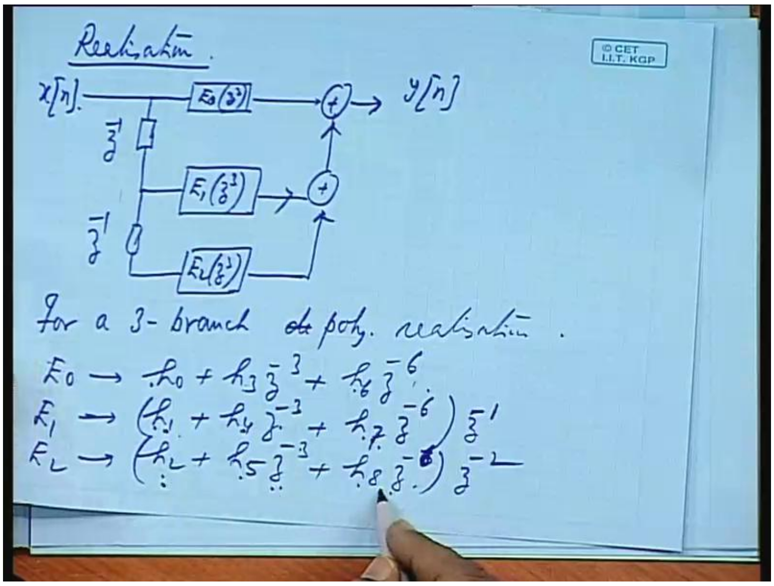


So, H z can be written as E 0 z squared, plus z inverse E 1 z square, if we have instead of taking up alternate terms, every third term we pick up, then H z can be written as E 0, what should I write, z cubed, do you agree, plus z inverse plus. So, generalizing this, if I have M such groups, H z can be written as E 0 z to the power M, plus z inverse E 1 z to the power M, last term will be M minus 1, minus. See, for every third term, if we pick up, that is, if it is having three polynomials, then z to the power minus 2 and if it is, the index here is 3.

So, here it will be 2, it will be ending at E M minus 1, z and z to the power, the same number, M minus 1, so we can write the polynomial, any polynomial, we can have any number of such groups. So, suppose L is the number of polynomial groups, what will be the terms of this? Any general term, say E k or E m, z, what will be this polynomial, h, if you have an infinitely long sequence.

Let us consider a non causal system, that is, n extends also, up to minus infinity, then and we choose the terms like this, say every Lth term, then it will be L n plus m. Is it not, if you have taken ((Refer Time: 11:46)) say the every third term, so this will be 0 plus 1, 2 into 0 plus or 2 into 1, plus 1 like that, if you have taken 3, three terms. Then the coefficients of this or any of this, any E Mth term will be appearing like this, can put them in this manner.

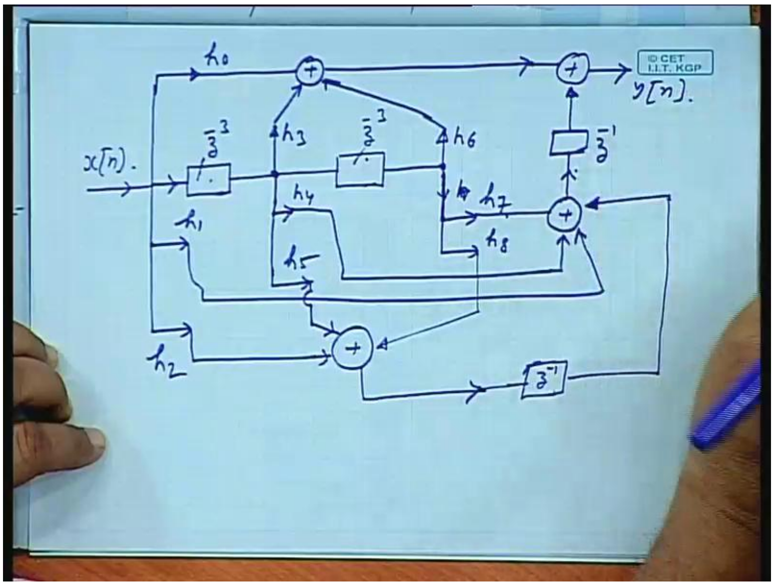
(Refer Slide Time: 12:22)



So, the realization for this first one is E 0, as it is E 0, suppose we take this one, ((Refer Time: 12:46)) every third term, so E 0 z cubed, next, there is a delay element, E 1 z cube, next E 2 z cube, this is y n. You can try, a few more structures, it is very simple, you can have fourth order or higher order realization, for a 3 branch, this is a 3 branch polynomial realization. So, we have got, E 0 as h 0 plus h 3 z to the power minus 3 plus h 6, z to the power minus 6. E 1 was h 1 plus h 4, z to the power minus 3 plus h 7, z to the power minus 6, whole thing into z to the power minus 1.

