Digital Signal Processing

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Lecture - 34
Multirate Signal Processing (Contd.)

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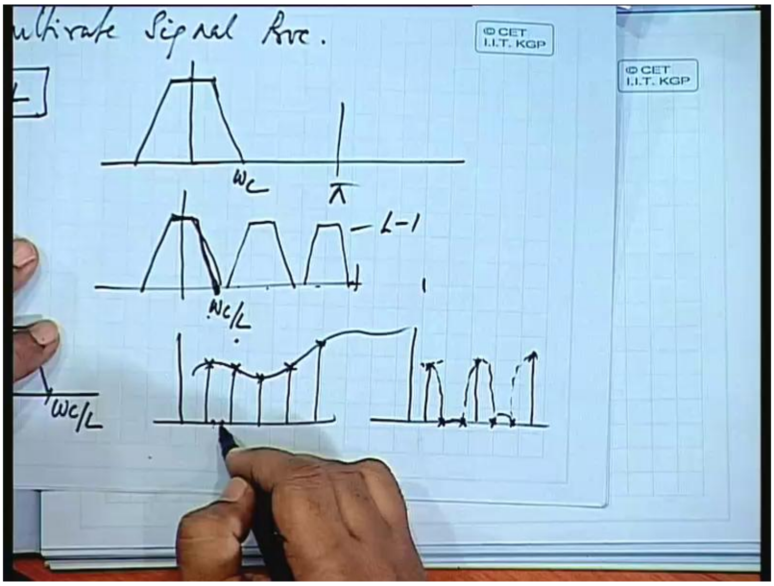

Last time we were discussing about in Multirate Signal processing, what are the effects of signal processing, what are the effects of down sampling and up sampling on the frequency spectrum. So, we have seen if we up sample it, if we up sample the system by a factor L, then they will be in the original frequency spectrum. Suppose it was like this, ending at a frequency omega c. Now, this was scaled down by a factor omega c by L, but then in the intervening space, we are having L minus 1 number of images, and so on up to pi.

Because it is now shrinking, suppose L is equal to two, then this will be halved, if L is equal to 3, this will be one third. So, omega c is replaced by omega c by L, so in this vacant space, we are having large number of images, so L minus 1 images will be formed in between. So, we do not want that; we want to remove this. These are basically high frequency bands. So, if we remove that by some kind of filtering will be left with this omega c by L, so such a filter, what it is doing is, it is trying to eliminate these. So, what is it basically doing on the data, suppose this was your original analog signal, these are the sample points originally, you have measured these values, now you want to adopt an interpolation technique, in between you are inserting zeros.

So, your new sample signal, with this over sampling, that is with this artificially inflated sampling, will have values like this, suppose I have put in, padded two zeros in between, then it will be like this, and so on. So, this profile is, now change to this profile, all right, so because of, the top of this will be still part of this, but then these points, which are otherwise here, they have been brought down all right, this is what is reflected here, so, you have to remove this.

What we are now trying to get is, passing this, now this is having a high frequency component, passing through a low pass filter. And these after filtering the, next set of points will be say somewhere here, somewhere here, here and here. So that, this will be partly, close to the original one all right, we trying to bring this, close to the original one. So this is basically an interpolated, so the filter, which removes, these images is also known as interpolation filter or interpolated.

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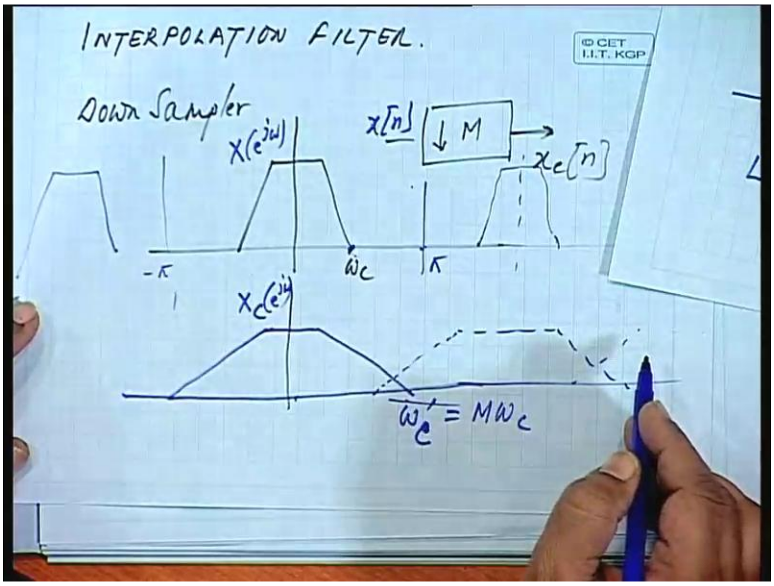

Similarly, the other one, that is when we are down sampling, we are flattening the base, suppose this is the original frequency spectrum all right, this again omega c. Now, if you are down sampling by a factor M, then the response of this, down sampled signal or compress signal will be, this is the first one is X e to the power j omega. So, X compressed e to the power j omega, will be this will be, this base will be going up by, M omega c dash, which is M times omega c, M times omega c.

Now, this was say pi, so it is a repetitive function is it not, with a period of 2 pi all right, about pi they will be a symmetric, similarly, what minus pi, they will be symmetric, so this is X e x e to the power j omega. So, X c will be having, identical features only thing this is multiplied by, this is getting flattened, by M times, so second one, if this overlaps with this pi, if this is greater than pi, then what will happen, they will be aliasing, to make sure, that this does not happen, before we put it to this compressor. We must ensure that omega c, if the compressor factor is M, there are we must see, that omega c multiplied by M should not touch pi is that all right.

