Digital Signal Processing
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Lecture - 31
Relationship Between Real And Imaginary Parts of DTFT Contd..

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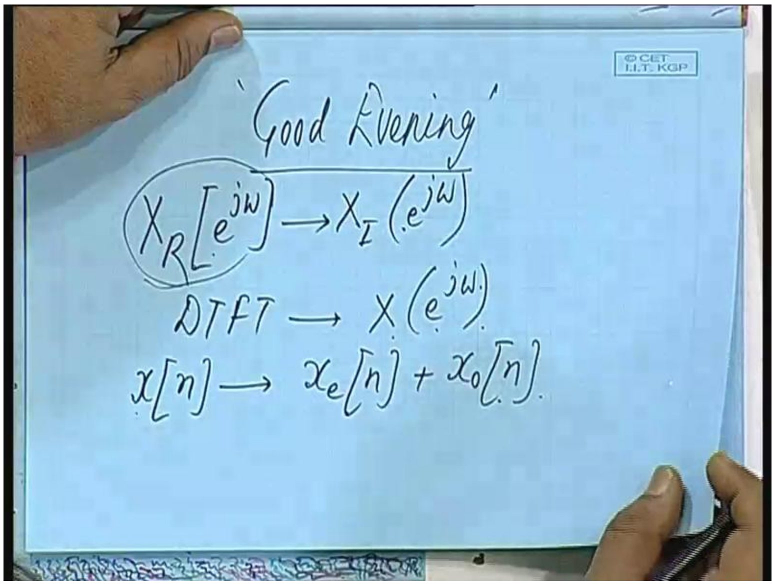

Good afternoon friends. Last time we were discussing about the relationship between two transforms. The real part and the imaginary part of the DTFT that is X e to the power j omega; that means if the real part is given how do you determine X I imaginary part or given the imaginary part how do you compute the real part. So, that was our problem and we established the relationship x n can be decomposed into an even part and then odd part. And the real part of X e j omega is related to the even part if it is a real sequence and the odd part is related to the imaginary part.

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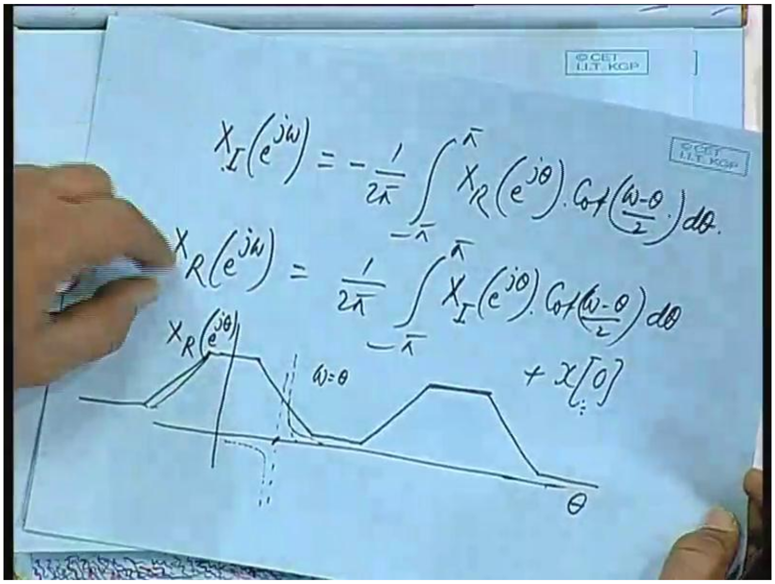

Now, with this last time we establish the relation X I e to the power j omega was minus 1 by 2 pi minus pi to plus pi X R e j theta cotangent omega minus theta by 2 into d theta that is given the real part of the frequency transform you can compute the imaginary part. Similarly, the real part was computed from the imaginary part from this X I e to the power j theta cotangent omega minus theta by 2 d theta plus, what is missing in the odd part, the initial value x 0. So, this is an additional information that has to be known and you can use it with imaginary part.

So, you discussed about this cotangent function suppose you are having you want to compute it for different values of omega. So you take theta, suppose X R e to the power j theta is like this may be like this I am just hypothetically taking a function like this. Similarly, it will have its image on this side and cotangent omega minus theta if I keep on varying omega take one value of omega, when omega is equal to theta this becomes cotangent 0 and cotangent 0 is infinity.

If it is minus 0; that means 0 from the left side and from the right side, then the function will be somewhat like this will be like this, one will be going to the positive side the other one will going to the negative side. Then, this is an improper integral you cannot compute this, because one part of it tends to infinity at a particular value, it is not analytic. In that case how do you evaluate this integral, if you assume this omega minus theta to be very small.