I mean, you want gets multiplied by z to the power minus 1, that is what I meant. And E 2 similarly, will be h 2 plus h 5 z to the power minus 3 plus h 8 z to the power minus 7 sorry, 8. And then z to the power minus 6, sorry minus 6, z to the power minus 2. So, all of them are having the same order, of course, if you have 9 points, then only it will be 3 each, if you have 10 points, then this will have 4 element set, others will be 3. So, the highest will be the 4 element set. Now, if you realize it, if you try to realize this filter, it will be like this.

(Refer Slide Time: 15:59)



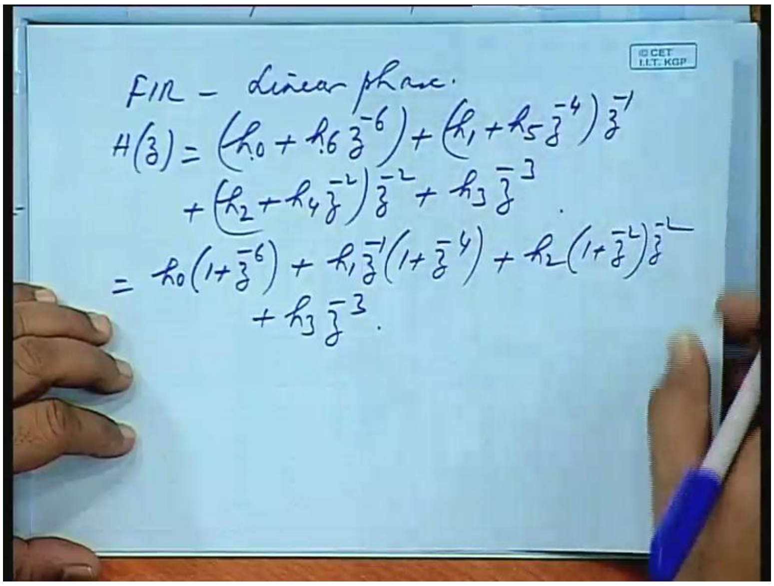
((Refer Time: 15:59)) x n, suppose we have a multiplier here h 0, then I can put, z to the power minus 3, then h 1, h 2, you can put another z to the power minus 3, then from here, I can have a multiplier h 6 ((Refer Time: 16:58)). Now you see, h 6 is multiplied by z to the power minus 6, added with h 0 and h 3 z to the power minus 3, so z to the power minus 3 is here, so I will take multiplier here, h 3 and add them here. This is the output, corresponding to the first phase.

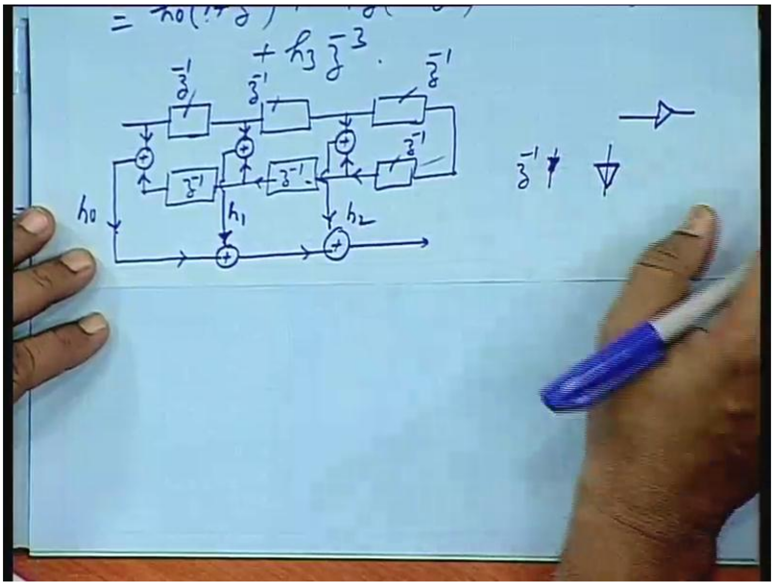
h 1, similarly will have h 3, h 3, I can take from here, h 4 and h 5, h 4 is added with h 4 z to the power minus 3, added with h 7 z to the power minus 6 and added with h 1, after that, the total sum is multiplied by z inverse. So, from here I can take h, let me multiplied this by h 7, this by h 8, so h 7, h 4, this is h 7 into z to the power minus 6, h 4 into z to the power minus 3 and h 1, this will be an output, which finally, if I multiplied by z to the power minus 1, I will get the second output.

And lastly, h 2 plus h 5 into z to the power minus 3 and h 8, these three will give me h 2 plus h 5 z to the power minus 3 plus h 8 z to the power minus 6, whole thing will be multiplied by z to the power minus 2. I will, I could have made use of this before addition, so I will put, z to the power minus 1 block here and then add with this and then this product is finally multiplied by z to the power minus 1. So, these are all getting multiplied by z to the power minus 1, this is getting this set, this sum is getting multiplied by, z to the power minus 1 and again z to the power minus 1.

