Digital Signal Processing
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Lecture - 33
Multirate Signal Processing

Good afternoon friends, last time we discussed about Hilbert transformer. Today, we shall be taking up Multirate Signal Processing. What is this Multirate signal processing?

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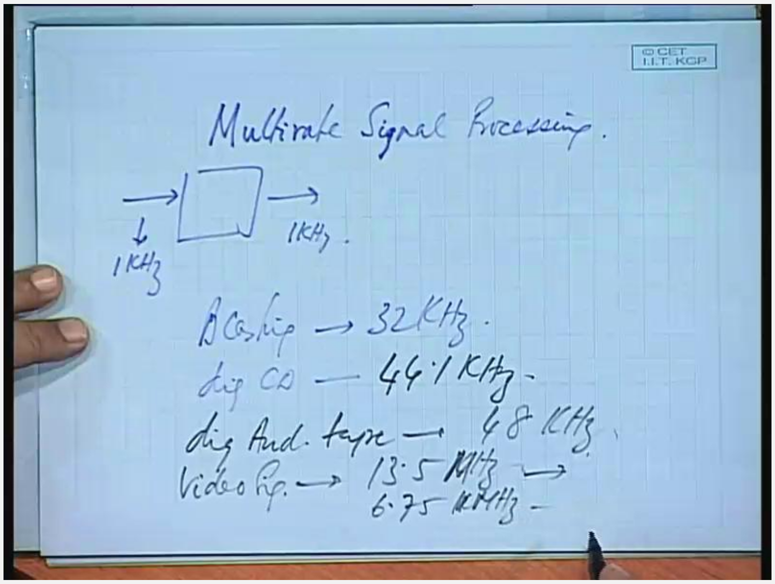

Now, so far we have used both the input and the output frequencies of any system, that sampling frequencies are same. If this is sampled at 1 kilo hertz, output is also obtained at 1 kilo hertz sampling frequency. So, the sample points, where at regular intervals at the same intervals both at the output and the input, sometimes we are working on systems, where there are say signals coming from different places, which have been sampled at different rates and we want to put them together.

We are required to convert the frequencies, sampling frequencies, from one value to some other value and that need not be always a sub multiple or multiple of the existing sampling frequency. For example, suppose you have sampled at 2 kilohertz and, now you are required to give a signal with 4 kilohertz or may be half of that 1 kilohertz. Then, what you do under, what conditions you can have a different sampling rate, are there any restrictions, this is what we are going to see and if at all you want to change the sampling rate how do you do that.

Now, as I told you from 1 kilohertz you can go to 2 kilohertz, you can go to 500 hertz just by dividing or multiplying by 2. Suppose from 1 kilohertz you want to go to 1400 hertz that is 1.4 kilohertz. So, it is not an exact multiple of 1000, so it may be a fraction or from 1000 we want to come down to 700 hertz not exact sub multiple 1000 divided by 2 3 like that it can be anything else, then how do you get that?

In broadcasting for example, the frequencies are 32 kilohertz, whereas digital CD, let me change the color digital CD it is 44.1 kilohertz. You can see the frequencies are varying so much and then digital audio tape it is 48 kilohertz, then video signals 13.5 kilohertz this is again for luminescence in video signal also. There are two components that we send one is luminescence the other one is a color difference signal. So, color difference signal that is at 6.75 kilohertz if this is in megahertz even telephony we normally in a telephonic message we take 8 kilohertz signal.

So, music system and telephonic signals they are not at the same frequency. Music system, you want to preserve different finer components of signals very high frequencies different instruments have been played. And in a telephone you just want to get the message just simple speech frequency is good enough, so we go for a very, very low sampling frequency.

So, there are different signals will different signals will have different frequencies different sampling frequencies, how to put them on a common platform when you require to transmit them in a particular channel or you want to mix them. So, compatibility of signals requires that, you should be in a position to transfer the frequencies of some signals from one base to another base.

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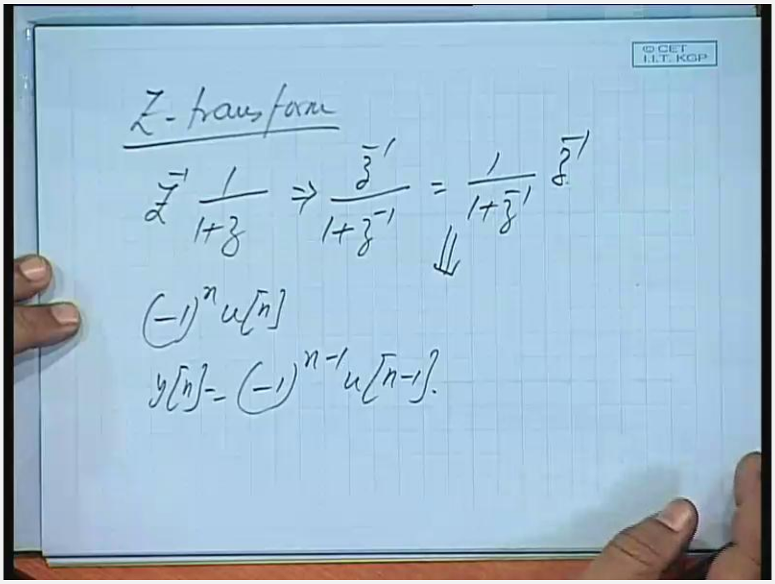

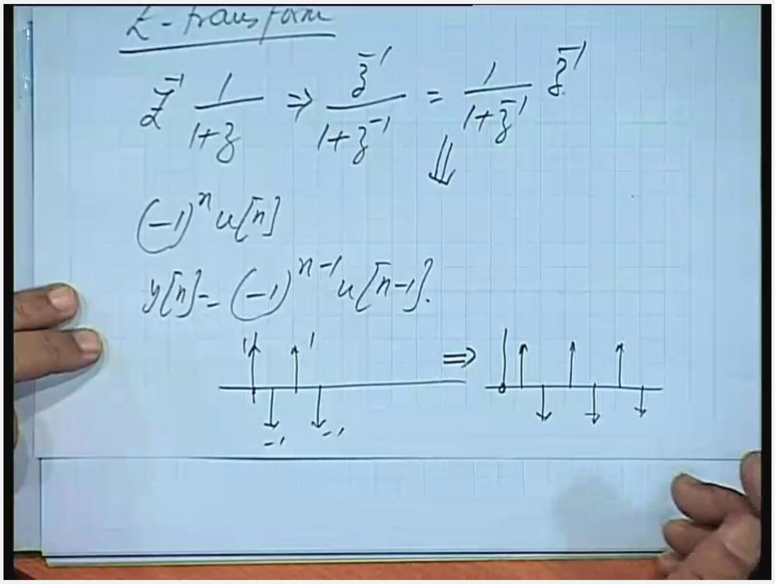

Now, before we go to the basics of this change of rate let us just brush through some other basic elements of Z transform, that you may be all knowing I want you to brush up your basics of Z transform. What is z inverse of 1 by 1 plus z, we take it as z inverse 1 plus z inverse, I can write like this, so this is delayed by one step.