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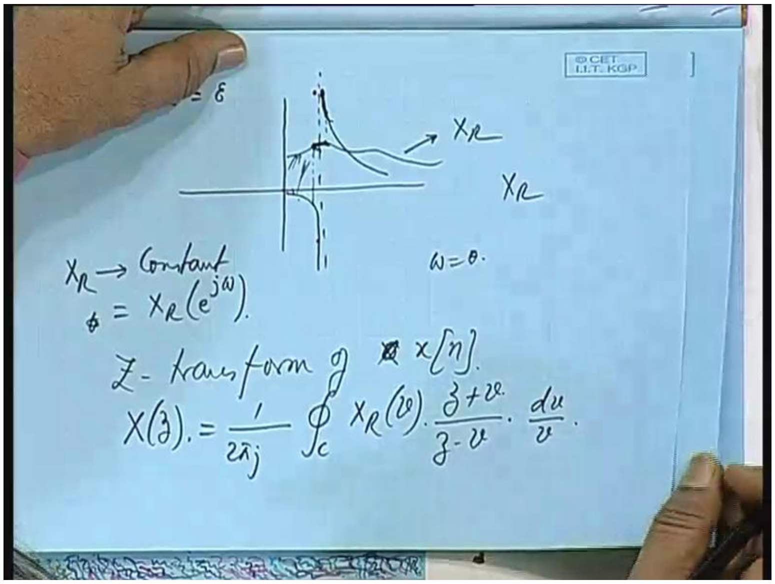

Let us, take omega minus theta as a quantity epsilon, so if you take a very small value of epsilon and from the positive side and from the negative side. You see a function is going like this going to infinity and another function say it is like this, we are taking the products, so this value multiplied by this. And this value multiplied by this, when it is tending to infinity it is tending to say plus 10,000 and at an equip spaced point here on this side it will be minus 10000. Cotangent function tending to when that argument is tending to 0 a very small value of epsilon, then it will be plus a very large value and minus a large value. And if you bring it very close epsilon is tending to 0 if you bring it a very close to omega.

In this range, we can assume the value of the other function that is X R to remain practically constant at that value. So, if you assume X R to be constant at that value of omega j that particular value of omega, omega is equal to theta at this point, then the plus very large quantity multiplied by this and minus this quantity they will cancel each other. So, by taking points at other than this location very close to omega equal to theta, you can evaluate the product and evaluate the area under the curve.

You can similarly calculate this live aside a very small zone you drop it you skip that one, you can have computationally you can take points by points summation. Integration means some of these products points by points and skips that points this is one way of evaluating an improper integral of this time. Then, you can calculate this but it is a tedious job can we surmount this problem by taking help of DFT that is instead of computing this over all values of omega we take discrete points.

Let us see, how we can do it, suppose we can do it before we go for that. Lt us see, what will be the Z transform of x for which the real part is given. It can be shown if we can make this convergent this summation this product of X R into this integral, if it can be shown this is an improper integral, but if you can evaluate it by this method. Then, we can since X I and X R, they are finite, now we can evaluate X z that will be equal to 2 pi j that relationship z plus v by z minus v d v by v.

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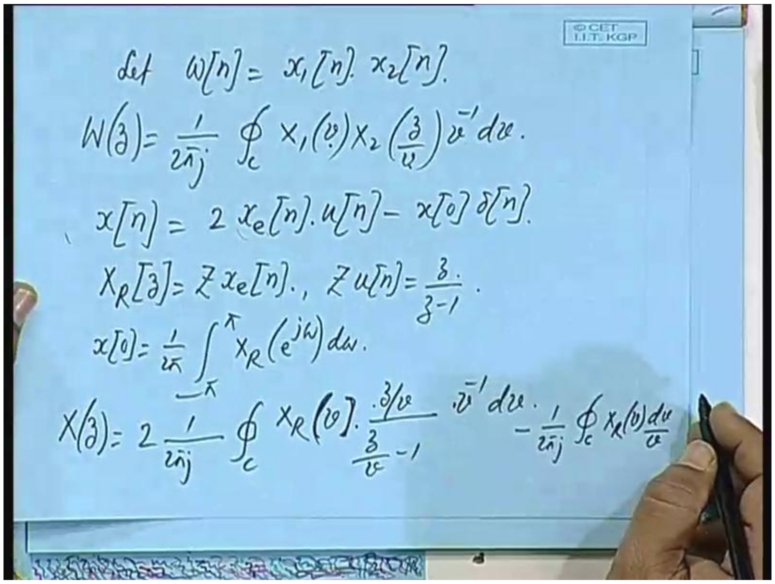

Now, how do you get this will prove it, let w n be the product of two functions x 1 n into x 2 n 2 sequences, then W z will be 1 by 2 pi j X 1 v. This is integration over this close path c X 1 v into X 2 z minus v v to the power minus 1 d v; that means if you know Z transform of X 1 and X 2. Then you can always write having a running variable v and taking the product you can calculate W z. It is basically convolution in the z domain, this is a standard relationship from here will establish this relation.

Now, in our case x n is 2 into x e n this is the relationship that we establish last time you remember in terms of the even function, we can write the original function x n. Therefore, X R z which is Z transform of x n z transform of U N is z by z minus 1 and x 0, we know is 1 by 2 pi minus pi to plus pi from the DTFT this is we known is it not, so these are given to you. X z will be 2 into I take Z transform of both sides so 2 into Z transform of this product Z transform of the product is given by this. Similarly, x 0 into delta n x 0 is a constant which is known and Z transform this is 1.

