Digital Signal Processing
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Lecture - 32
Relationship Between Real and Imaginary Parts of DTFT Contd…

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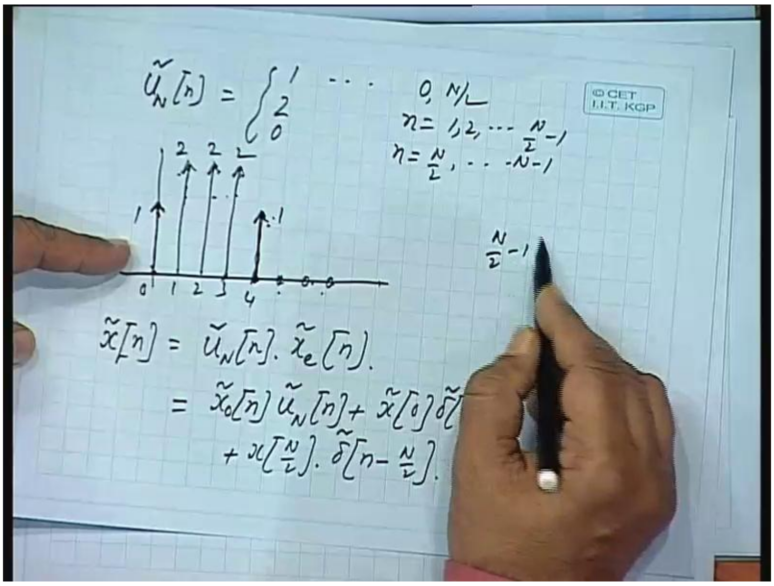

We were discussing last time about a new sequence. If, we define it like this 1 2 up to n by 2 minus 1 and then n is equal to n by 2 up to n minus 1, that is we are defining a sequence, which is 1 at this point, then 2, 2, 2. Say for example, for that 8 point sequence this is 0, 1, 2, 3, 4, 5, 6, 7 these 3 zeros, where all zeros is it not the last three were padded zeros. And this one there was a symmetry about the first point and N by 2 this point, so at this point the magnitude is 1 and at this point it is multiplied by 2.

So, if you define a function like this, which is 1 at this point and at N by 2 also it is 1 at all other points in between it is of magnitude 2, at this point it is 0 we defining like this. Then we can write the periodic function that we are considering as u n n into x e n is that, we could recover the entire function x n from it is even part where multiplying by this we saw last time or in terms of odd function. We can write x naught n x o odd this a periodic function again plus in terms of odd. You require the initial value also plus x N by 2 into delta for the odd function this value is not known this was zero you remember this was 0. So, here you have to add these two separately delta n minus N by 2, we will write these also as periodic functions, because we are considering a periodic function. So, these are to be multiplied at every period, so this denotes a periodic delta function.

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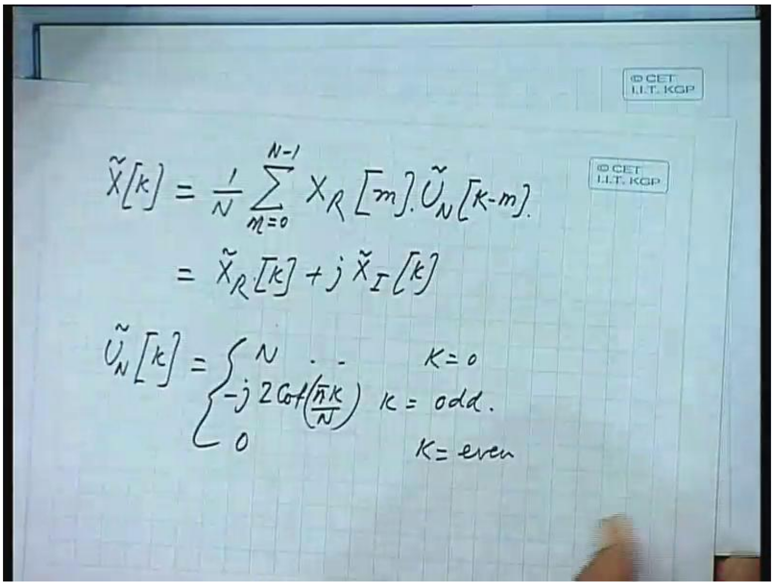

So, in terms of discrete Fourier series this will become if you allow me to write x tilde k it is a product of two functions. So, in the frequency domain it will be convolved 1 by N for the even part it will be X R real part X R. You can write m and I write capital U N to denote transform of this discrete Fourier transform of this, what should it be submitted over m and that suppose to be X R K plus j X I K.

So, let us see what this function is like U N K. I will first prove this is equal to N for K equal to 0 is equal to minus j twice cot of phi K by N for K odd and equal to 0 for K even will prove this, once you get that then after substitution will get very interesting simplified results. So, what we are having is basically a sequence 1, then 2, 2, 2, 2 up to what value this is N by 2 minus 1 this is N by 2, N is 8 0, so this N by 2, so N by 2 minus 1. So, you should have a sequence first starting with 1, then 2, 2, 2, 2 up to this term and then it is again 1, then all zeros.

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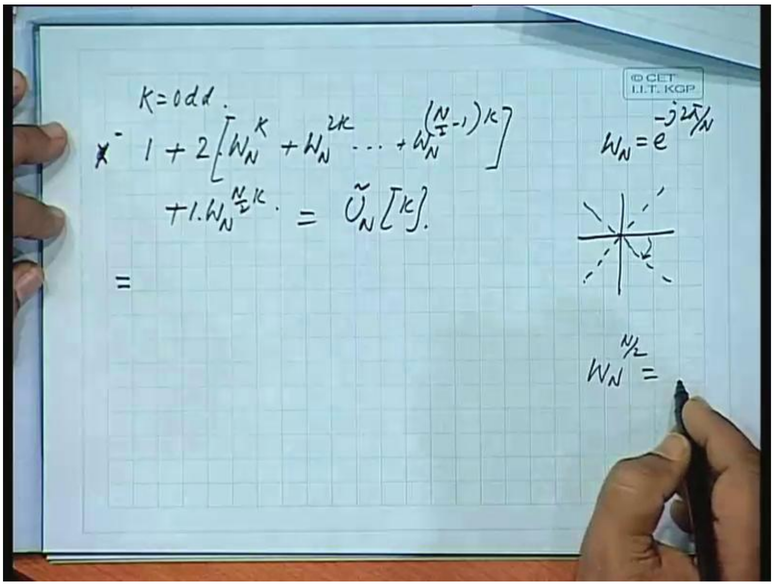