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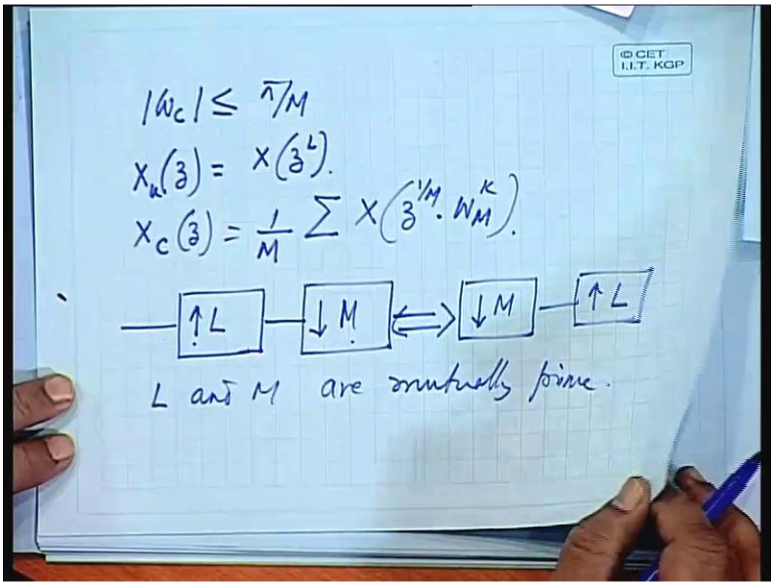

So, our restriction should be omega c, must be less than pi by M, if you are going to have a down sampler of M. M level down sampler, could require, that the original signals spectrum, this band must be, restricted to pi by M, then only you can have, a down sampler of this two, otherwise they will be aliasing. We have also seen X u z, that is when we are having, multiplication of sampling rate, X u z this was derived, last time similarly, X c z how much was it, 1 upon M sigma X z to the power 1 by M and W M k.

And, for cascade connection, I had asked you to verify, did you try that, all of you have got the results. So, there are to be necessarily mutually, prime, then only you can have a condition like this, this is equal to this, I can interchange L and M, provided L and M are mutually, prime. If they are not, if they are not, then what should, I do, first I should have a down sampler or up sampler, up sampler, first you must have more number of zeros padded expand it, and then you compress it, all right. So, up sampler and down sampler, so this will be, the right choice, irrespective of the factors L and M, whether they are mutually prime or not we are not concern, but if they are not, then you cannot use them, in this mode.

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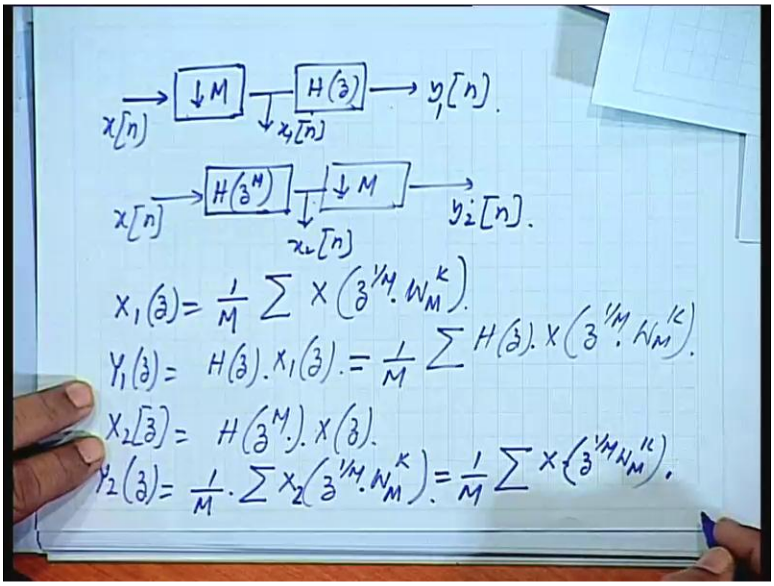

Now, there is another cascade equivalence, you can show M, if it is followed by H z, suppose this is y n, this is x n, this is also equivalent to H z to the power M, and then M. That is first I pass it through a filter, first I pass the signal through a filter, but then z should be replace by z to the power M. And then down sample by M or if I down sample first, and then pass it through this filter H z, they are equivalent, will see the proof of this all right.

Suppose this one is x 1 and this one is x 2 in the second case, the intermediate variable, this is y 1 and this is y 2. So, in the first case, what will be X 1 z, what will be X 1 z, if it is down sampled, then 1 by M, 1 by M this is the relation X z to the power 1 by M, W M k all right, and Y1 z will be, H z in to X 1 z all right. So, that is 1 by M sigma H z X z to the power 1 by M W M k, in the second case, what is X 2 z, what is x 2 z, H z to the power M in to X z, and what is Y 2 z, what is Y 2 z?

Student: ((Refer Time: 13:36))

Yes, 1 by M summation X z to the power 1 by M, then W M k.

Student: ((Refer Time: 14:04))

Yes, X 2 z is what, X 2 z is X z in to this, and this X 2 is getting sampled and a sample, I mean, you are having a compression, so what will be Y 2 z in terms of X 2 z, forget about this, what about Y 2 z in terms of X 2 z.

Student: ((Refer Time: 14:41))

Summation 1 by M summation X 2, I will write that, actually I was writing in one step, X 2 z, I did not complete it, X 2 z is that all right.

Student: Yes, Yes.

And that can be written as 1 by M, now you substitute for X 2 this one, summation X 2 is nothing but, this one. So, 1 by M X z is to be replaced by z to the power M, 1 by M, W M k and H has to be replace by, z has to be replace by z to the power M all right, z is replace by z to the power M. So, what should be this one, multiplied by,

Student: ((Refer Time: 15:37))

H, multiplied by, let me go to the next page.