So X z will be 2 into 1 by 2 pi j X R just x 1 into x 2, so this is x 1 this is x 2, so corresponding to this is X R. So, I will put v is that and then z by z minus 1 so z should be replace by z by v is it not. So, z by v divided by z by v minus 1 into v to the power minus 1 into d v, minus 1 by 2 pi j this integration x naught delta n delta n gives me 1 so this integration gives me X R v d v by v.

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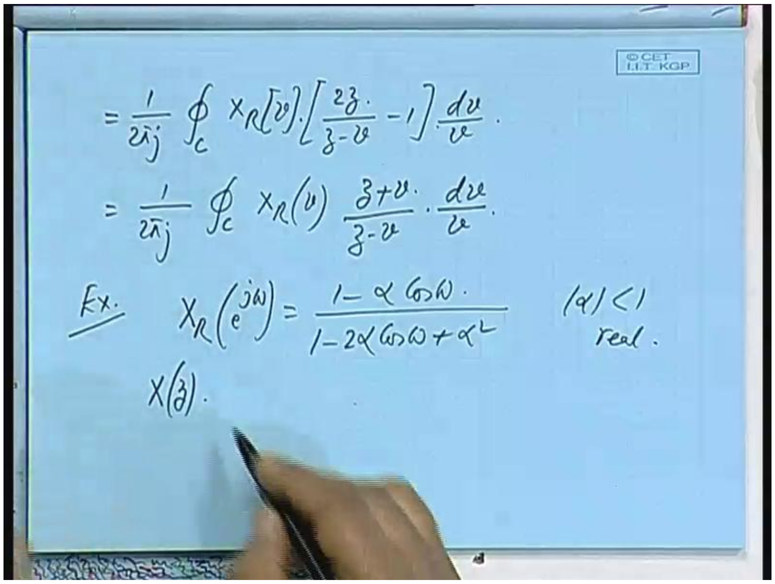

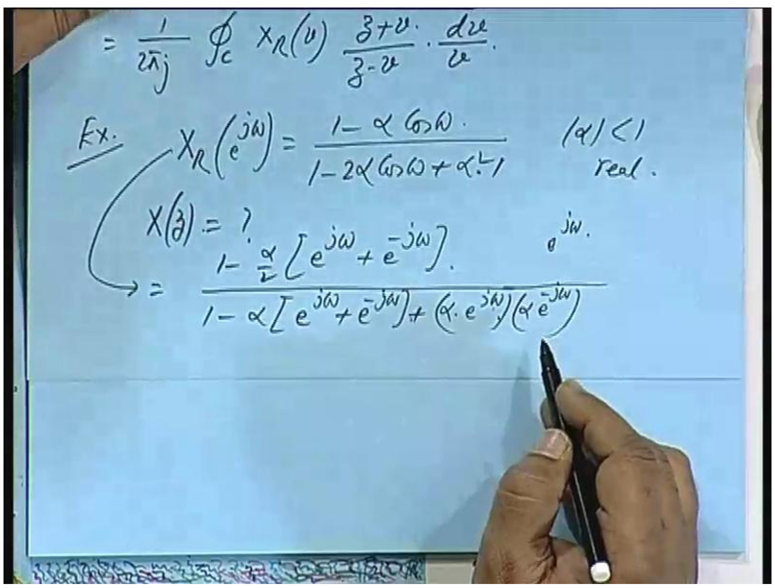

So, that is equal to 1 by 2 pi j X R v if you simplify that this one 2 into z, minus z minus v minus 1 into d v by v minus 1 comes from this term, this term X R v is common so this is d v by v only. That is equal to 1 by 2 pi j X R v if you simplify this, this will become z plus v by z minus v into d v by v this is what we wrote in the beginning, so this is a proof.

Let us work out an example, let X R be given 1 minus alpha cosine omega by 1 minus 2 alpha cosine omega plus alpha square alpha magnitude is less than one and it is real. Compute X z.

Now before we go for the for the evaluation of X z, let us express this in the exponential form of e to the power j omega cosine omega I can always write in terms of e to the power plus j omega e to the power minus j omega.

So let us do that, so the given expression will be equal to 1 minus alpha by 2 e to the power j omega plus e to the power minus j omega is that. So 1 minus alpha into e to the power j omega plus e to the power minus j omega, plus I can write alpha into e to the power j omega into alpha into e to the power minus j omega alpha square can be written like this, e to the power plus j omega and e to the power minus j omega will give me one, so I have written this alpha square into 1 in this form.