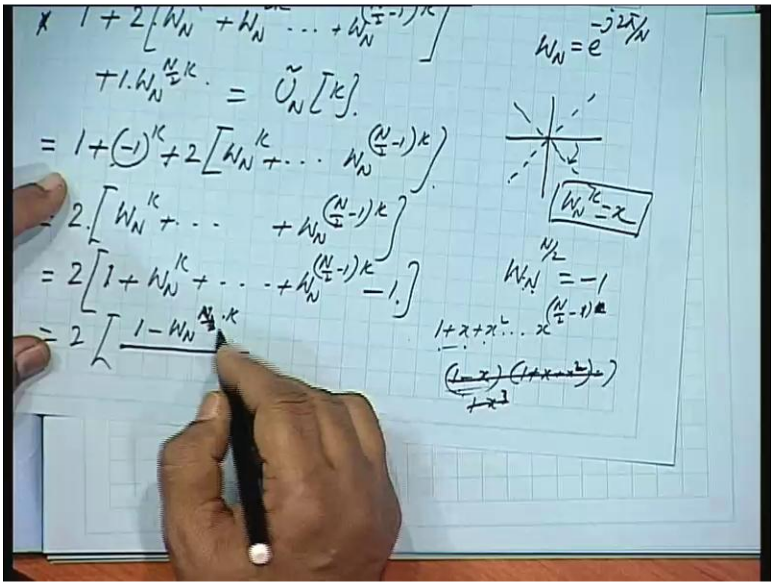

So, what will be the transform of this for k equal to odd say will be 1 plus 2 into if I write W N K, where W N is the usual term that we have used e to the power minus j 2 phi by N. In this case it is 45 degrees if you take an 8 0 sequence is it not? 2 phi by 8, so minus 45 degrees. So, W N K plus W N 2 K and so on, W N, N by 2 minus 1 into K correct me if I am wrong plus the midterm at N by 2 that is 1.

So, W N this is multiplied by 1 N by 2 K this is your U N K is equal to U N K, now how much is this I can write when any like when K is odd, what is W N N by 2 W N N by 2, you take any even number N. So, if I take 6 and this is 3, so the entire circle I divide into

6 equal parts I take 3 of them; that means, I rotate by hundred and 8y degrees, so W N N by 2 is always minus 1.

So, this 1 will be minus 1 to the power K, so if I club this with this 1 plus minus 1 to the power K plus 2 into this quantity W N K plus W N N by 2 minus 1 into K K is odd. So, this will be 0, it will be 2 into W N K plus W N N by 2 minus 1 into K, how much is the sum? How much is this? One may write like this it is 2 into I can add 1 and subtract 1. So, it is like this 1 plus x plus x squared up to x to the power N by 2 minus 1 into K I will not put K N by 2 minus 1 where W N K is taken as x .

So, this 1 plus x plus x square it is a gp series if I multiply by 1 minus x divide by 1 minus x it gives me. Let me write this I have just added and subtracted 1 and this series is like this, so it will be 1 minus W N if I multiply by N by 2 K is that, so are you writing that I am multiplying by 1 minus x. So, I will get 1 minus x cubed and divide by 1 minus x. So, if it is up to x square I will get something like 1 minus x cube by 1 minus x if it is for 4 3 terms, then I will get 1 minus x to the power 4 by 1 minus x, and so on.

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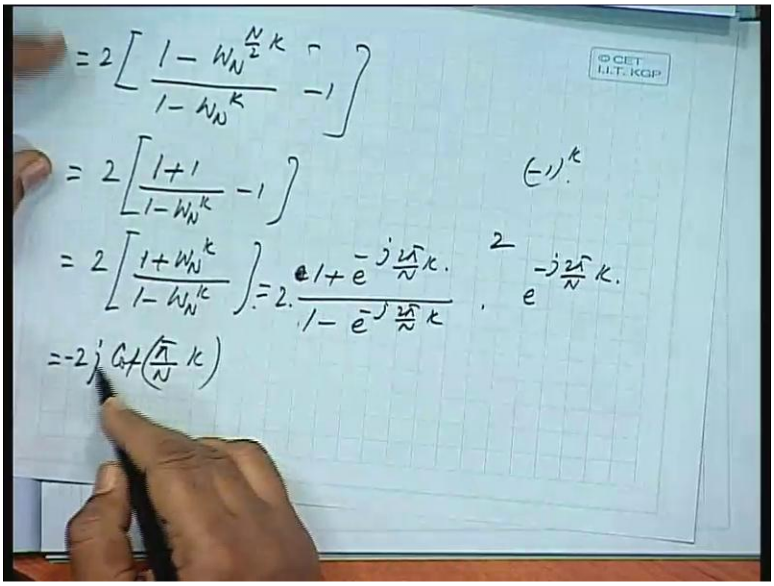