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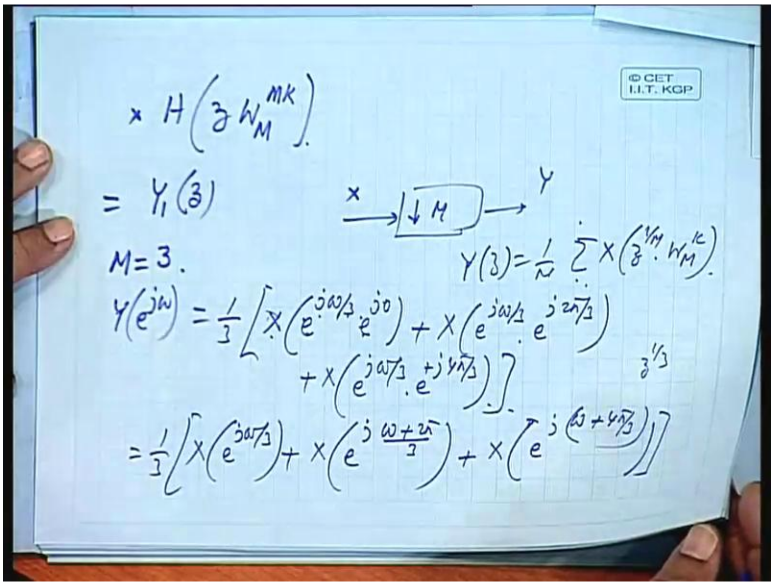

Multiplied by, H z to the power Student: ((Refer Time: 15:52)) M by 1 by M, and then W M.

Student: Student: ((Refer Time: 15:57))

No, and this is equal to one, so that is equal to Y 1 z, so this is another identity, ((Refer Time: 16:23)) that is M and H z, you must have studied in control systems, continue systems control system. If there is a g s and if there is a signal, if you want to transfer it, previously or transfer it, forward in a forward step, then you have to multiply or divide by the corresponding g s, it is almost similar kind of exercise, now it will be H z M.

Next let me take up an interesting problem, for a down sampler, suppose we have M is equal to 3, then this is X this is down sampled by M this is Y. So, what will be Y e to the power j omega like, what will be Y e to the power j omega like, Y z is 1 by M sigma X z to the power 1 by M W M k all right, k varying from 0 to 2, 0 to M minus 1. In this summation signs, ((Refer Time: 17:57)) I did not write, k varying from 0 to M minus 1, because it was derived last time, so I have left it blank.

So, this one will be, one third summation of these term, so X z to the power, that is e to the power j omega by 3, z to the power one third in to W M k. So, how much is that, e to the power j, k equal to 0 is 1, plus X e to the power j omega by 3 e to the power j 2 pi by 3, plus X e to the power j omega by 3 e to the power j 4 pi by 3 or minus 2 pi by 3 it is one at the same thing.

So, X e to the power j omega by 3, in to e to the power 0 that means 1, so that is equal to one third X evaluated at e to the power j omega by 3, plus X evaluated at. This is a complex quantity, e to the power j omega plus 2 pi by 3, plus X e to the power j omega plus 4 pi by 3 agreed, is that all right? What does it physically mean, you have calculated, this is what we have seen.

Here, this omega c gets multiplied, gets expanded, so X e to the power j omega by 3, it is reflects basically, say this part, then this represents shifted by 2 pi by 3, again shifted the second part comes, and third one, you can consider this, as a part coming here all right. So, that will be the overall Y output, that is the compressed X, we are writing as Y or X c, which one are the same thing, say that will be the effect, there is a, there is an interesting problem.

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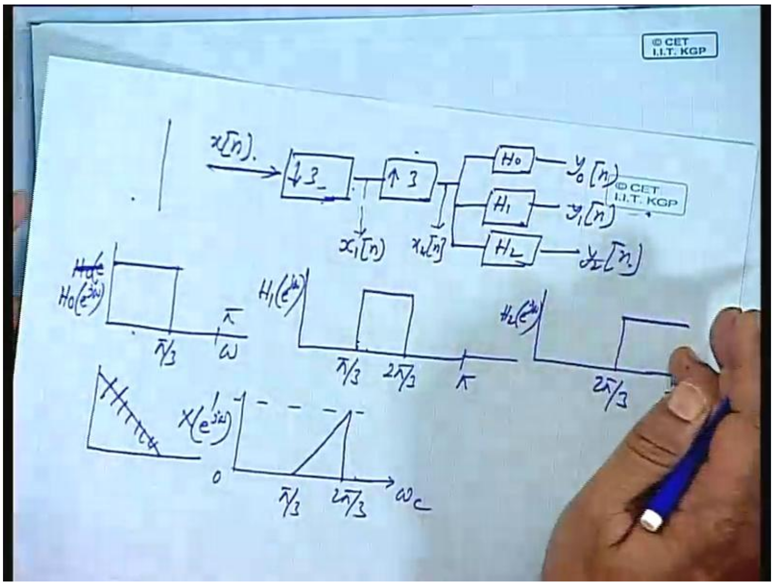

You are having, an up sampler and a down sampler, let me draw the diagram, there is a down sampler a value 3, and there is an up sampler 3. And then we are having three filters H 0, whose frequency domain characteristics, I will show, you H 0, H 1 and H 2 corresponding outputs are y 0 n, y 1 n and y 2 n, x n is the input. Now, the three, the three filter characteristics are given, H 0 is H 0 e to the power j omega.

I should write, more properly, H 0 e to the power j omega is the low pass filter, coming up to pi by 3, H 1 e to the power j omega is a band pass filter, all of magnitude 1, this is pi by 3 to 2 pi by 3, and H 2 e to the power j omega is 2 pi by 3 up to pi a. Now, the input signal has a characteristics like this, sorry, sorry the input signal is having, a characteristics like this, from pi by 3 to 2 pi by 3, it is magnitude 1.

This is X e to the power j omega for the input, what will be the outputs y 0, y 1 and y 2 in the frequency domain all right, apparently this is the input, a this is multiplied and divide by 3. So, this appears to be x n is it not, so we see, had it been x n, then I would have got this type of frequency spectrum for this signal here. And then then that filtered, by these three would have given me only, for this, that is the output only at y n, y 1 there would be no output here, or here all right.

Had I interchanged these two, had I interchange these two, first the up sampler then down sampler, then it would have been same, I do have got only this response at y 1, y 0 and y 2 would have been 0, is it not. If I pass through this filter, between 0 and pi by 3 it exist, so the this signal will not be pass by this, this signal will not be passed by this, it will be passed only by this in the same range.