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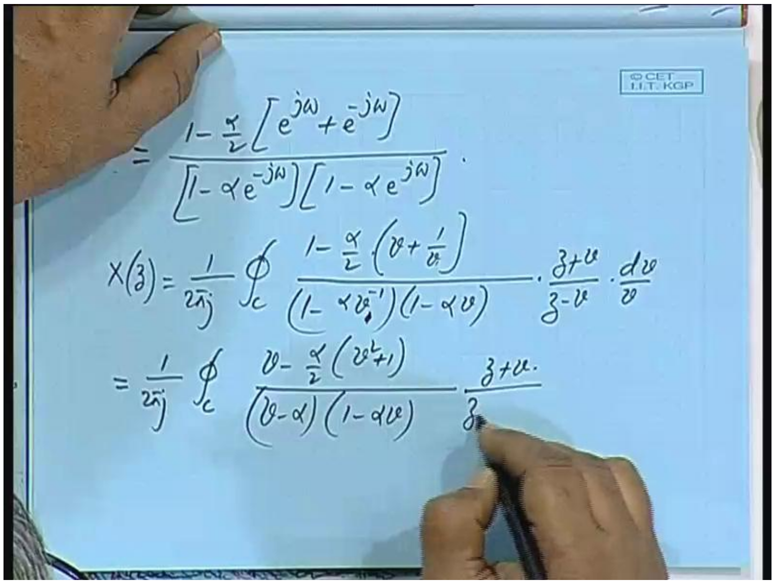

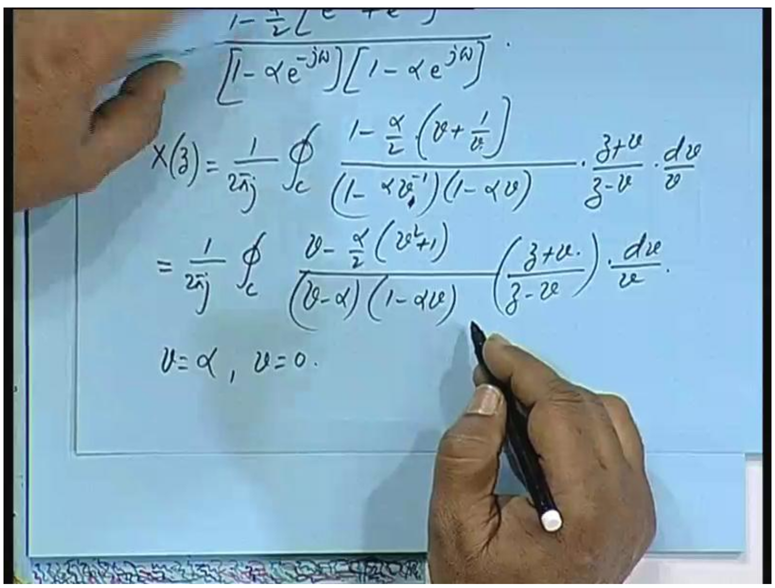

So, that I can write this as equal to 1 minus alpha by 2 e to the power j omega plus e to the power minus j omega divided by 1 minus alpha e to the power minus j omega into 1 minus alpha e to the power plus j omega. So, if you now compute X z it will be 1 by 2 pi j 1 minus alpha by 2 X R e to the power j omega is replace by v is it not.

So, I will write this as v plus 1 by v and this one will be 1 minus alpha v 1 alpha v to the power minus 1 into 1 minus alpha v into z plus v by z minus v into d v by v I have just use the earlier relationship that we established. So, 1 by 2 pi j if you simplify this multiply throughout by v the numerator becomes v minus alpha by 2 v squared plus 1 is that if I multiply by v this becomes v square plus 1.

If I multiply this by v then this becomes v minus alpha into 1 minus alpha v into z plus v by z minus v into d v by v, what are the singular. It is, v is equal to alpha with the unit circle v is equal to alpha is pole, when this function blows up, the other one is v equal to 0, because of this and these are inside the unit circle.

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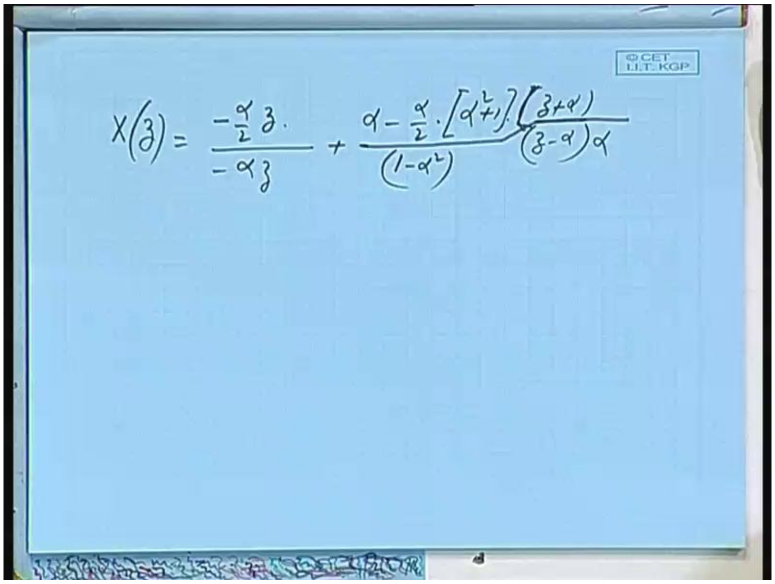