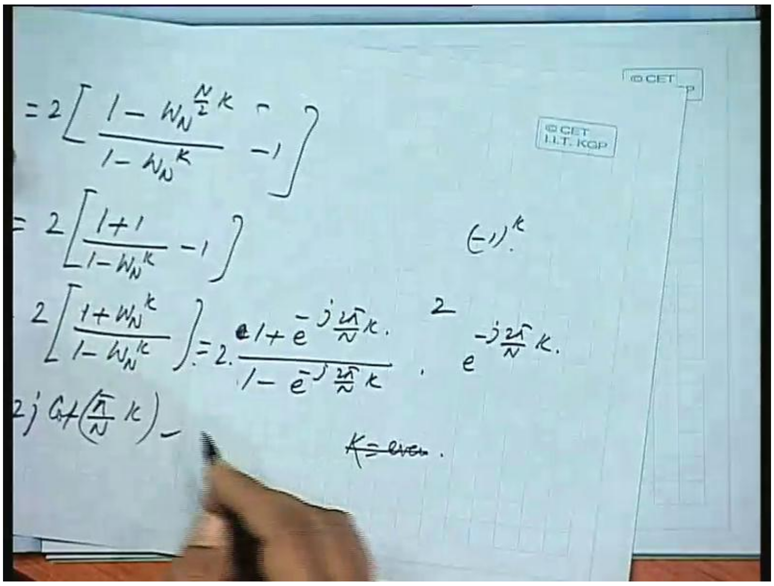

So, it is just the gp series that is, what I am trying to find out, so W N N by 2 will gives me 2 into 1 minus N by 2 K by 1 minus W N K minus 1. Since, K is odd this is minus 1, so minus 1 to the power K will be minus 1, so minus and minus will make it plus, so 1 plus 1 divided by 1 minus W N K minus 1 it is 2 into. So this is 2 subtract 1 will go, so 1 plus W N K by 1 minus W N K is that.

If I take W N K as e to the power minus j 2 phi by N into K, then it will be and if I take half of it outside. So, if I take I write 1 plus e to the power minus j 2 phi by N into K divided by 1 minus e to the power minus j 2 phi by N into K multiplied by 2. So, that is 2 this is, what the familiar cotangent form cotangent of this angle divided by 2 with a j term j comes in the denominator. So, that becomes minus j cotangent phi by N into K can always take e to the power j theta by 2 outside that will get cancelled, so this will become cos this will be becomes sin of theta by 2, so this is U N K.

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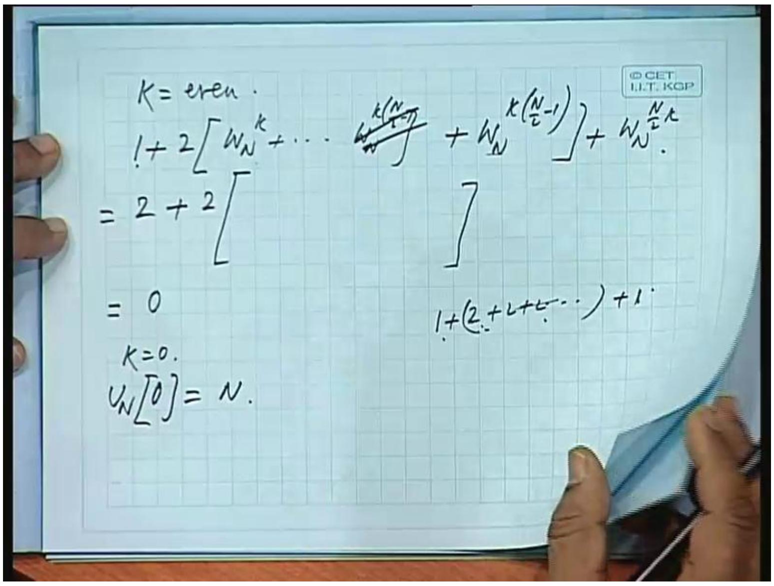

If, K is even, then what you get if k is even what is see can you see for yourself it is 1 plus 2 into W N K plus W N K into N by 2 minus 1 plus W N N by 2 K. So, this will give me 1 plus 1 2 plus 2 into you work it out yourself, you can see for yourself this whole thing will turn out to be 0. And, for K equal to 0 for K equal to 0, what will it be it is 1 plus 2 plus 2 plus 2 up to… So for example, for 4 0 Sequence for an 8 0 Sequence that we are considering there are five such terms 2 are going as ones, so 3 are left and these 2 ones, will make another 2’s, so basically it is N by 2 into 2. So, this will become this summation will give you N.

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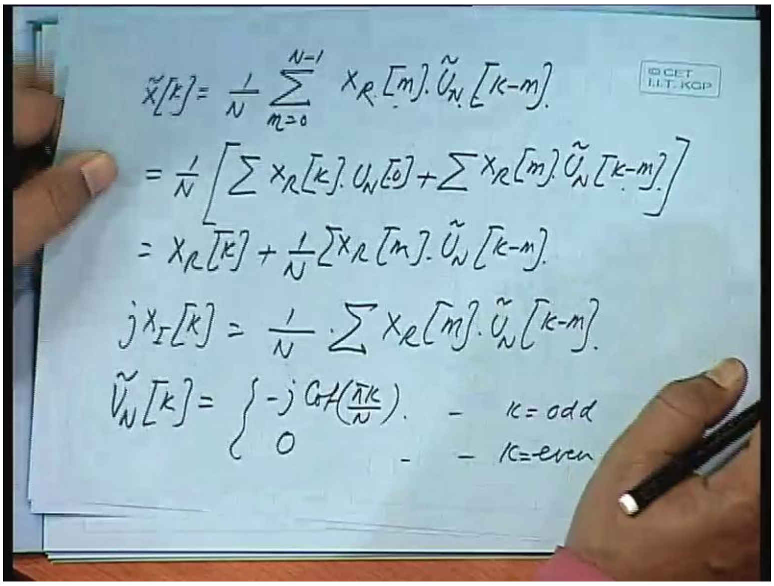

So, we can write X K from these as 1 by N summation say sum m equal to 0 to N minus 1 X R m into U N K minus m, and you can have any running variable. And that is equal to 1 by N I can take out X R K into U N 0 plus X R m U N K minus m, when K minus m will not become equal to 0, when submitted over N for all other values of N.

And, what is this un 0 is n, so N will get cancelled X R K is a constant summation U N 0 is N, so N will get cancelled. So this will be X R K plus 1 by N, this time this term X R K plus summation X R m U N K minus m I just taken out from here the even 0 term and that corresponds to finally, X R K. So, X K is equal to X R K plus something and what is that something then the imaginary part see these are complex quantity, which has the real part plus some imaginary part and that I am writing as the real part plus something, then that something must be equal to the imaginary part. So, j X I K is nothing but this 1 by N summation X R m U N K minus m we define another function some V N K as equal to minus j cotangent phi K by N for K odd and equal to 0 for K even for K even.