So, let us see, what will be the change pattern of this, when I consider the outputs here, so let me call this as x 1 n and x 2 n. if x 2 n is different from x n, in the frequency characteristics, then of course the outputs will be different. So, let us see, when I down sample this, what happens to the base, it expands.

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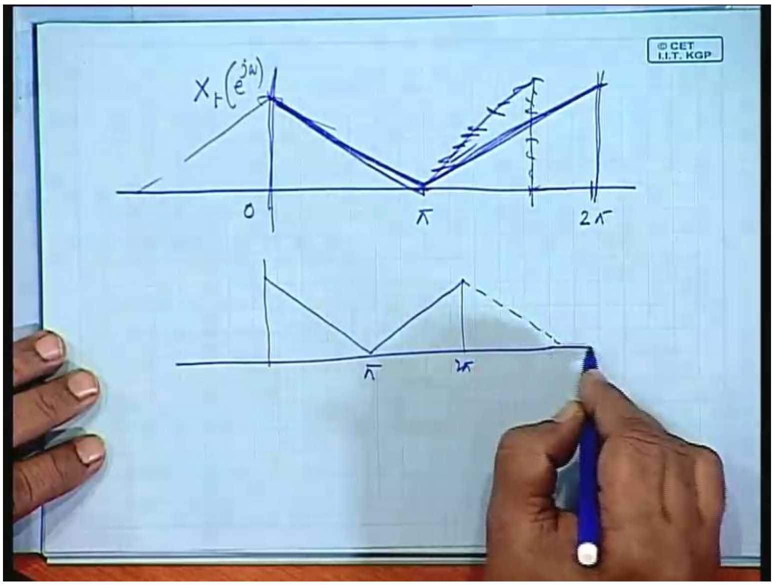

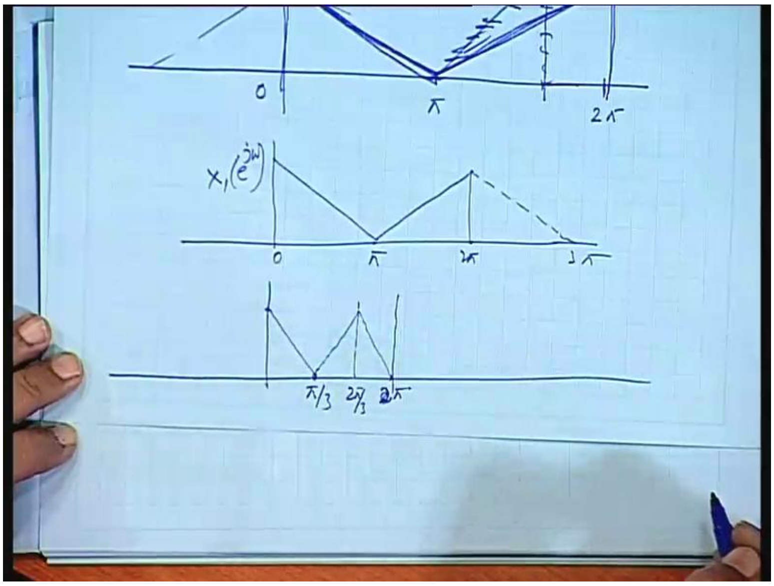

So, what will it be like, 0 ((Refer Time: 25:25)) it is multiplied three times all right, three times means, so this will be pi to 2 pi, you mean this, then I have not shown it, actually 2 pi should be, uniform same distance. So, this should be 2 pi, so it will be like this, pi to 2 pi, so whatever is coming between pi and 2 pi, the mirror image will be between 0 and pi, is it not. So, X 1 this is X 1 e to the power j omega, will be like this, is that all right.

It is obvious, because you see I have shown this part, means after pi, after pi, the counter part is like this, is it not, or if you take its image on the other side, it is having a counterpart like this, is it no? I have shown only one part between 0 and pi. So, you can take the images, and then you expanded and you see for yourself, so this will be, the nature of function x 1 n. So, x 1 n is pi, 2 pi, 3 pi and so on, what will be therefore, X 2 this is X 1 e to the power j omega, what will be X 2 that is divided by 3, so what will it be like, 0 to pi by 3.

Student: It may be 0 to pi.

Then pi to 2 pi, everything is now squished, is it not, so when you are down sampling, when you are down sampling, this are created all right. And now, when you are up sampling, they will be, the whole thing is all the quantities are reduced by one third all right. So, it will be pi to 2 pi by 3.

Student: ((Refer Time: 28:34))

This one will be, sorry, this one will be,

Student: 2 pi by 3

2 pi by 3 thank you, this will be pi, now this is the signal that we are giving to the filters, so what will be the outputs, now tell me, and first one low pass filter will be giving me this one.

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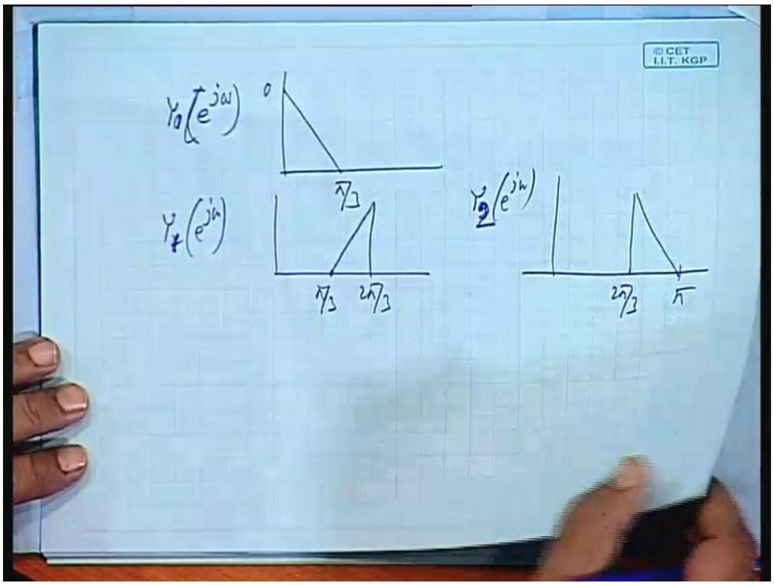

So, Y 1 n, Y 1 e to the power j omega will look like, this Y 2 will look like, what all right, and Y 3, what we started off with was, this one, which corresponds to only Y 2. So, this gives me, sorry, I have writing Y 1 Y 2, rather it should be Y 0 Y 1 and Y 2, please correct it, Y 0 Y 1 and Y 2. So, the middle one, gives me the correct signal, the original signal, so there are two ((Refer Time: 30:18)) signals, which will be coming in these two channels. If you first down sample, then up sample, if you interchange them, this will be 0, this will be 0, only this one will come.