So, I can write z inverse of this and then I will give a shift of one step so this z inverse can be kept aside and, what is this z inverse of this minus 1 to the power n u n and that gets a further shift of one step. So, actual output if I call it y n will be minus 1 to the power n minus 1 u n minus 1 is that, what will it be suppose it is a causal system, then had it been only minus 1 to the power n u n will be when n is 0 it is plus 1 then minus 1 then plus 1 then minus 1 and so on. And, since there is a shift of one step, so actual output will be this is the z inverse of this and multiplied by z inverse will give me 0, then plus 1 minus 1 plus 1 minus 1 plus 1 minus 1 and so on is that.

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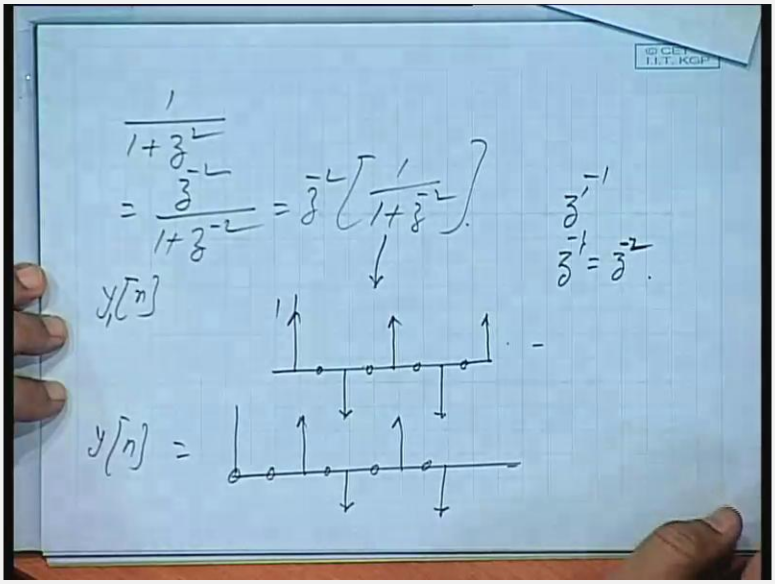

Now, suppose it is 1 minus z squared 1 minus z squared, then what you do or for that matter 1 plus z square I have plus z square and minus square. Once again I can multiply by z to the power minus 2 divided by 1 plus z to the power minus 2, since z to the power minus 2 plus 1 its 1 on the same thing. Now, I will take z to the power minus 2 out I will give at the end shift of two steps and determine the z transform of this z inverse of this, so what will be y n corresponding to this, let us call it y 1 n, what is the z inverse of this, what you do.

Now, if you consider the operation of z inverse is equivalent to just one shift, then z to the power minus 2 is equivalent to two shifts. So, treat this as some z dashed to the power minus 1, where z dashed is equivalent to two shifts two normal shift I am considering as a single shift in a new scheme.

So, z dashed is basically z square or z dashed to the power minus 1 is basically z to the power minus 2. So, what will be the result corresponding to this could someone tell me forget this, what was the result corresponding to 1 plus z inverse it was minus 1 to the power n u n it was like 1 minus 1 1 minus 1. So the same pattern will be followed with a gap of 2 this gap is, now 2 2 sampling time. So, I will have one this is one unit and so on, this will be corresponding to this do you all agree, so what will be the z transform of the whole product y n, it will have shift of two steps, so it will be 0 0 1 0 minus 1 0 plus 1 0 minus 1, and so on, is that is that?

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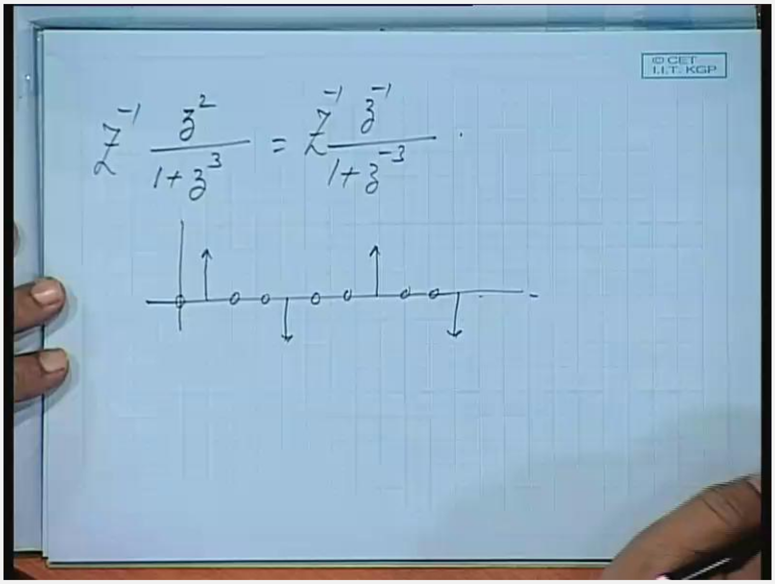