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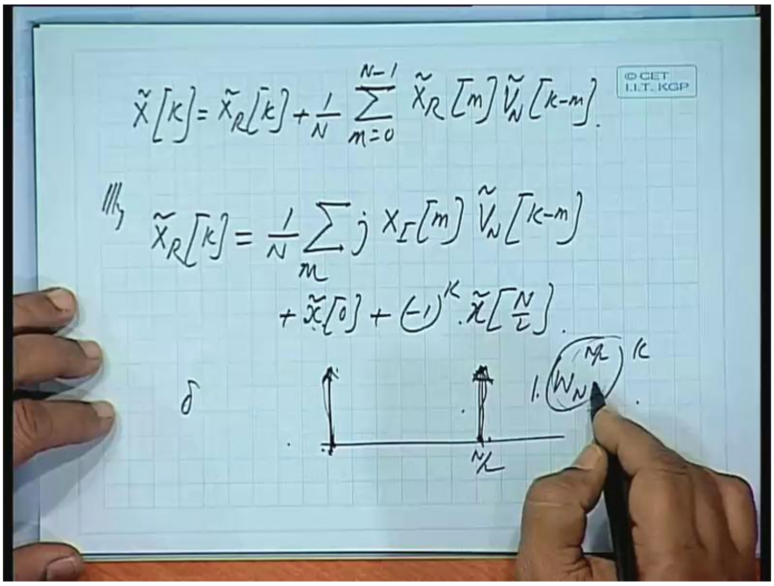

Then it is written in a more simplified form X K is equal to X R K is same X K is equal to X R K plus 1 by N this expression. I can also write as m equal to 0 to N minus 1 X R m in all this earlier cases I forget to put the tilde; that means, they are basically periodic sequences, there is not much of a difference if you take just one period of a dfs that is dft the relations are similar.

So, we have got X I similarly X I we have already got that expression here, similarly you can also find out the real part in terms of the imaginary part it can be shown it will be X I M V N K minus m anything else submitted over m anything else. When I am talking in terms of the imaginary part there are two missing values remember at N is equal to 0 and N is equal to N by 2 k is equal to 0 and K is equal to N by 2.

So, I have to add those values plus minus 1 to the power K x tilde N by 2, why minus 1 to the power K you remember you have to multiply by delta N and delta N minus N by 2 is it not? You have to add these two values, where the odd function was becoming 0 odd function was symmetric about this, but their values are 0, is it not?

So, these two are additional terms and if when I take the transform of this function that is equal to 1. So, 1 into delta x not and this is shifted by N by 2 steps for these delta shift delta what is the transform will be W N N by 2 into K into delta N by 2, which is 1 and this is what minus 1. So, that is why minus 1 to the power K into x tilde N by 2, so whatever is the value of x tilde at that 0, let us be multiplied by this.

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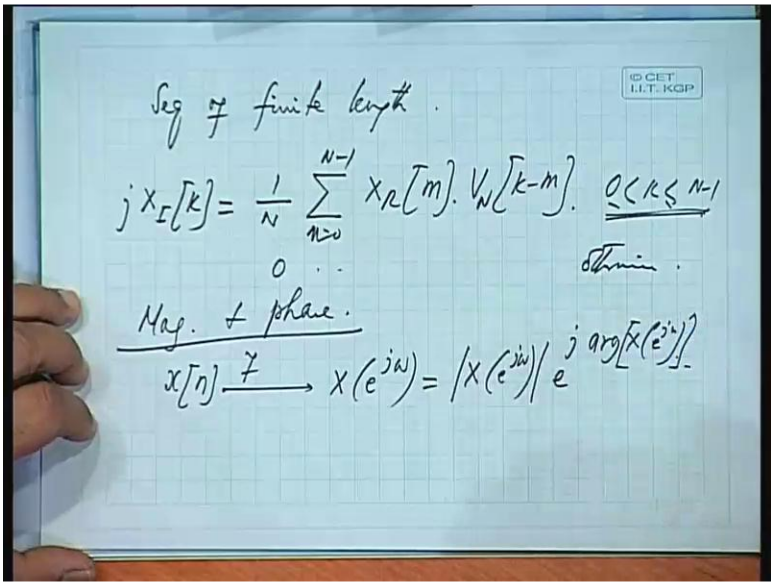

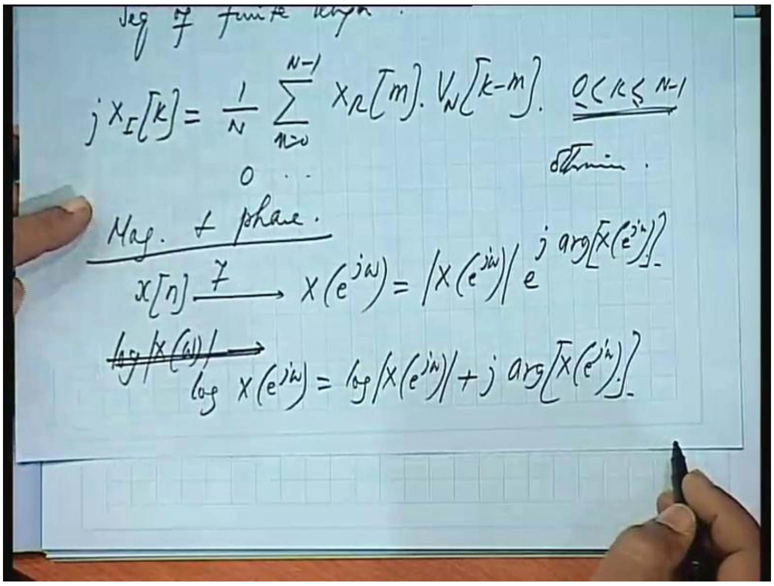

So, if you have a sequence of finite length if you pads on zeros get an end 0 sequence such that from N by 2 onward up to N minus 1 the values are all 0, then if you take the discrete Fourier transform. Because, we are not having basically a periodic sequence we defined a periodic sequence, but our periodic sequence is a pseudo periodic sequence, because it does not have any negative power, because ours is a causal sequence. We defined a new kind of a function causally periodic, but a periodic function cannot be causal.