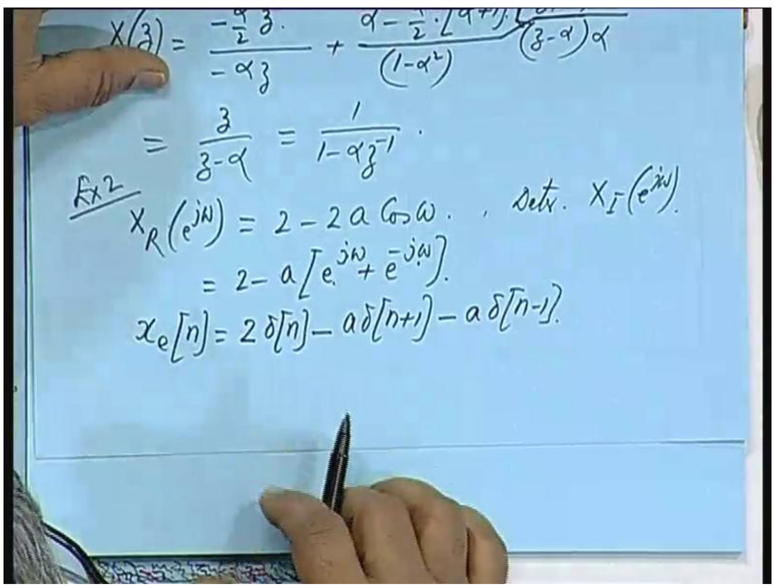

If they are within the unit circle then X z, calculate the residues at those poles, then it will come out to be minus alpha by 2 z by minus alpha z plus alpha minus alpha by 2 into alpha square plus 1 into z plus alpha by z minus alpha into alpha divided by I will put it here into 1 minus alpha square. I am putting alpha is equal to v so this becomes 1 minus alpha square, I am calculating the residues then first you multiply by v minus alpha that function and then put v equal to alpha is it not. So, v equal to alpha if I put so I get from here 1 plus alpha square 1 plus alpha square and so on, so these are the some of the residues.

So finally, if you add up will get z by z minus alpha which is nothing but 1 minus alpha 1 by 1 minus alpha z inverse, so this is X z after adding all this you will find lot of terms will get cancelled and will be getting a reduced form.

Let us take another example X real e to the power j omega is 2 minus 2 a cosine omega, determine the imaginary part. Now, this I can write as 2 minus a into e to the power j omega plus e to the power minus j omega. What would be the corresponding even sequence, what is its inverse, what is its inverse DTFT, two will give me 2 a constant will give me 2 into delta n is it not? Minus a into e to the power j omega, so that will be a into delta n plus 1 minus a into delta n minus 1 is it not?

A delta function, what will be its DTFT, one if I shift it by one step that is delta n minus 1 will be e to the minus j omega the sequence is 0 1 0 0 and so on. What will be its DTFT will be e to the j minus omega if I shift it by two steps it will be e to the power minus j 2 omega and so on. If I shift it on the other side it will be plus j omega so it will become delta n plus 1 corresponding to this term it is delta n plus 1 corresponding to this term it will be delta n minus 1 is that, so this is X e n.

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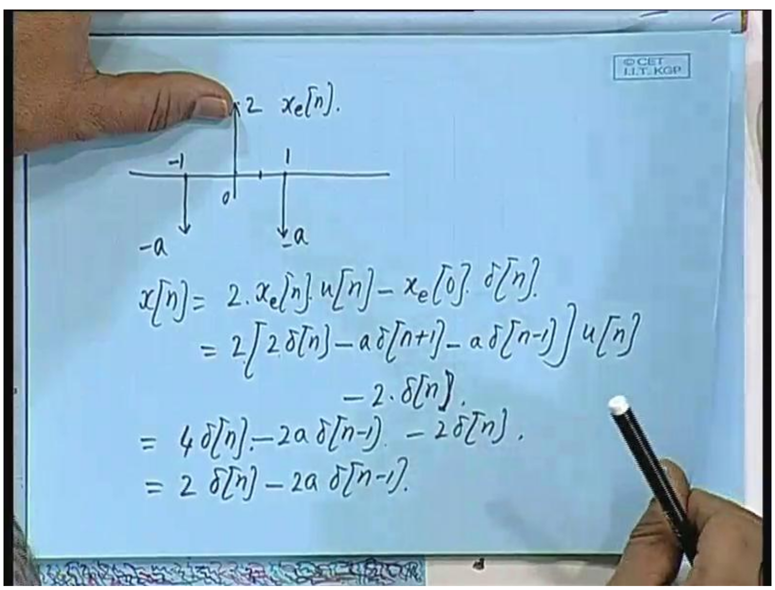

Now x n, so if I draw X e n it looks like this 2 into delta n this is 2 minus a delta n plus 1, so at minus 1 it is minus a and minus a delta n minus 1 at 1 it is again a magnitude of minus a, this represents this function. Mind you even function last term last time we discussed even function and odd functions will be extending on both sides is it not. When you take a real causal sequence only defined in the positive range positive region of time discrete time. The even part and the odd part if you compute from there even part is 0, the original sequence is having 0 values in the negative region of time. So, if you take the average you will get some values on the other side so even part will be X e n will be extending on both sides, similarly x odd.