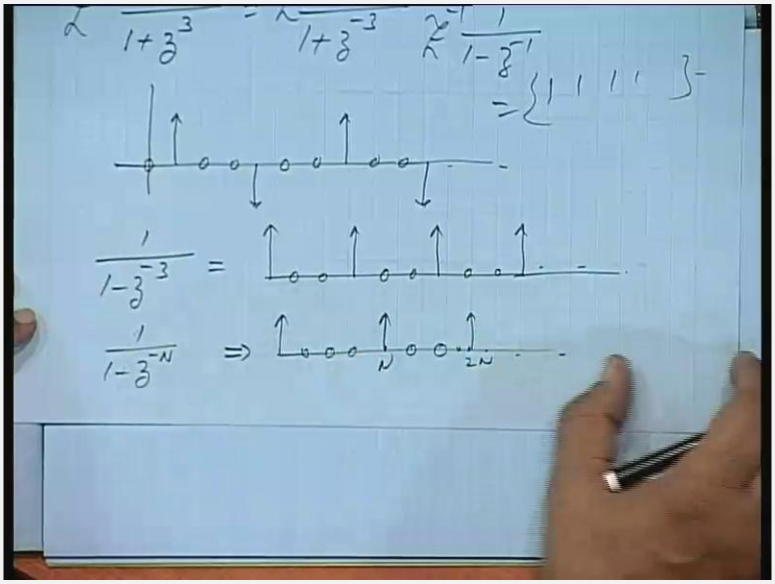

Now, let us take z by z square by 1 plus z cubed z square by 1 plus z cube, what will be the z inverse of this function, you can write this as z to the power 3 if I take out from here and multiply by z to the power minus 3 this will become z to the power minus 1 divided by 1 plus z to the power minus 3 is that so. So, corresponding to this could you please tell me the answer first will be 0, because it will be finally shifted by one step and, what is the result of this 1 by 1 plus z to the power minus 3 it is 1 0 0 minus 1 0 0 plus 1 0 0 minus 1 and so on.

Now, if I have 1 by z to the power minus 3 instead of plus if I have minus you know the z transform z inverse of 1 minus z inverse is unit step 1 1 1 1 so on is it not. So 1 minus z to the power minus 3 will be giving me 1 0 0 1 0 0 1 0 0 1 and so on is that. In general will be having a sequence like 1 0 0 0 at N again it will be 1 0 0 again at 2 N it will be 1 and so on 3 N and so on . If it is plus then, it will be alternative in sign, if there is any z to the power minus 2 3. So many shifts will be there in the entire sequence will be making use of this property at some stage, so I just wanted to brush up this.

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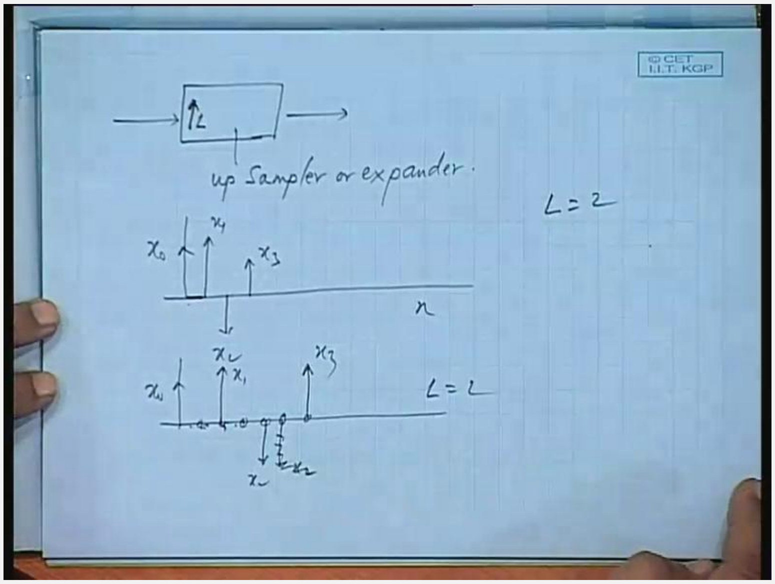

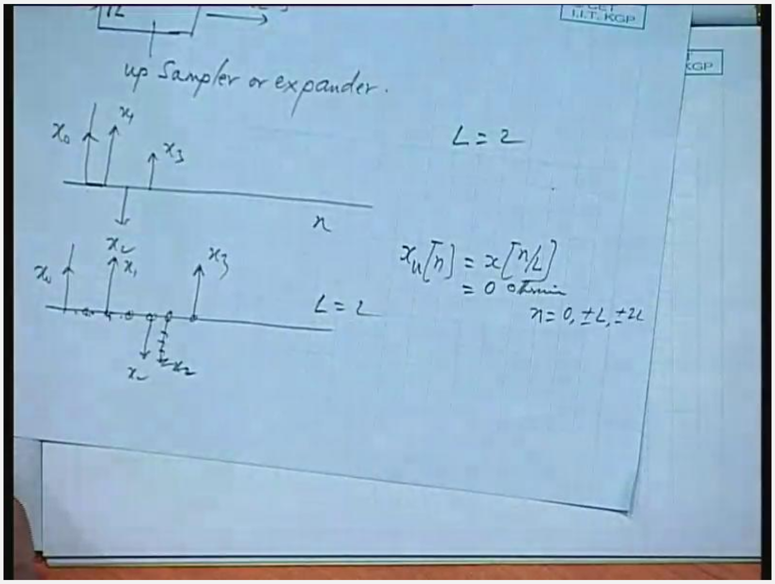

Now, we call a device which alters the sampling frequency, which increases the sampling frequency by a factors L you write like this an upward arrow we call this as expander or up sampler up sampler or expander, which expands the signal in the time domain. So, if you are having x 0 x 1 x 2 x 3 like this a sequence in the time domain it will be expanded x 0, then I push in a few zeros. So, if L is 2 I expand it by 2 that means instead of one segment difference there will be two segment gap.

Now, this will be x 1 0 0 this will be x 2 0 this will be x 2, then 0 this will be x 3 and so on if L is equal to 2 if L is equal to 3, then there will be 2 such gaps and 3 such blocks that means two zeros inserted and so on. So, what will be the mathematical expression for this I will write x up sampler output if I call this as x u n this is x n x u n will be x n by L n is equal to 0 plus minus L plus minus 2 L and so on, and is equal to 0 otherwise is this.

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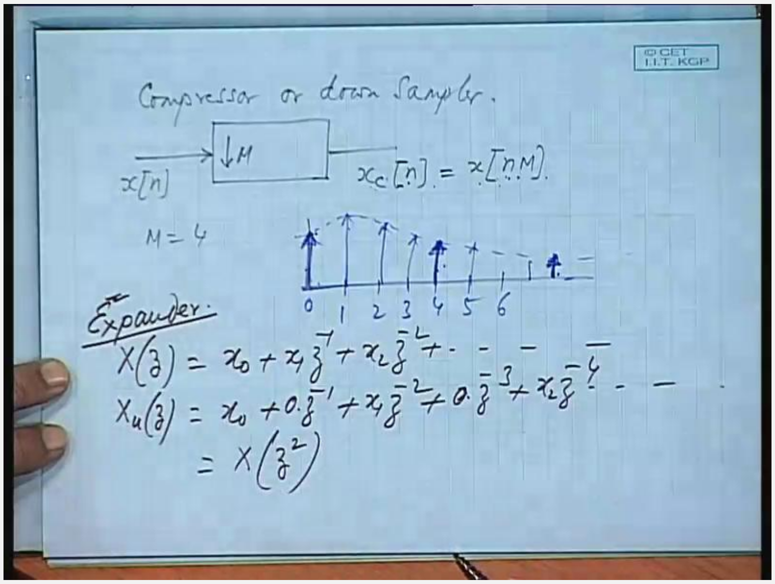