So, it should it was a hypothetical function, what you wanted to say is 1 period of that if we consider, then it can be represented as a causal sequence and hence instead of Fourier series you take Fourier transform. So, the same results you will get X I K instead of a tilde, now I will write just X I K in terms of X R whatever we wrote X I K j times X I K is equal to this was tilde earlier.

So, it will be X R m is U N K minus m or you can write V N K minus m, so j X I K will be X R m V N K minus m submitted over N equal to 0 otherwise; that means, it is only for a finite set it is a Fourier transform not series. Similarly, for the imaginary part given imaginary part you can write the real part, now relation between magnitude and phase we have, so far related the real part with the imaginary part with the Fourier domain in the frequency domain. Now, if you are given the magnitude and phase how do you relate them, so if you given X N, you take Fourier transform you get X e to the power j omega I can write this as x j omega magnitude and an argument e to the power j times argument X.

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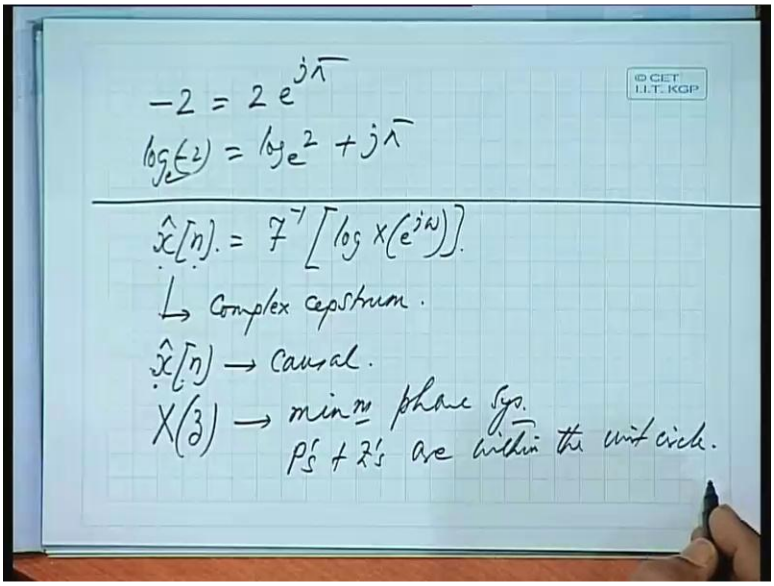

If I take log of this if I take log of this whether, what is log of minus 2 log of minus 2 you can calculate is it not? So minus 2, I will write as 2 into e to the power j phi principle value I can calculate, so log of minus 2 to the base e will be log of 2 plus j phi, so it is a complex quantity. Hence forth anything can have logarithmic value will not have the conventional notion that you cannot take logarithm of negative quantities I can take log of any complex quantities.

So, log of X omega log of log of X e to the power j omega will be log of x magnitude plus j argument X. We define a sequence x hat n as Fourier inverse of this quantity, this is a real and imaginary part, this is having real and imaginary part this a complex quantity. So you can take Fourier inverse of this inverse transform gives me a time domain response something like a time domain sequence it is not the original sequence x n. So, this is known as cestrum complex cestrum.

Now, it can be shown if x n is to be causal, we have establish for a causal sequence how to get the real part from the imaginary part by that Hilbert transform relation or imaginary part from the real part all right. That was for a causal sequence to utilize the periodicity property the DFT property we suggested and hypothetical sequence, that is a pseudo causal a pseudo periodic sequence, but it is meant for a causal sequence. Similarly, here if this is a causal sequence, then we can find out a relation between the real part and the imaginary part all right. That means, this imaginary part is nothing but the phase real part is basically log of the magnitude, so I can relate magnitude with phase.

So, this part can be related to this part provided x hat n, x hat n is causal and it can be shown x hat n is causal if X z is a minimum phase system we are not going to the proof of this it can be proved if it is minimum phase system. What is a minimum phase system all the poles, and zeros are within the unit circle that is poles and zeros are within the unit circle in the z plane. That means, if you have a sequence if you have a sequence x n whose z transform gives me poles and zeros within the unit circle within the unit circle. Then I can find out from the magnitude the phase of that sequence in the frequency domain or from the phase I can calculate the magnitude with a certain scaling factor, because the initial value x not is not known. So, similarly here the real part the initial value may not be known, so that has to be scaled by a factor.

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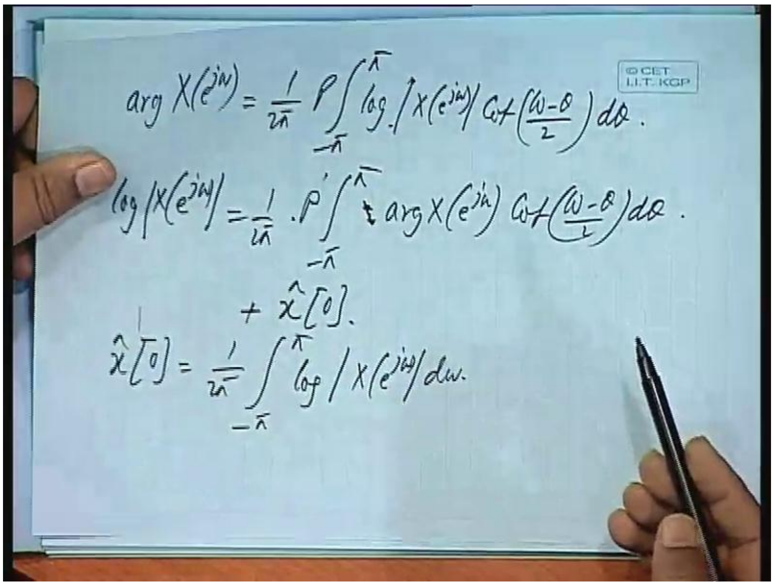

So, argument X I can write as 1 by 2 phi principle Cauchy into value log of real part is log of X e to the power j omega magnitude cotangent omega minus theta by 2 into d theta if want DTFT. If you want in the continuous domain you can do like this if you want only some discrete values of the real part and the magnitude and the phase, then you can use the other discrete domain description that we have just, now derived

Similarly, log of X magnitude will be 1 by 2 phi P, P denotes Cauchy principle value that is when you integrate by subtracting and adding the portion, which is very close to that critical 0 where omega is equal to theta. When it blows up infinity plus infinity and minus infinity discussed, we can always take a zone we can skip that portion where we can assume that positive side and negative side the areas will be cancelling.