So, this is the final output, this plus this, y n, now, if you have a linear phase filter, what is the advantage of linear phase filter, h 0 suppose there, there has to be a symmetry from the 2 ns ((Refer Time: 20:57)). Suppose, it is symmetric filter, then h 0, h 8, h 1, h 7 like that, you have symmetry of elements. So, you need not have different values, h 0 is same as h 8, so you will have same multiplying element and you see, the total number of elements will be much reduced, you can try for an anti symmetric filter also. So, you try at home, the final structure for an anti symmetric filter and a symmetric filter, having three or may be four levels of decomposition.

(Refer Slide Time: 22:01)





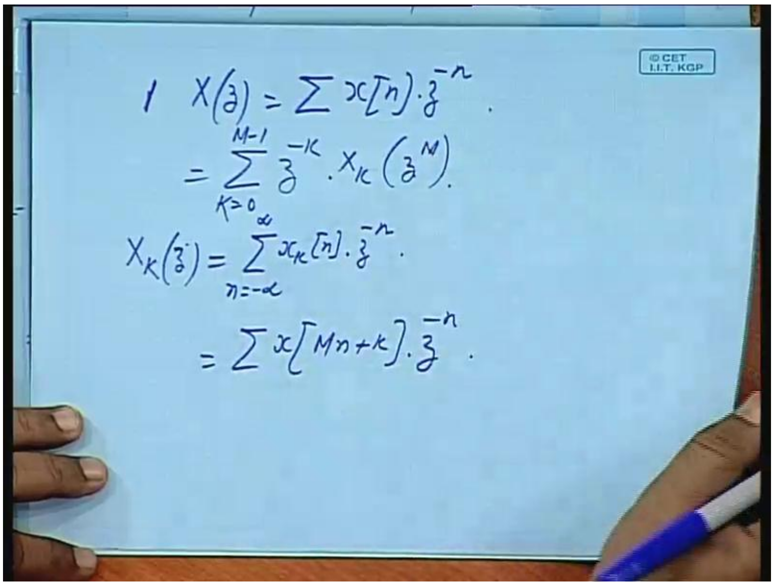
So, we can have, if it is an FIR linear phase, I will take up a simple example, you can try a little more complex structures. Suppose we have, h 0 plus h 6 z to the power minus 6, I have coupled a seven element linear filter, I have coupled the terms h 1 plus h 5 z to the power minus 4 into z to the power minus 1 plus h 2 plus h 4 z to the power minus 2, into z to the power minus 2 plus h 3 z to the power minus 3.

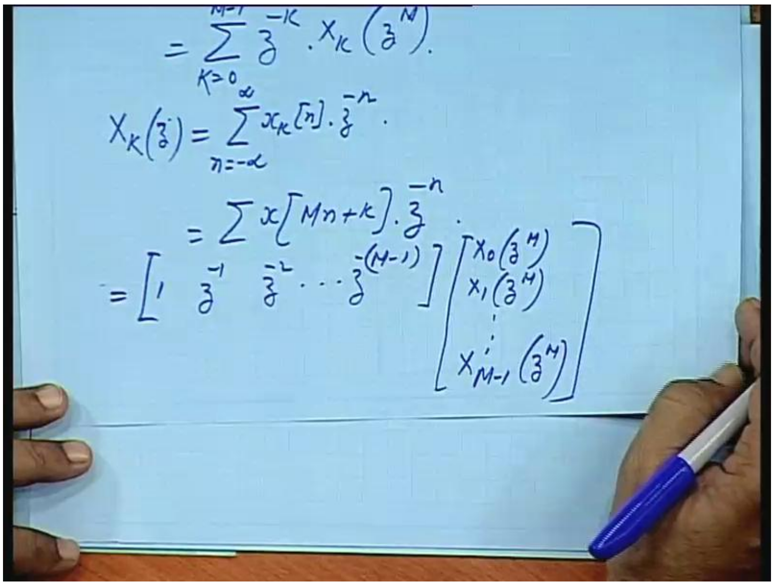
I can write like this and h 6 is h 0, so it is basically h 0 into 1 plus z to the power minus 6, plus h 1 z to the power minus 1 into 1 plus z to the power minus 4, plus h 2, 1 plus z to the power minus 2, z to the power minus 2, plus h 3, z to the power minus 3. So, this one can be realized in a very interesting way, you have each rectangular block is a delay element z inverse, z inverse, z inverse, again z inverse. I have just put the delay elements in chain.

Now, see what I am doing, you take h 0, it has to be added with, I say 1 plus z to the power minus 6, so there are six delay elements, so I will take the output here, add them together, I have done this addition, take this out, multiplied by h naught. So, can I suggest, what should you do next, it is z to the power minus 1, and z to the power minus 5. So z to the power minus 1 has already been taken and this is z to the power minus 5, so add them here and then this output, you take here, what should be the multiplier, h 1.

In signal flow, we use just the arrow for the multiplying constant, so I need not always write a constant with any block. Similarly, this one, I add with this output, this is to be multiplied by h 2 then it is better to show the direction, by an arrow like this and a constant by a little bigger arrow, in analog gains you show, multiplier like this, so it is like this. In many books, they use a small arrow, both for direction z inverse or any multiplier that is all allowed.