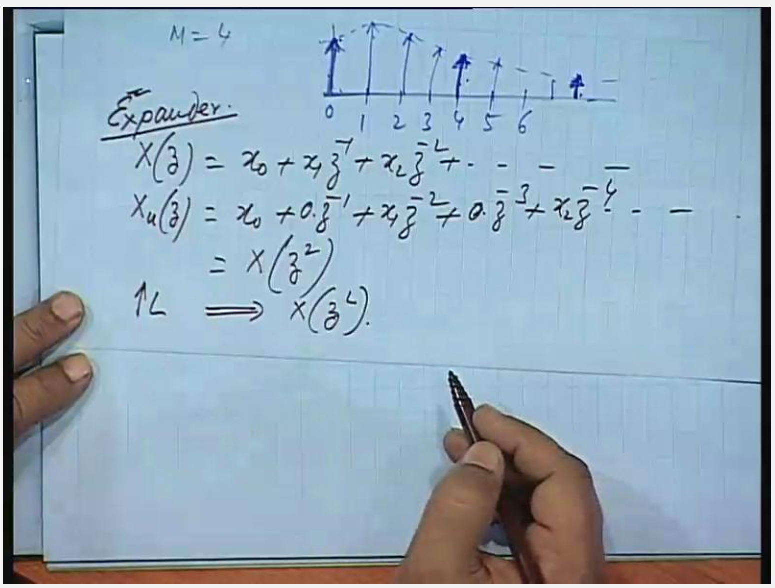

Similarly, we define a down sampler or a compressor or a down sampler like this M and this is x n this side I call it x compressed n. So, this x compressed n will be x n M I will show it like this suppose M is equal to 4, then the first value of down sampler n is the first value of the original sample first value actually. We normally write as 0 0 is the first value, second value will be x 4 next value will be x 8 and so on.

So, the series that you get here will consist of the values of x original sequence x, which are occurring at 0 4 8 and so on is it not. So, in down sampler you are actually dropping the values at regular intervals excuse me, so 0 1 2 3 4 5 6 and so on. And suppose these are the values this is say original analog function the continuous domain function and we are sampling it at regular intervals initially at this points.

Now, what you are doing in down sampling you are changing the sampling rate; that means, you are taking this point, then again you are taking this point, then again you are taking this point. So, I will darken this the points that you are picking up when M is equal to 4 it is like this; obviously, lot of information is lost, because from this point to this point there has been a change I have missed it out it is possible.

So, will see very soon what are the restrictions on the rate at which, we can down sampling it we can down sample it at any rate, now there has to be some restriction, if you want to return the information, then what should be the rate of down sampling. Similarly, do you get anything unwanted if you up sample it; that means, a signal is there it has a certain characteristics you sample it you over sample it actually you do not over sample it, you have already samples it is already sampled.

Now, you are playing with a data by injecting some zeros you are injecting some zeros; that means, you are inserting some values is it different from say injecting suppose I want to expand it by a factor of 2. So, instead of inserting 0 suppose x 1 and x 2 these are the 2 adjacent values I insert a value x 1 plus x 2 by 2 and somebody else insert a 0 will it be equivalent. So where is the problem we would like to see that also very soon.

Let me change the pen its drying up, so x z in an expander or up sampler suppose originally X z is x 0 plus x 1 z inverse plus x 2 z to the power minus 2 and so on and x up sampler that we are considering there is a 0 in between. So it is a sequence is x 0 then 0 then x 1 then 0 and so on. Suppose you have expanded factor of 2, so it will be 0 into z to the power minus 1 plus x 1 into z to the power minus 2 plus 0 into z to the power minus 3 plus x 2 into z to the power minus 4 and so on. So what is it x drop the 0s instead of z I have actually made z square is it not. So, in general if it is up sample by a factor of L, then this will be taking the form X z to the power L, what would be its frequency response.

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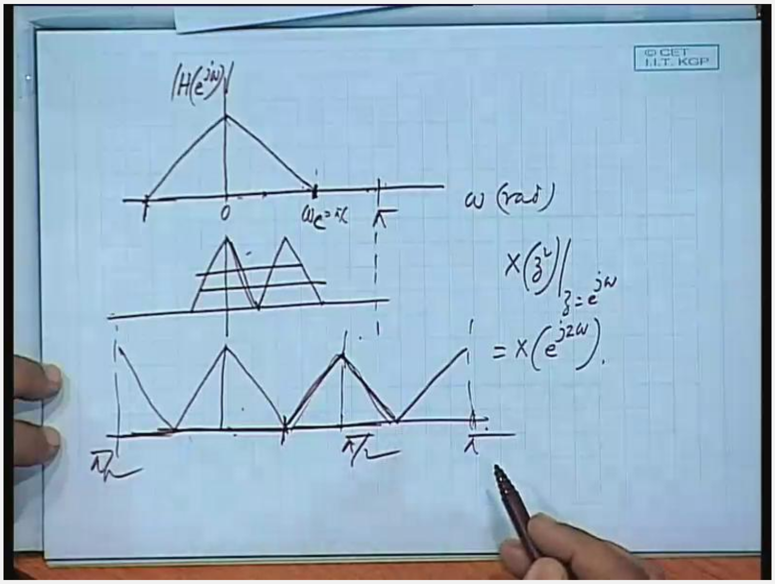

Let us see this is the discrete frequency in radiant is suppose I am just considering a very simple frequency characteristics this is your H omega magnitude 0 this is omega c and this is say pi it is well within omega, omega c is well within pi that is the sampling frequency the frequency is covered. Now, I have just doubled it I have doubled it, what will you get X z square if I write X z square and z is equal to j omega e to the power j omega then what do I get x e to the power j 2 omega.

So, whatever values we are getting at omega c by 2 at omega c earlier I will start getting at omega c by 2 so this will be this base has been reduce to half. But, then if I keep on increasing omega this gets repeated also, so within pi it will start reappearing. Also I have not drawn very good diagram it will be like this suppose it is just pi by 2 if it is pi by 2. So one segment is covered in pi by 4 in pi by 2, then 2 such segments will be covered in pi.