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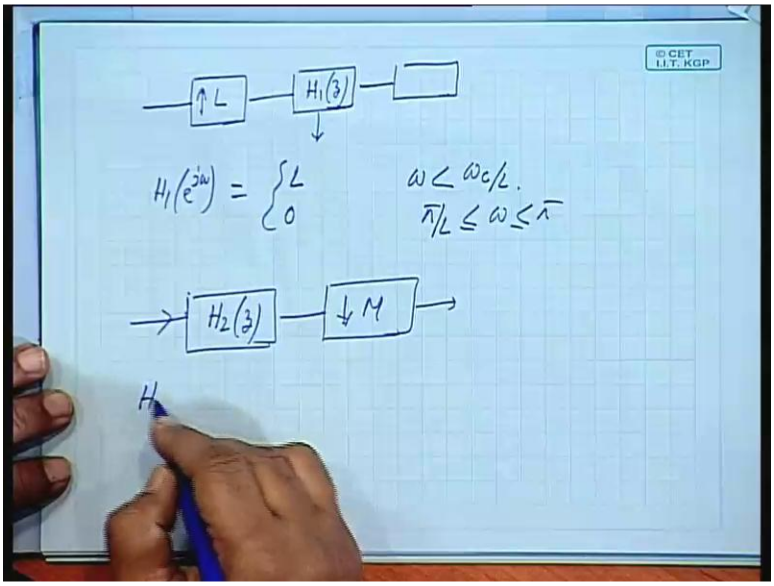

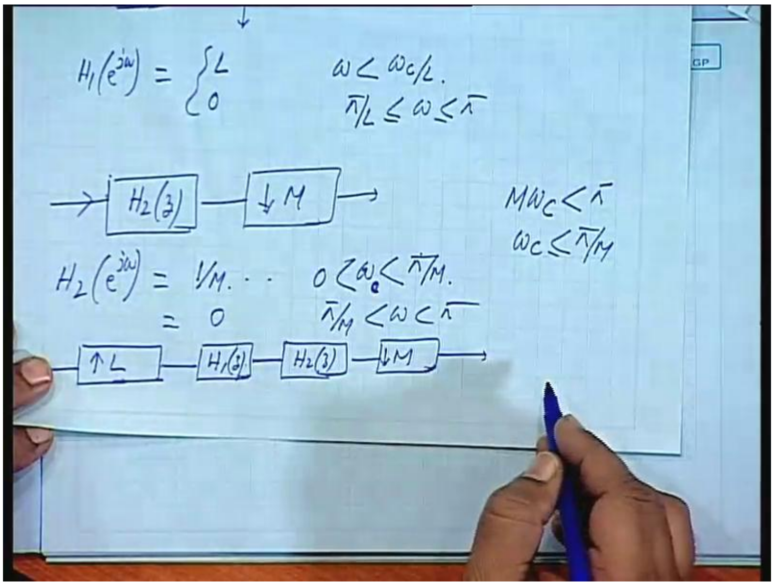

So, that is the need for filtering, now what should be the specifications for the filter, I need suppose I want to change the sampling rate, I would first go for up sampling, the moment I up sampled there will be, images, so I have to put a filter. And what should be the band for this, what should be the requirement; I will call it H 1, so H 1 e to the power j omega should be, it will be multiplied by L all right. The magnitude is multiplied by L, and rather, because the actual magnitude comes down, omega less than omega c by L.

I do not want any images, and equal to 0, between pi by L omega pi, basically this is the stop band, requirement this is the stop band all right, magnitude gets modified, so that is why we are putting, a multiplier L. So far we are not, talked about the magnitude part, we are all the time, talked about normalized gain, but the gain actually comes down. Now if I put a factor M, if I want to down sample it. If I have a down sampler, then first I must restrict, that after the aliasing has taken place you cannot do anything.

So, you have to have a pre filtering, so anti aliasing filter H 2 z, and then only you can down sample it, is it not, and what was the condition for this, H 2 e to the power j omega, what should be the magnitude, 1 by M, 1 by M. Let us see, actually this magnitude is not, so much important, what is important is the frequency band, what should be the frequency, omega should be, we just now derived, it should be pi by M times omega c, should be less than pi, is that so.

So omega c must be less than pi by M, omega c, so this gain should be between 0 and pi by M, and equal to 0. Beyond this so pi by M to pi it should be 0. In that case, if I want to have an expander L, I put H 1 a filter that is an interpolation filter, it is a low pass filter; both are low pass filters all right. And then I will put for a compressor, I will put H 2 z first, and then put M all right, then only I can change the rate, H 1 and H 2 can be put together, club together, two ideal low pass filters.