(Refer Slide Time: 26:59)

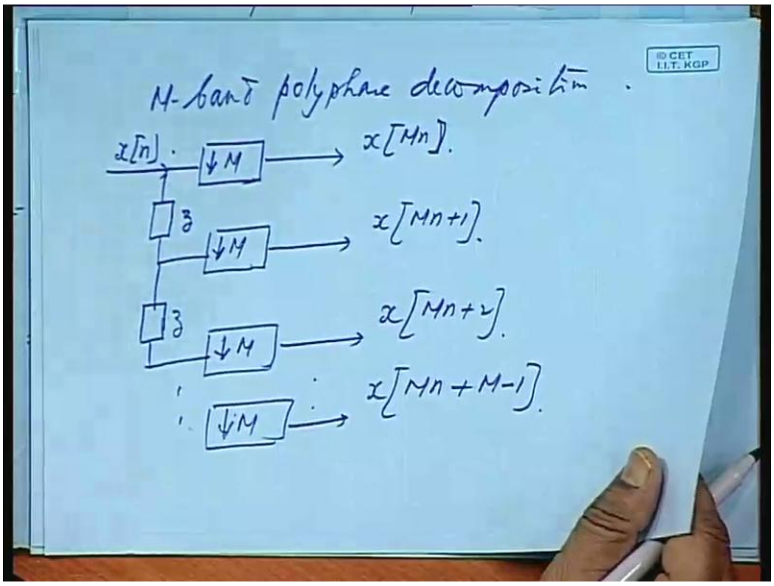




So h z, any signal X z, I can write therefore, in the decomposed form as sigma z to the power minus k, x k, z to the power M, k equal to 0 to M minus 1, is it not. If there are, M such sub groups, then k will have maximum M minus 1, the polyphase components, this is the polyphase components, x k z will be, say x k n, z to the power minus n, n tending to infinity. And what is this, in terms of the original sequence x n, from the original sequence, we are picking out say, either every third term or every fourth term or every Mth term. So, it will be x, M n plus k z to the power minus n.

Mind you, this is the polynomial, when I am writing in terms of z to the power minus 1, polynomial of z to the power minus 1 and this will be expressed, in terms of polynomial of z to the power n. So, in a matrix form, I can write this as 1, z to the power minus 1, z to the power minus 2, z to the power minus, M minus 1, into X naught z to the power M, X 1 z to the power M. Similarly, the last one X M minus 1, z to the power M, this summation I have written in this form.

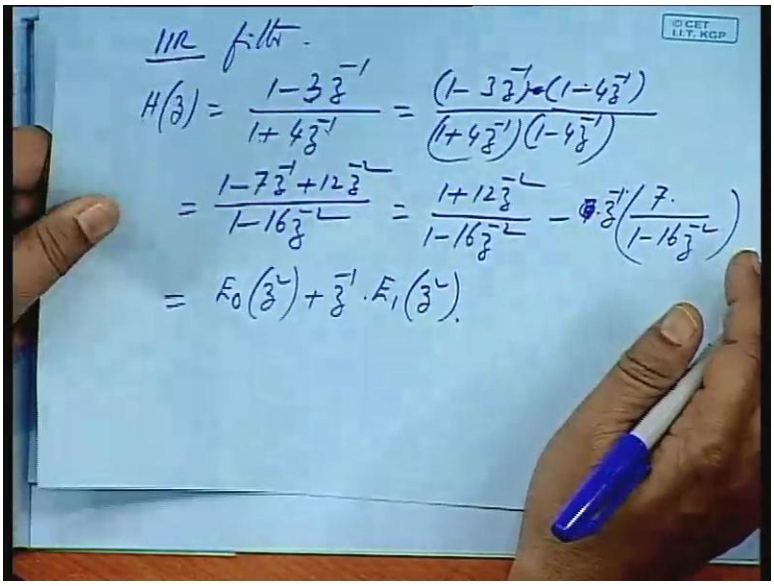
(Refer Slide Time: 29:38)



So, for the M band, polyphase decomposition will look like this, I give a signal x n, I have a down sampler M, down sampler does what, you are picking up only the Mth terms in the given sequence. So, we are down sampling by M, this will give me the terms x M n, every Mth term, again down sample by M, this will give me x M n plus 1, mind you, this is plus 1, we are generating the next set, that is why this is, z not z inverse, again down sample by M, x M n plus 2 and so on.

Lastly, it will be x M n plus, M minus 1, n is any number varying from minus infinity to plus infinity, integers, is this structure, it is like any of our previous forms. So, like this ((Refer Time: 31:48)) this series is h 1, h 4, h 7, this series is h 2 h 5 h 8. So, the general term is 2 plus 3 into n, 1 plus 3 into n. So, the starting value is 1 2 3 etcetera and this is M n.