So, it will be let me take this as pi by 2, so half of this base basically this is from minus pi by 2 to plus pi by 2. So this is pi so it will be somewhat like this, this entire range this is half the segment this is half the segment that makes one full segment. And this is one segment two segments will be covered if in a range of pi one segment was covered.

Now, in a range of pi once again this will be pi by 2 this will be pi by 2 and pi will be here somewhere, so it is doubled so till pi, now it is things will start appearing like this. Now, this is the original one and this is a false image; that means, at a frequency within pi, but higher than this frequency you will have another false control, what is it due to let us see.

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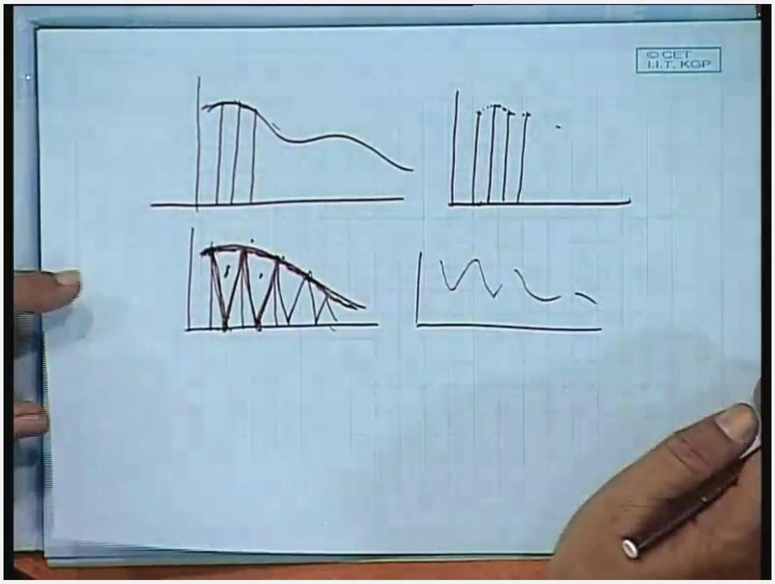

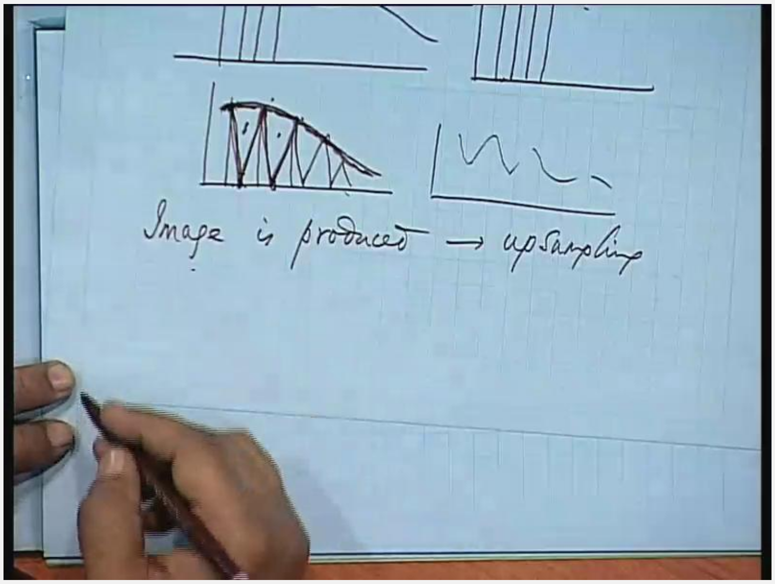

Let us physically understand the actual phenomena in terms of the signal itself its original signal is like this you have sampled it at this point, at this point, at this point. So; that means, your sampled values are these I if I keep on joining them joining the tapes by straight lines even that will be describing very closely the original analog curve.

And now, what you are doing by this expander, what you are doing is this one then is 0 then this one then is 0, we have inserted is 0 in between I am just showing it in an expanded form. But basically what you are doing instead of connecting them now you are bringing it here, and then going up there, bringing it here going up there.

So; obviously, this function will have an envelope which corresponds to the original analog signal. But it has very high frequency components; that means, you are bringing about a very fast change in the signal value from here it is falling to 0, then again shooting up to a high value. That means, there is a very fast change that is taking place and that will give rise to a high frequency image, which has to be got rid of. So, if I have a band pass filter, a low pass filter I will allow only this and drop this, then I can reclaim the original signal is that.

Basically, if you for example averaging is a low pass a low pass filtering suppose I take 3 part 3 point average, so this plus this plus 0 divided by 3, then this will be brought up somewhere here, similarly these 3 0.s if I take the average it will be brought up here. So, if I take the averaging if I do the averaging of this then finally I will get something like this will be somewhat better than the earlier one. It all depends on what kind of filter averaging is, one kind of filter I can always have other varieties of filter, where finally will come back to this profile.

If I filter this it will give me this set of data will give me an output whose frequency transform through frequency transform you will find which will be describing more or less the original analog signal, which this was described by this practically. So, the image is produced in an up sampler by this is up sampling process and this you will have to see you have to insert a filter. Later on in a down sampler, you are dropping some values, so you are losing some important information’s.

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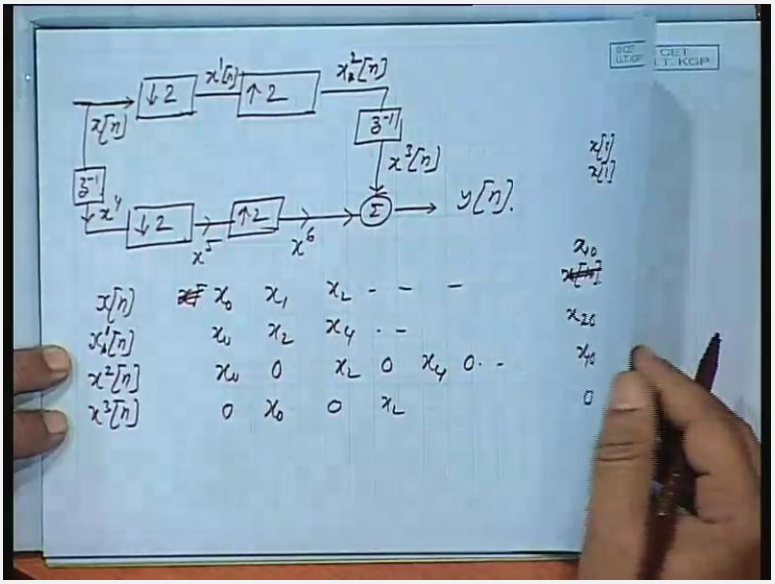