So, this is X e n, now how do you recover X n, we know is twice x e n into U N minus x e at 0 into delta n is it not, this we are using quite often. So, how much is it 2 into x e n is 2 into delta n just now we have got 2 into delta n minus a into delta n plus 1 minus a into delta n minus 1 whole thing multiplied by U N minus x e 0, X e 0 is 2, so 2 into delta n correct me if I am wrong is that.

So, how much is it 2 into 2 4 into delta n mind you it is multiplied by U N, so delta n plus 1 does not have any chance, because it exist in the negative region of time and it is multiplied by U N which is having a 0 value here, so this will be blocked out. So, it is 4 delta n minus 2 a delta n minus 1 minus twice delta n. So how much is it? 2 into delta n minus twice a delta n minus 1 this is the sequence, if you want you can calculate X z also from here, what is the imaginary part and what is the DTFT of this.

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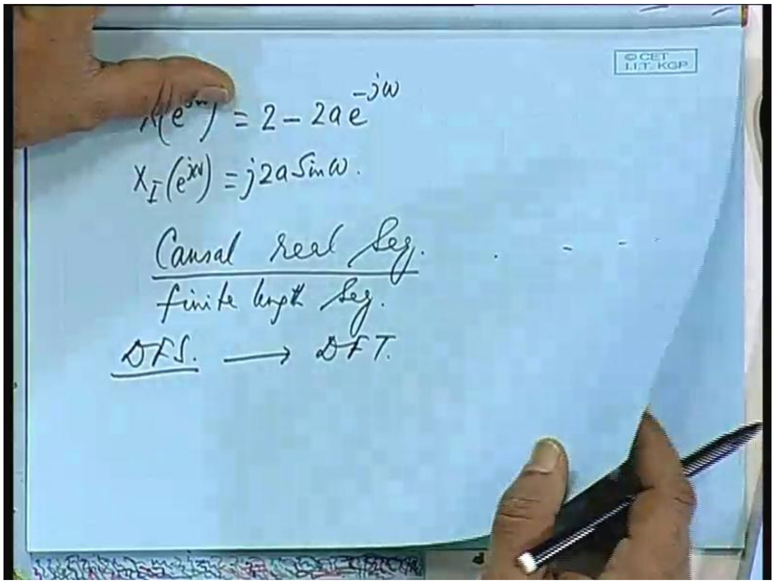

Therefore, X omega will be 2 into delta n so 2 minus twice a delta n minus 1 so that will be e to the power minus j omega so this is x and what is its imaginary part, you can compute from here. Imaginary part will be only from here, it will be minus and minus will make it plus twice a j times twice a sin omega is that.

Now so far, we have considered a causal real sequence, now this can be applied to a finite length sequence causal sequence. Now, we can very easily since it is a finite length we can very easily represent this by a periodic sequence taking only one period of this as this sequence. So we will create an artificial sequence an artificial sequence where one period will be embracing only this particular part along with some padded zeros.

Let us have a stretch sequence that means the given sequence we stretch it in time domain by putting some 0 values and then that is made one period and let us repeat that. We take advantage of a periodic sequence where discrete Fourier series can be computed and discrete Fourier series is equivalent to discrete Fourier transform. If you consider one period the expressions are identical, what is a difference between a discrete Fourier series and a discrete Fourier transform, what is a difference between Fourier series and Fourier transform in the continuous domain.

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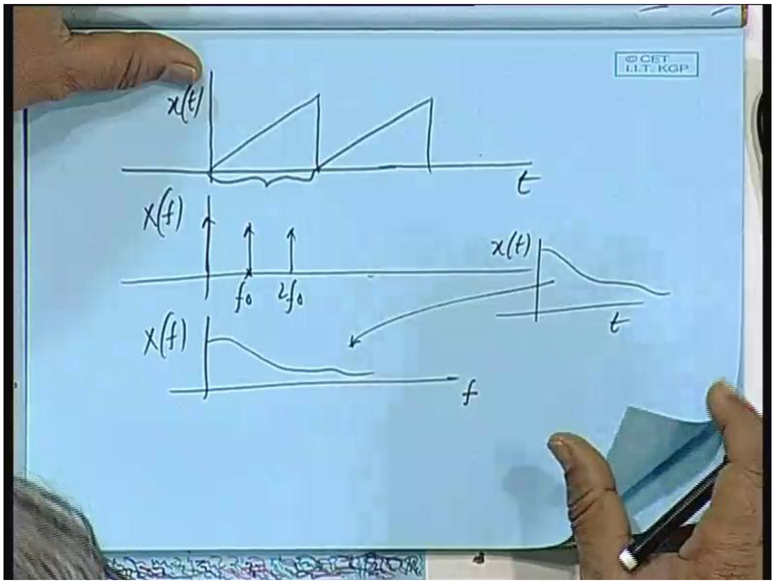

Fourier series you take for a periodic function, if a function is periodic say like this then we go for Fourier series, you get certain lines if this is x t t I get x f at some values of f corresponding to this period we call it fundamental. So, you get if that is called f naught then we get some values f naught twice f naught thrice f naught etcetera we call them harmonics in periodic function. In case of periodic functions, if this is an periodic function then x f is no more discrete it is a continuous function, all possible frequencies are present. When x t is and a periodic function this is x t so corresponding to that you have a frequency transform which is continuous not lines.