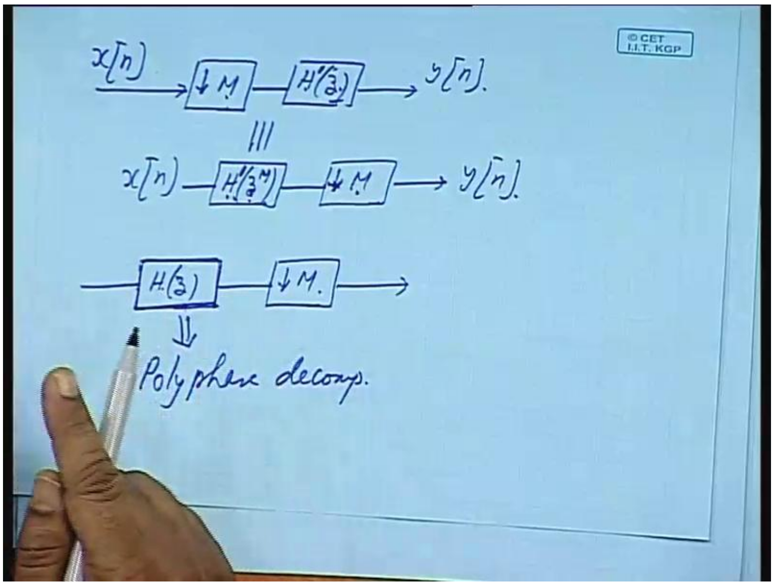
(Refer Slide Time: 32:20)



For an IIR filter, Polyphase decomposition is rather difficult, for IIR filter it is not straight forward, a simple example will give you, so 1 minus 3 z inverse, divided by 1 plus 4 z to the power minus 1. Suppose, this is an IIR filter, how do you decompose into such terms, say in terms of z square, say if we take a polynomials of z square, so I reduce this, if I multiply by 1 minus 4 z inverse, multiply both sides by the same factor. So, the denominator becomes 1 minus 16 z to the power minus 2.

So, I have got a polynomial in, z to the power minus 2, this one, 1 minus 7 z inverse, plus 12 z to the power minus 2. So I can write this as, 1 plus 12 z to the power minus 2, by 1 minus 16 z to the power minus 2, minus z inverse into 7 by 1 minus 16 z to the power minus 2. So, it is like our first, E 0, z square plus z inverse into E 1, z square, this is z inverse multiplied by this function E 1. It was so simple, because we have taken a very simple polynomial, reducing a polynomial, to a polynomial of quadratic terms, will not be so simple, had it been a little more involve, then it will be difficult.

(Refer Slide Time: 35:01)

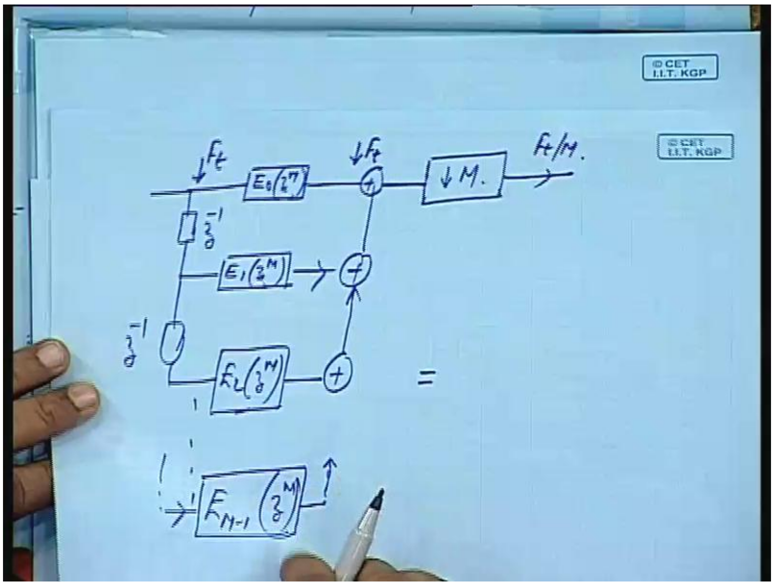


So, earlier we saw x n, if I down sample x n and if I have a filter here H z, giving me an output y n, this is equivalent to x n, H z to the power M. That means, if I have a down sampler followed by a filter H z, I can interchange their positions, but then the filter has to be changed, z has to be modified to z to the power M, then we will get identical output, this we had discussed earlier. Now, when you are going for a decimeter, you require a low pass filter, this we saw earlier, why do you need a low pass filter, that is an anti aliasing filter, for down sampling you require, a low pass filter.

Let this be some H z, mind you, this H z has got nothing to do with this, this was only an equivalent solution, now we are talking about, a decimeter, whenever we want to decimate the sequence, by a factor M, we must check whether aliasing takes place or not. So, we have a band limited filters, so this is a low pass filter H z, this H z, suppose we write in a polyphase structure of z to the power M, then what we get and then we will be using this relation, to be more specific.