So, this will be phase cotangent omega minus theta by 2 d theta plus you need the initial value x hat 0 is nothing but that logarithm of x at that is if I take the value at omega equal to 0. So, x hat 0 is nothing but 1 by 2 phi minus phi to plus phi log of X e to the power j omega magnitude d omega this is our earlier derivation that we did you remember that was real part. Now I am taking the real part as a log of magnitude the imaginary part I am taking as argument, so in the logarithmic domain we are trying to find out this.

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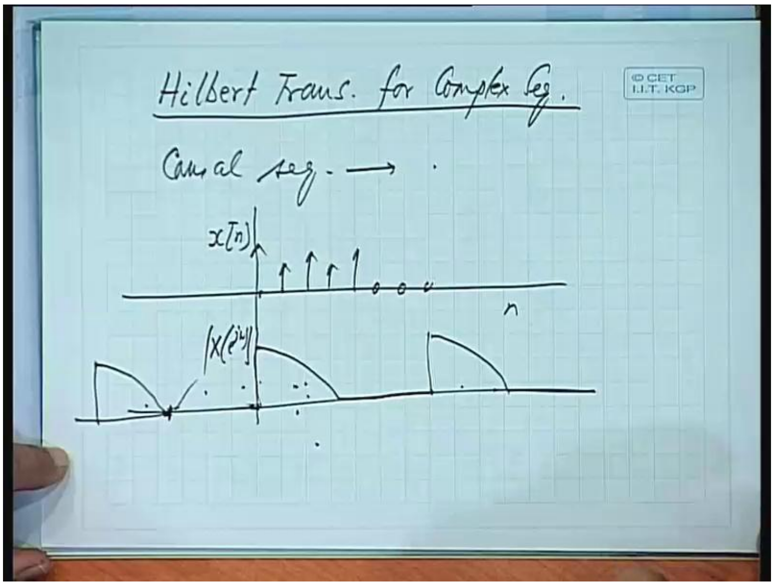

Now, we I will take up the Hilbert transform relation for complex sequences it is little difficult to conceive complex sequences. Now, we have seen for a causal sequence we have got in the frequency domain we have established the relation between the real part and the imaginary part, now if the sequences itself complex.

We can establish a relation provided we impose some conditions like, what we have imposed on the actual causal sequence in the real case, what was the condition imposed we wanted the sequence length. Such that it is periodic it is periodic, but half of the second half of the period is always 0, we padded with zeros to make a periodic sequence where from N by 2 onward up to N minus 1 the values are all 0.

So, similarly if you can black out some values in the Fourier domain for the Fourier transform; that means, the Fourier transform is made such that it exist it is just a converse problem. You see earlier we had an x n from a periodic function we have taken just 1 period, which will be existing, which will be having some finite values say in an 8 0 sequence first five are finite, then the second half was made 0. So, in the Fourier domain we could establish a relation between the real and imaginary part.

Now, we are demanding if you have in the Fourier domain a function like this say like this and then it is 0 it is like this it is 0, you cannot make 0 on the left hand side, but at least for one half it is missing, then it is like this. If you can get a Fourier transform like this for any function, then the corresponding sequence is complex and you can relate the real part of that complex sequence with an with it is imaginary part. Normally, for a normal sequence a positives as a real sequence, what is the Fourier transform. It will be symmetric about the origin is it not? For j omega plus j omega and minus j omega, you get identical values of magnitude, here this magnitude is 0 have you seen.

So, on this side I have made half of the period 0, so it is a very peculiar type of Fourier transform for such transforms it cannot be totally 0 on this left hand side. So, for such Fourier transforms, you can get sequences you can establish a relation between the real part and the imaginary part of such sequences; let us see what it means.

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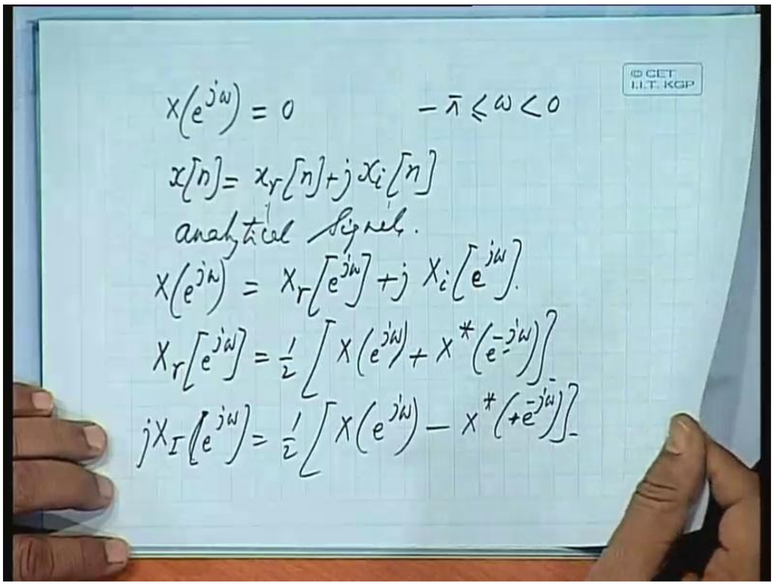

So, we are demanding a particular type of function X e to the power j omega, which is 0 between minus phi omega and 0, you can also take first part 0 and then it is non 0. Now, let us have a sequence xn is equal to x real n plus j times x imaginary n these are 2 real sequences. So, it is a plus j B these are called analytical signals in the complex domain they are known as analytical signals.

So, X e to the power j omega will be if I take the Fourier transform of this will be X r plus j times X i e to the power j omega X r e to the power j omega will be half of X e to the power j omega plus X star e to the power minus j omega. If you remember for complex sequences if we take the Fourier transform of the real part and the imaginary part they can be related with the original Fourier transform with it is complex conjugate in the with frequency minus j omega. Similarly, X I e to the power j omega j times this will be half of X e to the power j omega minus X star e to the power minus j omega.