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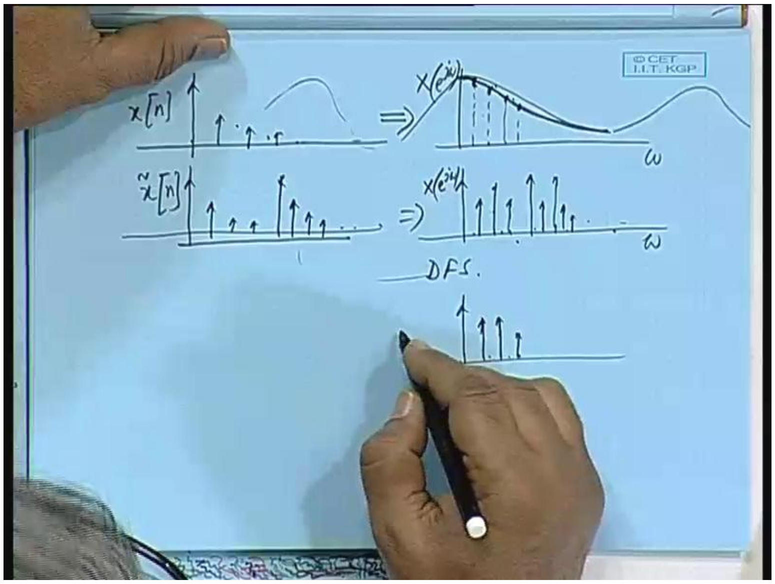

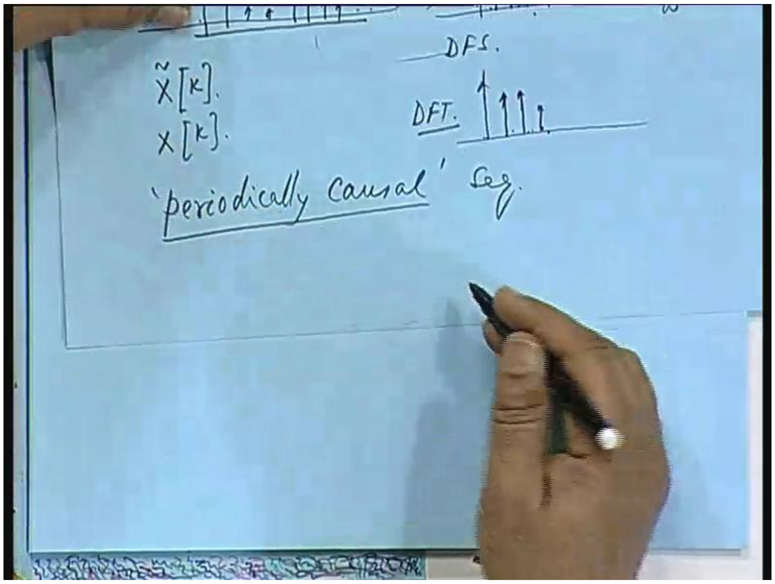

Now in the discrete domain, if you have a discrete sequence of finite length suppose this is the sequence after that it is all 0 it is not periodic. Then, what will be its frequency transform, x e to the power j omega in the discrete frequency domain, frequency domain we get a continuous function at if this is discrete and periodic if this is we show periodic function by this delta.

So, if it is like this, then we get on this side also a periodic function which is discrete, it may be anything we do not know this will be basically discrete versions from here, these values again repeating. So, this is a periodic function in the discrete domain corresponding Fourier series is called discrete Fourier series in the frequency domain also will get only discrete frequencies and coming periodically.

In a normal Fourier series you have because it is continuous, you have only the lines going up to infinity and when it is discrete instead of a continuous function we get this periodic this will be coming periodically. When this is discrete and periodic this will also be discrete and periodic, so these are the types of functions that we get so this is known as discrete Fourier series.

Now, if we do not have a periodic function if we have just one sequence, then this can be represented instead of representing it by in the frequency domain by periodic functions like this we can represents this by only that many frequencies may be like this there are four points. So, you will get only four frequencies that is good enough that will be describing basically the values at this points.

And at this points if the values are known, then you can describe with these four points this entire function these four definite points in the sequence. Basically, we need so many frequencies to represent the original time function original function is discrete it is a four points sequence, so I do not need more than four frequencies, I do not need so many frequencies to represent this, so this known as discrete Fourier transform.

So, discrete Fourier series and discrete Fourier transform these magnitudes are identical, only thing here it is periodic. Here it is just one set, one of the periods we are taking, so basically the relationship is so far as the mathematical relationship is concern they are identical. So, for discrete Fourier transform, you just put a tilde k, k equal to 0 1 2 3, if it is a four points sequence and for DFT's we write X k but the expressions are identical.