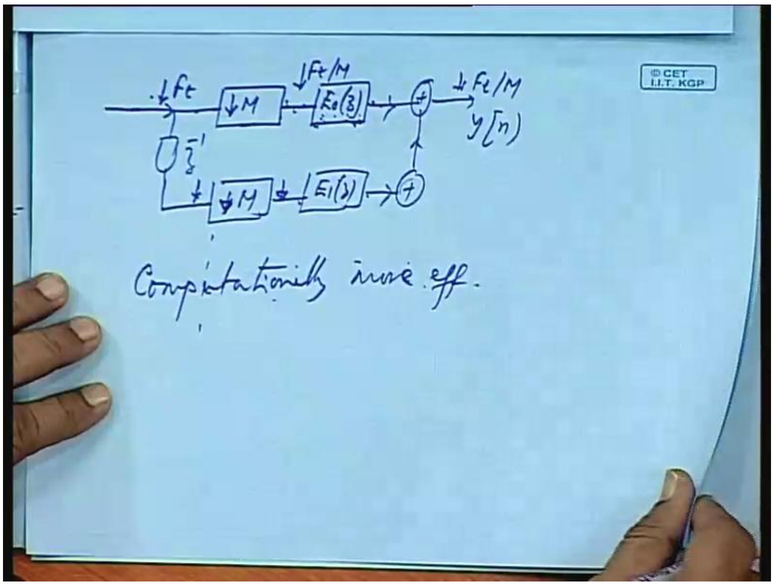
To avoid confusion, let this be, written as H dashed z, to avoid any confusion in this H z, so this is a relation that we have established earlier. Now, we are going to make use of it, at a later stage, suppose you have to design this H z, now we decompose it, in a poly phase form. So, if we decompose H z, it will look like this, just now we have seen the decomposition, the earlier one, anyway, we will have it here.

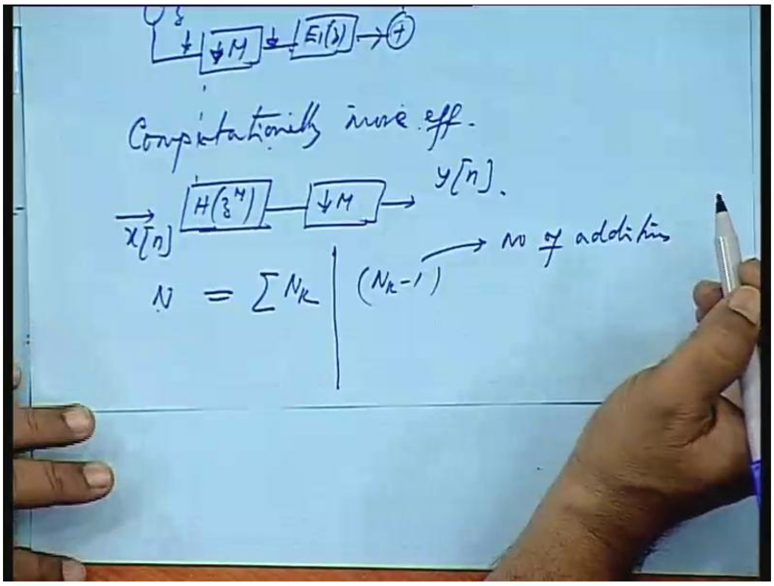
(Refer Slide Time: 38:12)



E 1, z to the power M, E 0, z to the power M, again a delay element E 2, z to the power M and so on, is it not. E M minus 1, z to the power M and this is H z, ((Refer Time: 39:20)) the center block is H z and then we put the down sampler. Now, each one of them can be treated separately, is it not and if I put a down sampler, that means, I can put the down samplers here and then I can interchange, by this solution. So, if a down sampler is having previous to this, H z to the power M then I can put H z only. So, this will be equal to, I will take this page, this will be equal to, what was the frequency here F t and this is also F t, the frequency here is, F t by M.

(Refer Slide Time: 40:35)





And now we are having F t, I can put the down sampler first, then this will be, what will be this one, E 0 z, z to the power minus 1, E 1 z and so on. And keep on adding them, I will get the same output y n, what about the frequencies, now, it is here F t by M and here also F t by M, down sampler has change the frequency, so will be at these points also. This is computationally more efficient, why can you justify, this is computationally more efficient, see if you have a single H z (Refer Time: 42:03)).

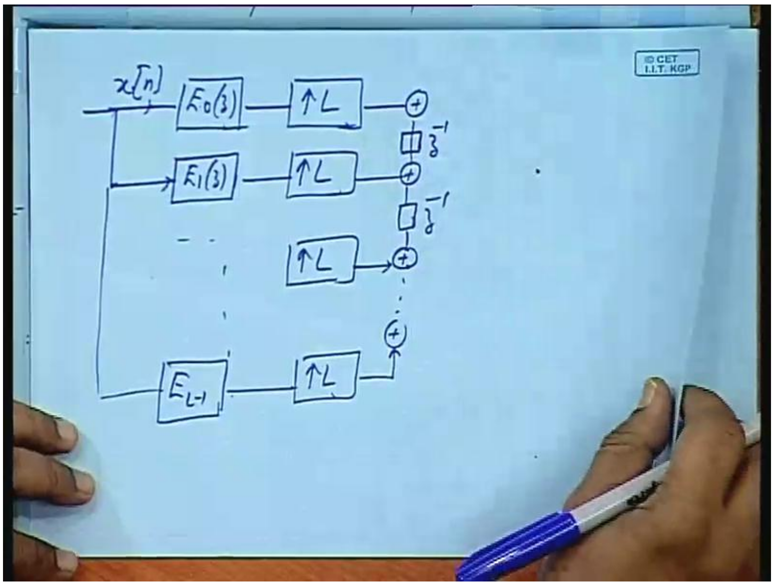
Suppose there are ten coefficients or say nine coefficients and there will be, nine multiplications and then eight additions, it will take time, unless the entire cycle of…. For each stage, I too have the input sequence, computing all these multiplications and additions will be taking place here and from here. We will be picking up only a few of them, all of them are not required, but when the chain comes, all the computations will continue, is it not