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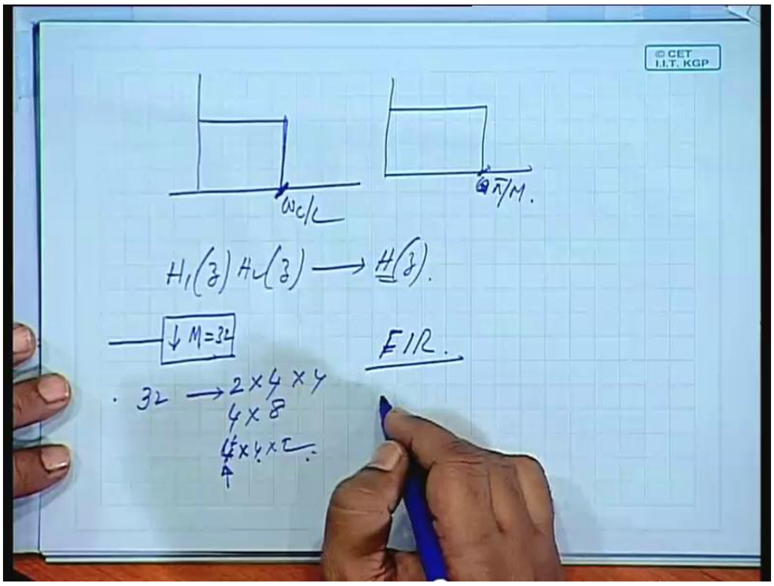

See two ideal low pass filters, one is having a frequency corresponding to the first requirement omega c by L and the other one is pi by M, omega it is cut off is pi by m, all right. So, whatever is the value of m or L given, which one, whichever is lower, we should take that, so H 1 and h 2 can be club together, in a single low pass filter. I can have a single low pass filter, where whichever is less will be considered, for the design.

Now, there is a question, if I have, suppose I have to down sample by a factor M is equal to 32, 32, stepping down by such a large step, large amount, means this filter H z, will have a large number of coefficients. It have a large number of coefficients all right. So, to avoid that, we want to go for an economic design, because suppose you want to down sample it, you have to store the values, pass it on, to the next stage. So, the memory requirement is very high, the overall performance will be very slow.

So, we go for multistage down sampling, that is 32, we try to factorize, different combinations, what are the possible combinations. I can have it 2 then 4, then 4. One may go for 4 into, if I go for 3 stages, if I go for two stages, then I can a 4 in to 8 and so on, or 4, 4, 2. So, the computational requirement, will be much, much reduced, if I have, such factored forms, with a multiplier of 4, I am in a compressor of factor 4. And once again 4 and 2 the number of total number of elements required, will be moisturize.

We are going to have it, for an FIR filter, because FIR design is very straight forward, so we can take an FIR filter, the number of coefficients required will be much less, and the errors will be less. So, we shall be, taking one or two simple examples, before that I would like to mention the steps.

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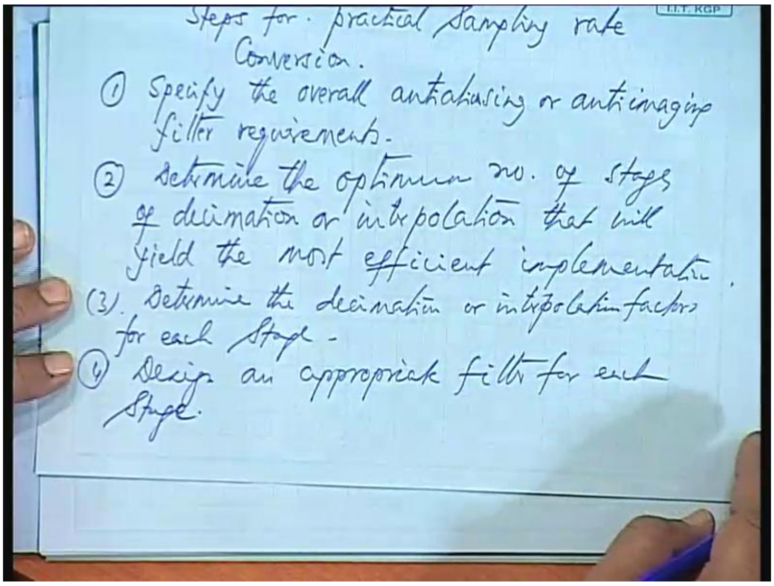

A steps for, practical sampling rules, sampling rate conversion, specify the overall anti aliasing or anti imaging filter requirements, suppose I tell you, it has to be reduced by a factor 32 all right. So, this is the specification, then determines the optimum number, determine the optimum number of stages, of decimation or interpolation that will yield, the most efficient implementation, the most efficient implementation. Finally, what we are interested in is the number of elements required, the multiplier etcetera and the storage.

So, which will be, giving me minimum number of, such hardware elements, and that will also, involve the total delay. Determination a, determine the decimation or interpolation factors, for each stage and then design, and appropriate filter for each stage, design and appropriate filter for each stage. So, I hope you have, done some design in dsp class earlier.

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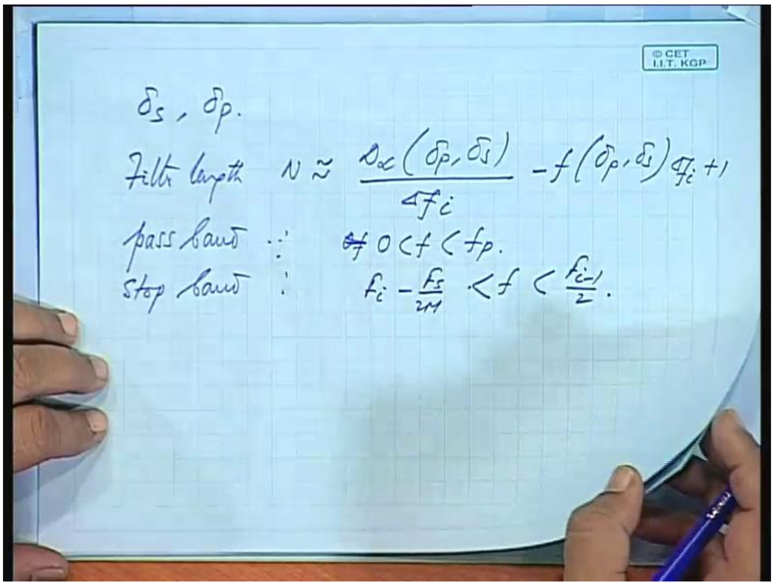