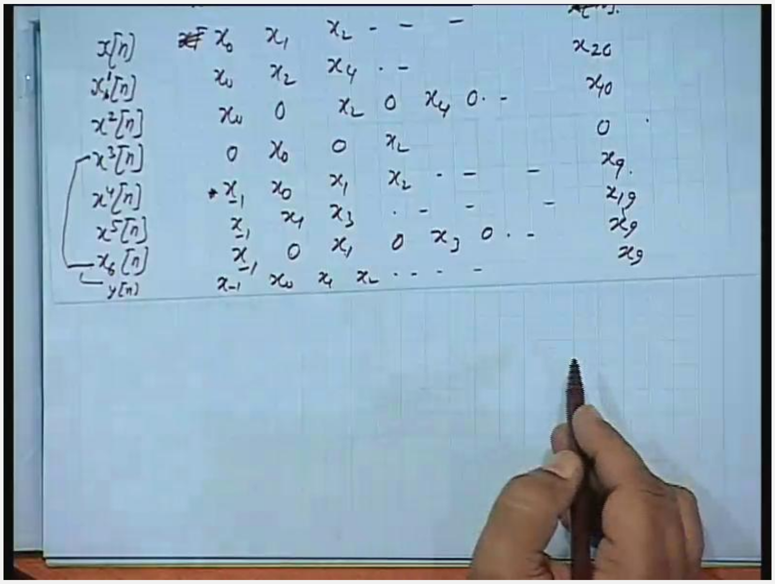

Now, let us see a simple example I will take a fresh page a down sampler, which will be down sampling by 2 and an up sampler by 2. Then I delay I get an output and the same input is given to another channel, where do you have a delay, then a down sampler, then an up sampler and then add it, let us see what this y n will be like.

Let us take this as some x 1 sequence x 2 I am writing superscripts then x 3 similarly x 4 x 5 and x 6 x n is the sequence x 0. You can write like this x 0 x 1 x 2 it means basically x 0 x 1 every time I am not putting this is. So, let me write suppose x 10, then what will be x 1 n x 1 n this sequence like I have down sampled by 2, so it will be x 0 x 2 x 4. Similarly, x 20 x let me stick to the same convention x 10 and x 20 what will be x 2 n this sequence will be like x 0, now I am up sampling it 0 x 2 0 x 4 0 and finally x 10.

Then, that gets delayed by one step, so what will be x 3 what will be x 3 like it will be x 2 n shifted by one step, so it will be starting with a 0 then x 0 this whole thing is shifted by one step 0 x 2 so this will be a 0 x 10 will come later. I am just observing one particular column there are data there will be data points on this side also x 4 this one is x n delayed by one step. So, it will be shifted 1 step if it is a sequence of data coming then it will be something like x minus 1 x of minus 1, then x 0 x 1 x 2 x 9.

The entire thing has been given a shift x 5 will be x minus 1, then it is down sampled x 1 x 3 like that x 19 and x 6 again fill with zeros, so x minus 1 then how much is it 0 x 1 0 x 3 0 and so on so x 9. Now, if I add x 3 and x 6 x 3 and x 6 if I add here, what will be the output y n it will be x minus 1, then 0 and x 0 then x 1 x 2 you will get x 9, you will get back the original sequence except for this one shift you get the shifted version of the same input sequence.

So, their complementary it to each other individually you see there is an up sampler and down sampler by the same factors there is a delay element there is a delay element here also, but they do not produce the same sequence x 6 and x 3 they are not same. Though, I have taken the down sampler and up sampler in the same sequence same order I have just change the positions of these two delay elements and they generate two different sets. That means, this entire operations is not linear it is linear it is not time invariant in the time domain they are not invariant.

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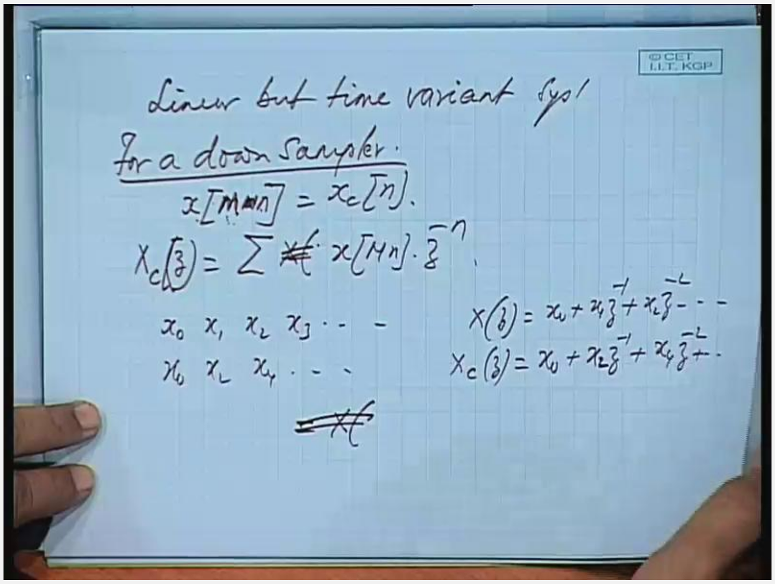

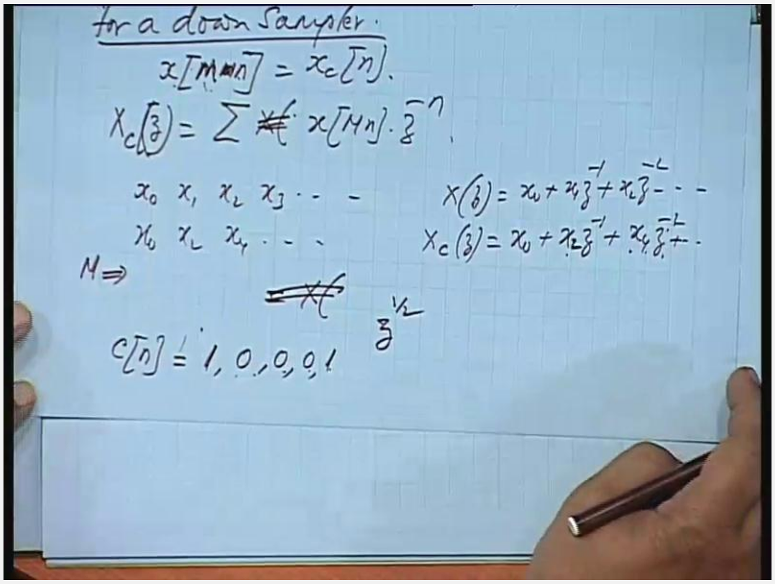

So, it is linear but not time invariant, so time variant both these up sampler and down sampler. Now, let us take for a down sampler now, you have got x m n capital M n as the output of the compressor is it not every meth value we are considering. So, what will be x compressor z the z transform of this I will not write x now it will be x M n z to the power minus n, because it is x c n z to the power minus n and x c n is nothing but x original x capital M n, so it is like this. Suppose you have got x 0 x 1 x 2 x 3 like this, so x z is x 0 plus x 1 z inverse plus x 2 z to the power minus 2 and so on and now you have down sample by 2, then it will be x 0 next x 2 x 4 and so on.