And from here, you are selecting only the Mth terms, when you are picking up Mth terms, so first term you are picked up, you are waiting, the computations are going and you are not interested in it. So, there is a lot of ideal time for many of the elements and unless the entire cycle of computation is over, the next set of data cannot be taken. Now, what happens here, you are down sampling the data right, in the beginning, you are choosing the data, you are making a shorter length and the computations are going on simultaneously in parallel channels.

So, all the arithmetic operations are going on, nobody is sitting ideal, so all these elements will keep on computing and they are having selected coefficients. Now, out of nine coefficients, here only three computations will, three terms are there, so three multiplications and two additions. Only a few additional, addition terms will be coming here, but in each of this blocks, the number of computations will be drastically reduced and all of them will be done in a parallel mode.

So, this is computationally more effective, very efficient, so we have got x n, H z M, M and y n, this structure, we have reduced to this form. So, if there are say, N points and N k points, per parallel branch, so the number of computations per branch will be, N k multiplications and N k minus 1 additions, is it not. So, summation of N k will be equal to N, these are number of additions in each block, number of multiplications, either it was totally N and here, in each section it will be N k and sigma N k is equal to N. Because number of coefficients you are not reducing, if there are nine points, you are picking up alternatively three, I mean every third one and you are making three groups, so number of coefficients will be same.

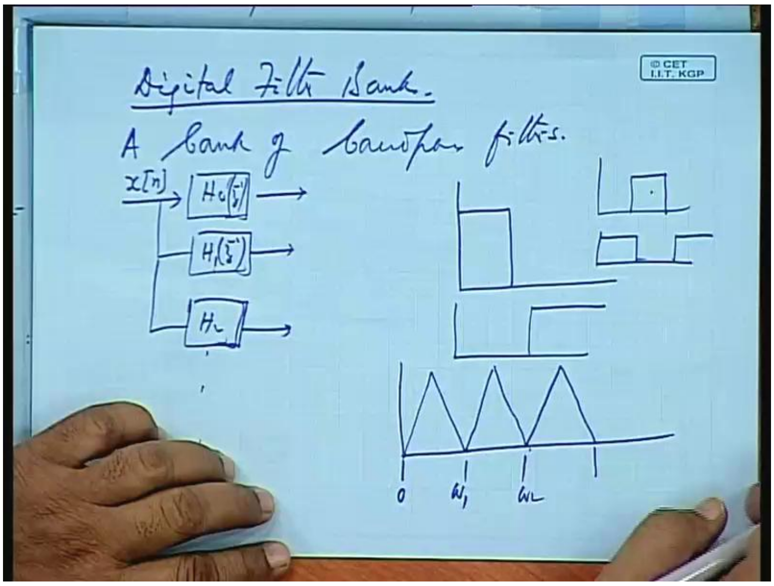
(Refer Slide Time: 46:09)



Now, let us see an up sampler, if we have E 0 z, L, E 1 z, L, and so on and you put a, z inverse term here, a multiplier here. And you have one such parallel blocks, last one will be L minus 1, L and this will be the output, so you can feed to E 0 z, E 1 z, etcetera. If you remember, for a decimeter, we had what was the relation for a decimeter, we have seen, if it is decimated first and then if it is H z to the H z.

If we interchange the position of the decimeter and the filter, then the filter will have to be, you know, z inverse will be replaced by z to the power minus M, reverse will be the case, in case of frequency multiplier, what is it called, up sampler. So, for an up sampler, it will be E 0 1, E 1 z etcetera and then it will be up sampled later, if you put in this fashion, so mind you, this is z, index of z. So, this will be put, for a down sampler it will be put on this side, for an up sampler you just will put on this side, now will go to digital filter banks.

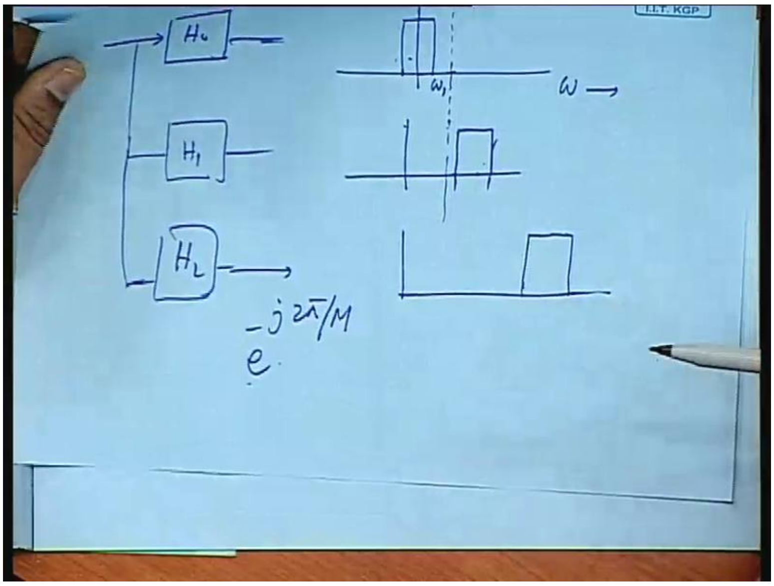
(Refer Slide Time: 48:37)