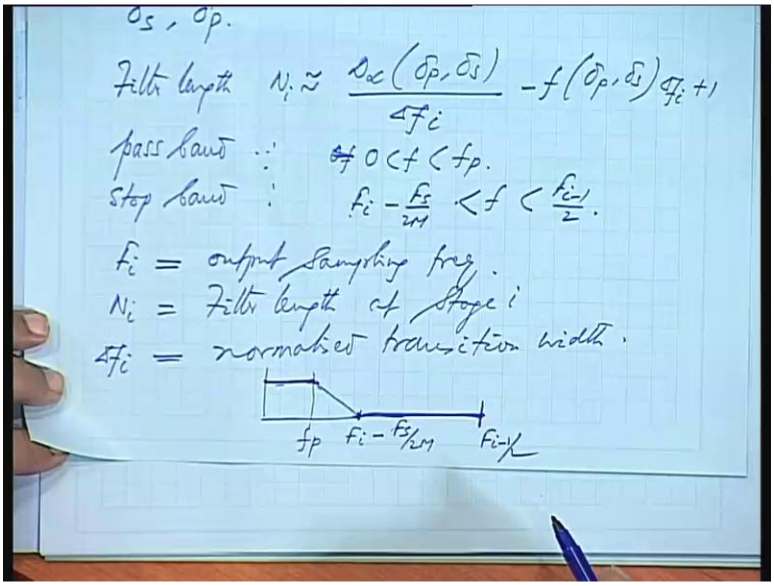

So, write the specifications are given, in terms of stop band and pass band repels in decibel, and there are specifications given. So, from there, you can design, from the standard relationship the filters, I will just take up one example, then it will be clear, Filter length, it is an FIR filter length is given by, you calculate the filter length, for different stages. D infinity delta p delta s, this is a standard relationship, that you will get in all the text books, minus f delta p delta s in to delta f i plus 1, where pass band is 0, f, f p stop band is F i minus F s by 2 M, f and F i minus 1 by 2.

These are the, you say at the i'th stage, this is the i'th stage frequency, and this is the i minus 1'th stage frequency all right. So, if these are the two limits, then this minus Fs by

2 M should be, the lower limit this will be the upper limit, so F i is the output sampling frequency, N I, this is actually N i, is a filter length, filter length at stage i. And delta f i is a normalized transition width, that is if this is f p all right. This is F i minus Fs by 2 M, and this is F i minus 1 by 2. So, this is your stop band, this is your transition width delta f i, and this is the pass band.

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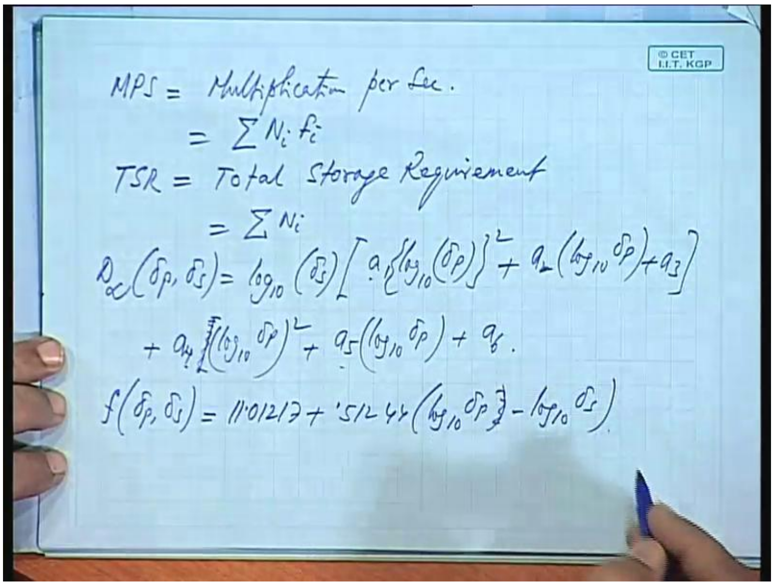

Now, multiplication per stage, Multiplication per stage, per second is equal to sigma N i in to F i that is at the i'th stage, if there are n number of elements all right, n number of elements. Then N i in to F i output sampling frequency, that will be the total number of multiplications involved, at every stage you are having, so many multiplications. And, TSR is total storage requirement, these are the two indices, by which we judge, the performance of the filters.

And D infinity del p del s, I am taking it from a standard text book relation, that is given in almost all the books, a 1 log delta p to the base 10 write squared plus a 2 log delta p to the base 10 plus a 3 plus a 4 times log delta p to the base 10 squared plus need not put this bracket, plus a 5 times log delta p to the base 10 plus a 6. Now, a 1, a 2, a 3, a 4, a 5, a 6 these values are given, in the hand books or any standard design book, they will be supplied to you, in case we have to compute this.

And f delta p delta s is equal to for example, I am taking just a standard value, 1 7 plus 0.51244 log of delta p to the base 10, minus log of delta s to the base 10, this is a function, that describes, this quantity. So, it is a, that is total number of elements required, will be also dependent on the, repel that you are having here, the, repel that will be allowing here, the repel width all right.

This is the difference basically, this is log of del p by del s is, it not repel width in the pass band by, repel width in the stop band. So, that ratio counts a lot, so that ratio also decides ,this and that gets multiplied by delta f I, this factor multiplied by the transition width, that will give you the number of elements. More number of elements you have, more is the transition, transition width, sorry this is the other way it is less, because it is subtracted, more number of elements if you have, you have a smoother transition. Otherwise, if you truncate it, with a smaller length, then the repel will continue, it cannot be contained, all right. So, let us see, if time permits will hurriedly go through one example, the sampling rate of a signal x n, at least I will state the problem.