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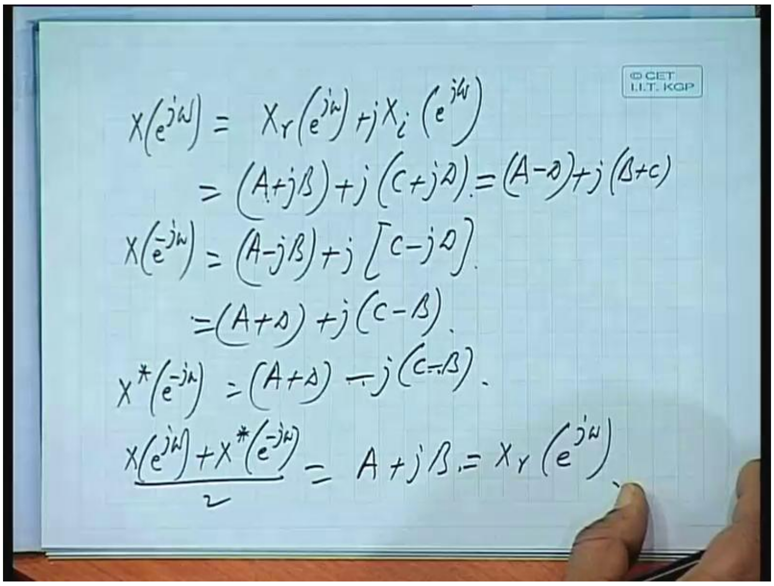

We can prove this for yourself we can see for yourself you see X e to the power j omega you say X r e to the power j omega plus X i e to the power j omega and suppose this is A plus j B and this is c plus j D. Then, what is X e to the power minus j omega if I put e to the power minus j omega this will become A minus j B, because it is a real sequence this is a real sequence.

Similarly, this also real sequence it gets multiplied by j, so plus j times C minus j D, which means in this case it is A minus A plus j B and this will be minus D plus j times B plus C and this 1 will be A plus D plus j times C minus B. So, if I take complex conjugate of this I did A plus D minus j C minus B, so if I add these 2 divide by 2, what do I get this and this D will get cancelled. So, I will get a similarly here I will get B minus j B minus and minus plus j B and plus j B. So I will get A plus j B, which is nothing, but corresponds to this X r e to the power j omega, so similarly by taking negative sign you will find the other relation this one.

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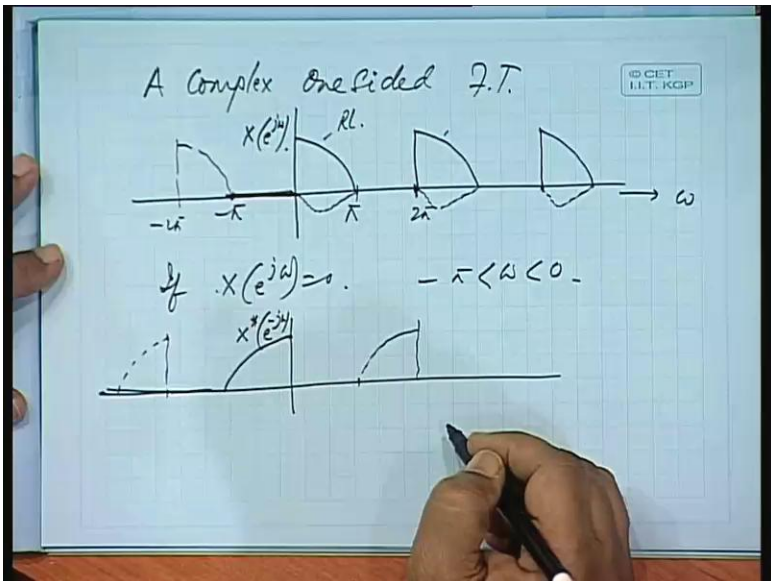

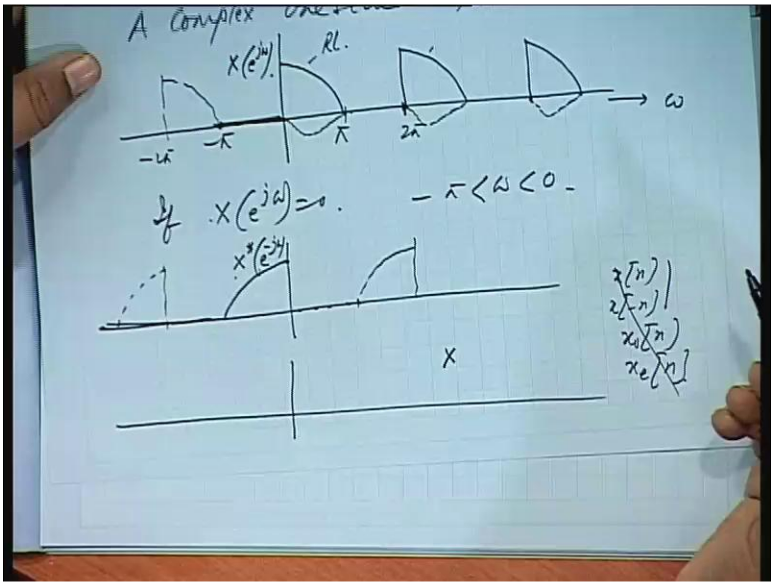

So, this is a one sided Fourier transform a complex one sided Fourier transform. Of course in the discrete domain will be something like this, so the real part something like this and if this is the real part this is 2 phi this is phi minus phi minus 2 phi and so on. And, the imaginary part may be something like this if you have X e to the power j omega equal to 0 for minus phi to omega to 0, so it is 0 here both real and imaginary parts are 0, so it is like this.

So, there is no overlap in the non zero position X star this is X, what will be X star e to the power minus j omega. If I put omega equal to minus omega and then complex conjugate, what will it be like, X star can you imagine you just rotate it about this, what you get something like this know think over it think. And then see you remember, when you are talking about x n a even part and the odd part, and then summing together we are trying to derive from x n given x n we plotted x minus n. And then we derived x even n and x odd n after adding these two you remember we had done in the earlier classes taking some sequences.