And, so what is this complex sequence z transform it will be x 0 plus x 2 z to the power minus 1 plus x 4 z to the power minus 2 and so on. So, what is it in terms of x z in terms of X z z, what can be what can replace z z to the power half there is something missing, no. So, let us search for that will it be you are getting x 2 z to the power minus 2 x 4 z to the power minus 2 and so on. So, z is getting compressed by a factor of 2, so z will be replace by z, but will that be sufficient there is something else missing. So, let us see what this will be like, let us define a sequence C n as 1 0 0 0 after M steps if M is the down sampling rate, then after M steps again 1 appears. That is it is having a excuse me its having a period of M it is a periodically appearing 1 this is a set.

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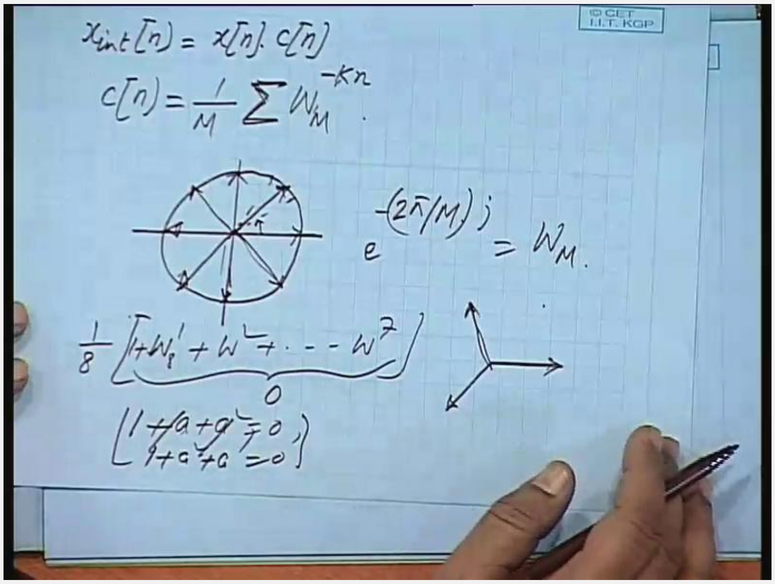

So, I define some intermediate sequence x n as xn into C n, where C n was this sequence. So basically x n is multiplied by zeros elsewhere x n is multiplied by at 1 at regular intervals and this is what is the intermediate sequence. So, C n can be written C n can be written as 1 by M sigma W M minus K n is it not? You have a similar thing we have done earlier, suppose there are this is 2 pi and we have divide it divided it by some M number of equi spaced points.

So, what will be per segment angular ship 2 pi by M, so W M is e to the power minus j 2 pi by M, which we have used in discrete Fourier transforms. So, this is W M, let us take M is equal to 8 then, what does it mean 45 degrees spacing like this like this. Now, what is 1 by 8 sigma of all these W M, you can take n is equal to 0 it will be 1 1 1, because anything to the power zeros one, so there will be 8 1is divided by 8, so it will be 1.

Then, if you take n is equal to 1 n is equal to 1 means this will be e to the power minus j 45 degrees 2 pi by 8 I am taking M is equal to 8. So it will be W1 I am not writing 8 every time W 2 w 0 is 1 up to W 7 and how much is it 0 all these components, if you add up its a balance set of vectors, so it will be 0.

Then, if you take n is equal to 2 if you take n is equal to 2 it will be 90 degrees and then multiples of that. So you will get again 0, so for the next 7 steps it will be all 0 again in the 8 step you will get all ones this is precisely I was discussing earlier. This is precisely, you get in case of a positive sequence negative sequence components in a three phase system if the system is balanced, then what you get is only one, when you multiply by alpha, alpha square of that balance set you get 0.

So, you get this terms is it not 1 plus a plus a square is equal to 0 if i take the double the value of that shift, so 1 plus a square plus a to the power 4, which is again a that is also zeros. So, very similar operation you take any n point sequence and these values will be always equal to 0 every nth step only it will be appearing as one.

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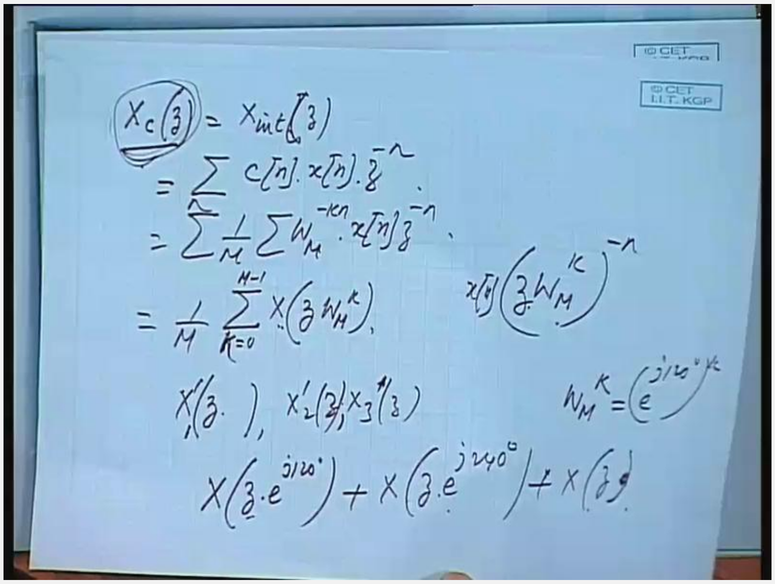

So, C n can be conceived as this, so if that is so then what will be X z x compressed sequence z, which is nothing but that X intermediate, which is summation of C n into x n into z to the power minus n okay and what is C n I have expressed in terms of 1 by m sigma this. So, I will put it as this is varying over n 1 by M sigma wm minus K n x n and z to the power minus n. So, it will be 1 by M sigma this W M K n can be clubbed with z n X, so z and this if I club together and this to the power minus n gets multiplied by x n so original z is replace by z into this is that.