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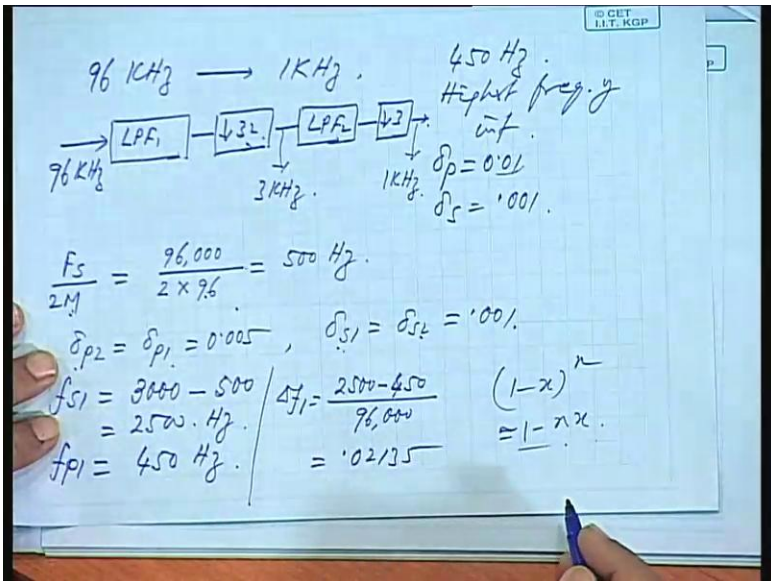

x n is to be reducing from 96 Kilo Hertz to 1 Kilo Hertz all right, so you have to reduce it, the highest frequency of interest is 450 Hertz, highest frequency of interest, this is a highest frequency of interest 450 Hertz. Assume that, an optimal FIR is to be used, you are going for an optimal FIR, with the overall pass band repel delta p is equal to 0.01 and delta s is 0.001, design and efficient decimeter. So, you have low pass filter 1, we can have a reduction say 32, then low pass filter 2, and then a factor of 3, 32 in to 3 is 96, one may go for this kind of decimation.

So, this is 96 Kilo Hertz, I will write the corresponding signal frequencies, this one will be, that gets divided by 32, so this will be 3 Kilo Hertz. Again, it is under sampled by, a factor of 3, so this will be 1 Kilo Hertz, is that all right, so F s by 2 M is 96 Kilo Hertz divided by 2 in to 96 all right, because the reduction is reduction is by a factor of 96. So, this 96 is the reduction factor, 2 M, overall M is 96, and F s is this frequency all right, because it is 96 to 1, it could not could have been 192 to 2, then this would have been 1 92.

So, this is 500 Hertz, so delta p 2 is equal to delta p 1 equal to 0.005 all right, delta s 1 is equal to delta s 2 is equal to 0.001. Why should it be, 0.005 we are having, we are having in different stages, the repel in the pass band distributed, repel in the pass band all right, it is like this, one percent, if you decrease, from the reference level it is 0.99. If you multiplied again 0.99 by 0.99, it becomes 0.98.

It is like this, 1 minus x to the power n is approximately 1 minus n x, binomial expansion if you approximate to the first order, that means, mean three stages, if I distributed. Then this will be 0.001, 0.01 by 3, here it is 0.01 by 2, there are two filters, some distributing it uniformly, whereas the stop band repel does not change, it is above 0 all right. So stop band repel remains same, this must remember, so f s 1 is 3000 minus 500 all right, so that is equal to 2500 Hertz all right, f p 1 is 450 Hertz, the first stage this frequencies 3 Hertz, 3 Kilo Hertz all right, so 3 Kilo Hertz minus 500 is f s by 2 M.

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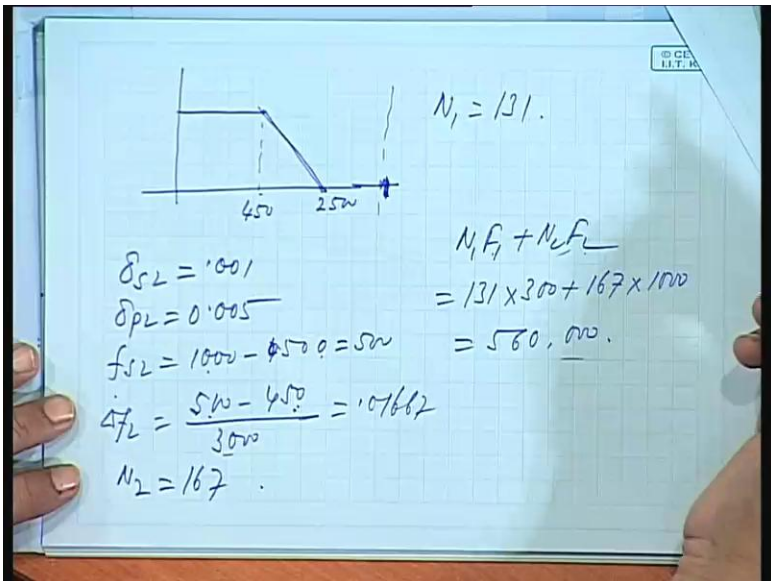

It is like this, this is 450 Hertz, this is 2500 Hertz all right, and this is 3000, and in the second stage, ((Refer Time: 56:03)) f p 1 in the first stage let me complete the computation, delta f 1 is 2500 minus 450 divided by 96,000 this frequency, and that is equal to 0.02135. The relationship I have already mentioned here earlier, so you are just using those relationship, and then finally, we are computing N i, N i comes out to be N 1 in the first stage, N 1 comes out to be 131.

Then, delta s 2 is 0.001, delta p 2 is 0.005, f s 2 is 1000 it is this frequency, again minus 450 minus 500 sorry, and delta f 2 will be 500 minus 450 divided by 3000, and that is 0.01667. Earlier it was, 2500 minus 450 divided by 9600, now it is 1 Kilo Hertz minus 500 that gives me 500, so this 500 minus 450, 500 minus 450 is the balance, this is del f p, basically what we are ensuring is this, is well within this frequency. And here it is 3000, in the second stage it is 3000, so compare to the base 3000, this is del f 2, so that gives me N 2 is equal to 167.

So, will stop here, this gives me N 1 F 1, plus N 2 F 2 as 131 in to 3000, plus 167 in to 1000, F 1 F 2 are the output frequencies, and that is ((Refer Time: 58:44)) approximately 560,000 all right. We will take in the next class, two or three more, such factors, instead of 32 in to 3, if we take other factors, what will be this index N 1 F 1 plus N 2 F 2 plus N 3 F 3, if that is less then we will choose that, this is how you optimize the design.

Thank you very much.