So, similarly here we are in the Fourier domain this kind of a function something like a zero padded values, you are getting here and you are taking X star e to the power minus j omega it will be like this. So, if you add them together divide by 2 that will give you the 1 that, just now we found, so it will be this plus this by 2. Similarly the imaginary part you try to find out sketch, what the imaginary part will be like and then for the imaginary part also, you can find out the relation.

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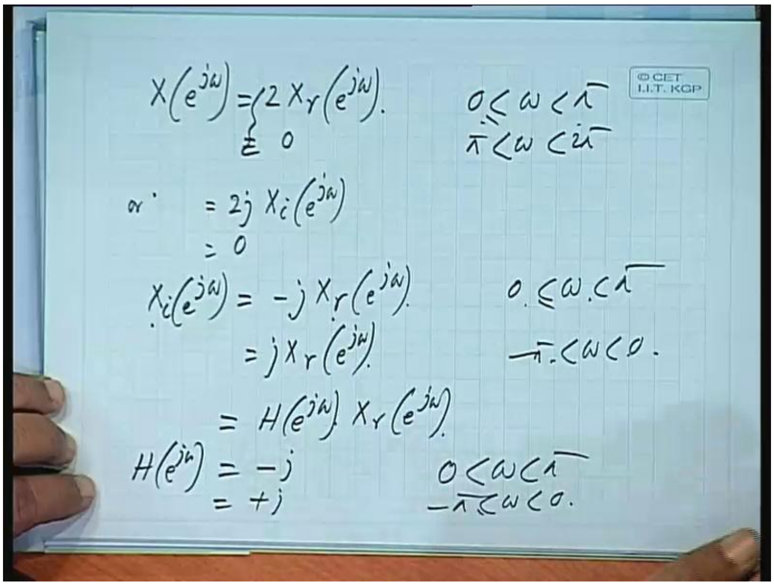

So, X e to the power j omega will be twice X r e to the power j omega 0 omega phi from here, if I want to derive this I will take two times this in the positive region. So, if I add this with this divide by 2 I will get half this magnitude here, is it not? This plus 0 divided by 2 that will give me the real part this plus this divided by 2 . So, it will be coming like this and take it over 1 period 0 to phi whatever is the value multiply by 2 I get this real part.

So, that is what we are doing and equal to 0 otherwise that means the even part whatever be the value here it should be 0. So, this function if I want to reconstruct from the real part, so I will multiplied by 2 and then there can be something here, there will be something here, because this will be giving me some value here. So this plus 0 by 2, so that will be giving me something like this, so this is the real part.

So, this multiplied by 2 gives me this only up to this portion after that I have to make it 0 is it not, so we define like this between phi and 2 phi. And, similarly we can show in terms of the imaginary part and equal to 0 for the same conditions or X i we can also find relation between the real part and imaginary part. So, real part multiplied by minus j is the imaginary part in this range of frequency and plus j into this in this range of frequency.

So, I can write this as some H e to the power j omega into X r e to the power j omega, where H e to the power j omega is minus j in this range and plus j in this range. So, I have a complex multiplier, which is multiplying this by minus j for some range and by plus j for the other range I get the imaginary part, so it is a simple multiplier.

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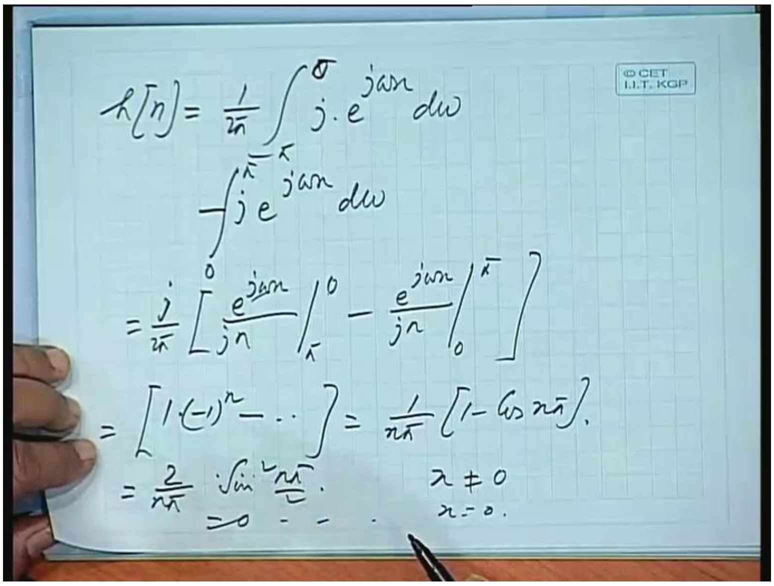

So, what is the corresponding sequence for this, if I have to get this is the frequency domain description of a function, what is it is, corresponding sequence take inverse transform, so h n will be 1 by 2 phi minus phi to plus phi j times minus phi to 0. Because it is minus phi to 0, and 0 to phi, the values are different. So, minus phi to 0 j times e to the power j omega n d omega, then plus minus j sorry put minus j into e to the power j omega n d omega integrated from 0 to phi is it not that will be the sequence.

So, that is 1 by 2 phi I can take j by 2 phi, then e to the power j omega n gives me e to the power j omega n by j n between phi and 0 minus e to the power j omega n by j n between 0 and phi, so j gets cancelled.

So, could you please tell me e to the power j omega n 0 is 1 and what will be the value if I put phi e to the power j phi n how much is it minus minus 1 minus n phi or minus 1 minus minus 1 to the power n minus 1 to the power n. Then minus this 1 you work it out, and you will get 1 by n phi j will get cancelled 2 phi they will be 2 coming out of this. So, 2 will also get cancelled and then will get 1 by n phi 1 minus cos n phi, which 1 may write as 2 by n phi into sin square n phi by 2 for n naught equal to 0, n is equal to 0 for n is equal to 0.

So, it will be a sequence something very close to a sing function, but there is a sin squared term, thank you very much will stop here for today. So, this is known as Hilbert transform all right, this is used for band pass filters for creating synthetic signals complex sequences otherwise will not exist as a signal this is artificial creative.

Thank you very much.