What you mean by digital filter banks, this is a bank of band pass filters, we have a common input and we have a large number of filters, say H 0, H 1, that means H 1, H 0 z inverse, similarly all these are there, H 1, H 2 and so on. Corresponding outputs you get in different bands, that means you have different bands of frequency, you have a band pass filter with varying bands and uniformly distributed in the range 0 to 2 pi.

So, for example, we normally talk about low pass filter like this, or a high pass filter like this, or a band pass filter like this, or a band gap or a band pass band pass filter like this, band gap like this. Now, same is the type of filter, instead of this kind of a rectangular ((Refer Time: 15:20)) characteristics. Suppose, we have a filter like this, this is one band, next we take another filter like this, next we take another filter like this. Obviously, you cannot have all the filters put together, so the first filter will be filtering frequencies from, say 0 to omega 1, next one will be, from omega 1 to omega 2. So, it is like this.

(Refer Slide Time: 50:56)

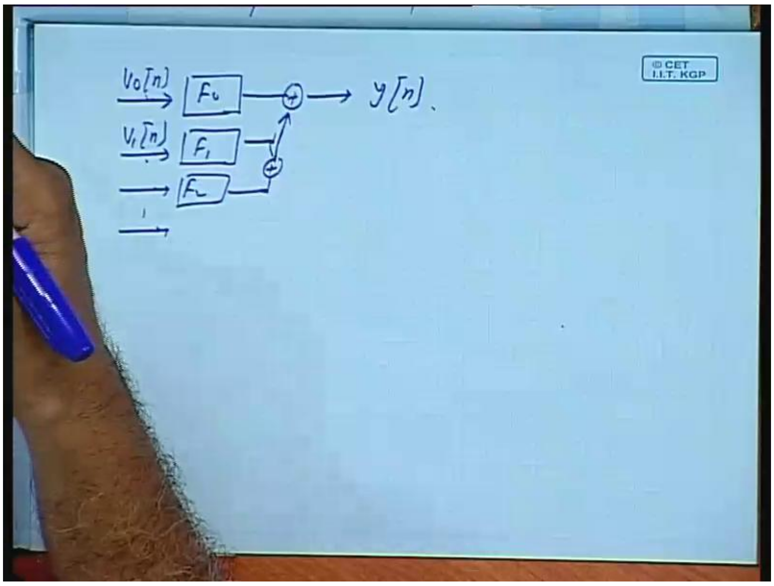


I have a channel H 0, which gives me a filter gain of say, say a characteristic like this, then I have H 1, this is mind you, in terms of omega, so this is say some omega 1, will see later on, it need not be symmetric on this side, need not be symmetric on this side. Why, when do you have absence of symmetry, when the terms or the coefficients are complex, for a real coefficient H 0, H 1, H 2, if you have real quantities then for plus, e to the power j, plus j omega e to the power minus j omega, the magnitude will be symmetrical.

So H1, anyway, for the time being I am taking as symmetrical structure, it will be, say from like this, so this is my band, the third one would be like this and so on. So, how do you design such a filter, all of them are basically this gain, having a shift of a particular angle. And what is that angle, what is that angular shift, if I have an M band, equal band, equal band width filter, then 2 pi by M, e to the power minus j 2 pi by M is the shift, it is shifting delayed by e to the power minus j 2 pi by m.

So, whatever is the filter function here, multiply by e to the power minus j 2 pi by M, you will get successive once. This is for analysis, given an input, we are trying to get what will be the outputs, these outputs will be corresponding to different band pass filters, this is what we do sometimes in mfcc, basically this is the type of filter bank you use.

(Refer Slide Time: 54:18)



And when you synthesize, you take different signals, say V 0 n, V 1 n and so on, and then you have this filters, say F 0, F 1, F 2. F 1, F 2, F 0 these are all similar filters and add them together, finally you get the output, from various inputs we get the output. So, when you want to filter out a signal, you have it is ((Refer Time: 55:11) response in different bands, take these and then moderate them, modify each band, whatever you want and then again add them, do you get my point.

In different bands, you are having different responses, so you moderate them. Suppose I want to suppress or I want to modulate certain ranges of frequencies, so that has been done and then after that you again add them. So, this is a synthesis filter, we will stop here for today, we will take it up in the next class how do you get the terms, for this kind of output of the filters.

Thank you very much.