So, you take the z transform of x if you know z transform of x in a down sampler you add suppose it is down sampled by three, then whatever x z is given you multiply z by you multiply z by W M K, where W M K it is it is submitted over K z W M K will be e to the power j 120 degrees to the power K. So z will be replace by that you take different values of K 0 1 2. So, three such terms will be added together that will be give you that will give you and of course divided by 3 that will give you x z is that this is submitted over KK varying from 0 to m minus 1.

So, basically it is a summation of z transforms if you call it call it as X 1 z X 2 z X 3 X, where I will it X1 these three, then what is X 1 z it is original X z multiplied by this factor e to the power j 120 degrees. If I write like this next one is X z into e to the power j 240 degrees third one is plus X z e to the power j 360 degrees, so that is x z itself. So, summation of these complex z transforms will be and divided by 3 that will be the overall z transform of X z is that, what will be its effect.

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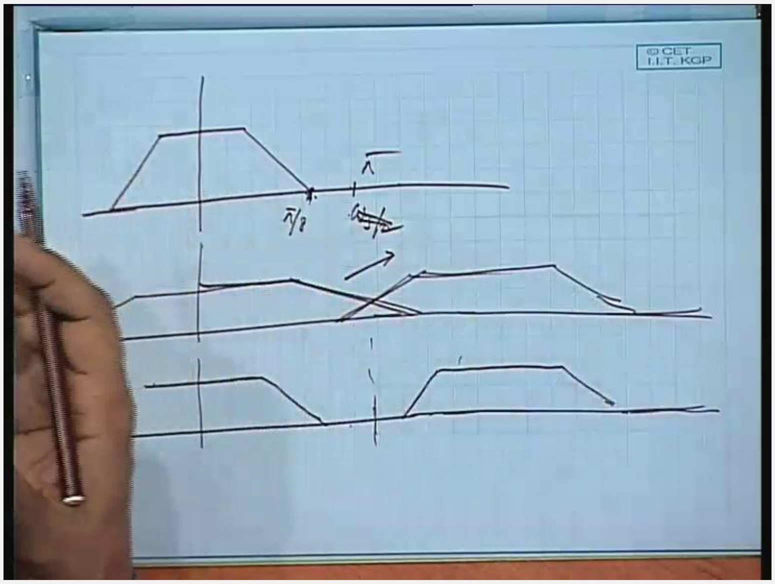

You, are down sampling means you are crossing into probably you are getting you are crossing the nyquist border; that means, frequency response say, which was like this was well within omega W s by 2 rather pi we call it pi. Suppose I have double the frequency I have double the frequency; that means, this frequency rather I have down sampled it by a factor of 2. So, this will be going into this it will be doubled, so the base will be expanding like this and then they will be aliasing.

So, you can down sample by a factor m such that m times this should be contained in pi suppose this is already given as limit is pi by 8, then I can expand it by a factor of 8. When it will be just touching pi if I down sampling down sample it by a factor more than 8, then there will be an effect like this. So, down sampling has this limitation number one number two suppose I have down sampled it by a factor of four, then this will go up to pi by 2 or pi if I have gone to pi, then it will be approaching this and then it will also start reappearing here and so on. So, will see what should be the restriction on the rate at which, we can down sample and what are the filters required both for up sampler and down sampler to a see that the signal the change signal carries the original information; that means, the frequency response does not get altered.

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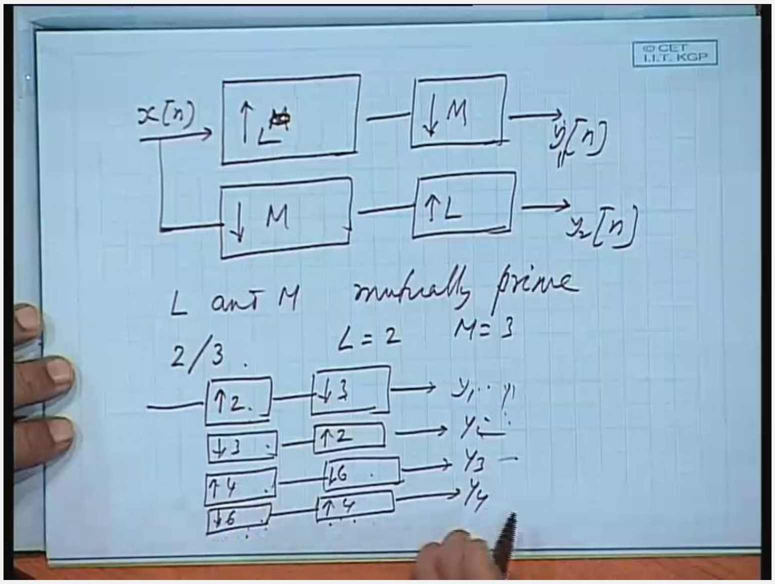

I will lastly touch upon this point suppose I have an up sampler M and then a down sampler we are using l and down sampler M this is an information x n I am getting y n y 1 n. Now, I have a down sampler M first and then l y 2 n under what condition both y 1 and y 2 will be identical it can be shown that if L and M are mutually prime if they are mutually prime they will be identical. Now I leave it as an exercise you try at home take say 2 and a factor of 3 L is equal to 2 M is equal to 3, so in one case you experiment you have an half sampler of factor 2 and a down sampling rate be 3 observe y 1.

Then, you down sample by 3 up sample by 2 observe y 2 up sample by 4 down sample by 6 observe y 3 down sample by 6 and up sample by 4 observe y 4 and then see y 1 is equal to y 2. You verify for yourself whether y 1 is equal to y 2 in this because this is 2 and 3 mutually prime whereas with 4 and 6, which 1 gives you the correct output that is whether y 3 is equal to these 2 or y 4 is equal to these 2. Then, that will decide what should be the order of appearance of the down sampler and the up sampler intuitively, you can guess if a down sample first, then I will lose out many information's then packing them with 0s will not really rest out the original signal is it not?

I can always infuse some zeros, I can always put some zeros in between and then I can down sample some other zeros, you will get dropped, some other information's will also get dropped in any case while down sampling we are doing that; but while up sampling we have already we have expanded the data. And then only we are reducing. It is better to do that instead of doing the down sampling first, and then up sampling this, but if they are mutually prime, the order does not really matter. So, I would request you to verify all these four conditions four outputs.

Thank you very much. We will stop here for today.