So, we are considering a discrete Fourier series that is we are taking one sequence, padding it with sufficient number of zeros will see how many zeros are required in one and then we are creating a periodic sequence. Now, we started off with a causal sequence, now we are going to a sequence only a part of it is this sequence, so that one is a periodic sequence.

So, it is a pseudo causal sequence or periodically causal sequence, it is a misnomer periodically causal a causal sequence cannot be periodic means it exist also in the negative region of time whereas a causal sequence will be always in the positive region of time. So, we are calling it periodically causal or pseudo causal sequence periodically causal sequence.

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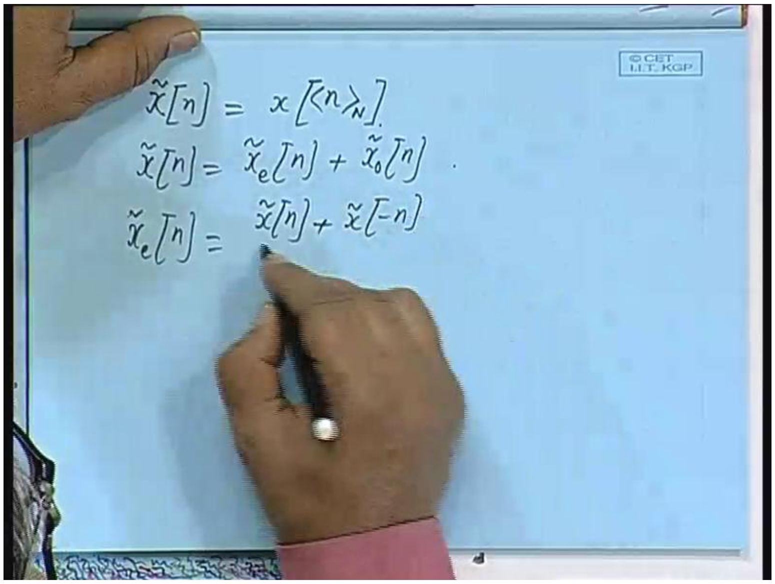

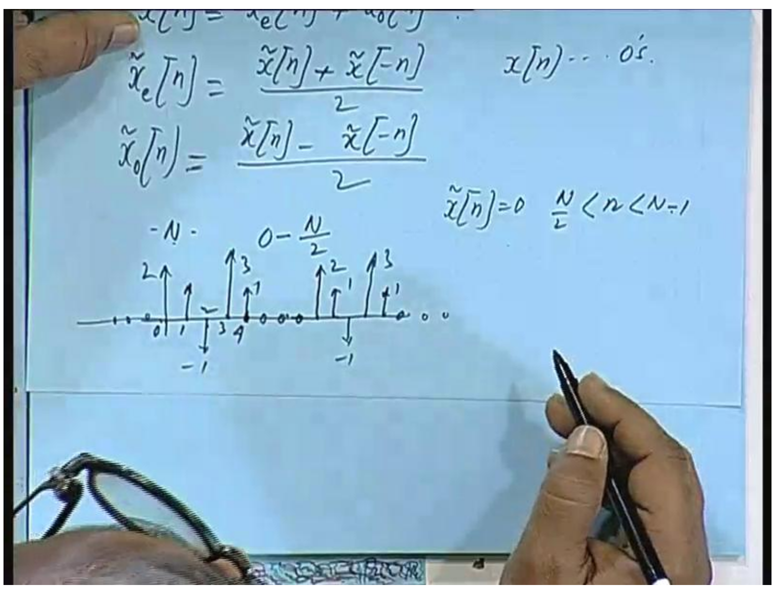

And let us see, what is that sequence like after appending with zeros, we take this periodic sequences as x n modulo n the moment I write modulo n means its periodic you keep on separating n, the magnitude is repeated after every n steps. So, I can write x tilde n as x even part and an odd part where x even part x tilde is x n these tilde marks means they are all periodic functions.

Similarly, the odd part is x tilde n minus x tilde minus n by 2, so you append the sequence x n original given sequence x n by a few zeros such that we get a length of sequence length of N, we assume N to be even and up to N by 2 it exists. And, that means x n is made 0 between N by 2 and N and 0 to N minus 1 and 0 to n by 2 it is having the same value as the given sequence.

So let us see, what x n will be like, suppose you are given a sequence value say 2 1 minus 1 and 3 1, so I will try to go to an even number such that this becomes half this 0 becomes half of this is 0 1 2 3 4. So, 4 multiplied by 2 8, so 5 6 7 this is an 8 point sequence, so I create an 8 point sequence by appending 3 zeros 8 divided by 2 is 4.

So, the index number 4 that means on this side there are 5 non zeros and 3 zeros mind you they are not equal in number. It is up to N by 2 N by 2 including 0 means it is more than half by 1 and this is less than half by 1 is that, the index number is 4 8 by 2. So, I have an 8 points sequence this is our one period if this is the sequence given then we create an artificial sequence with padded zeros and this is one period. Now this keeps on repeating 2 1 minus 1 3 and then again 1 this is 1 so and so on again zeros, similarly on this side zeros and so on.

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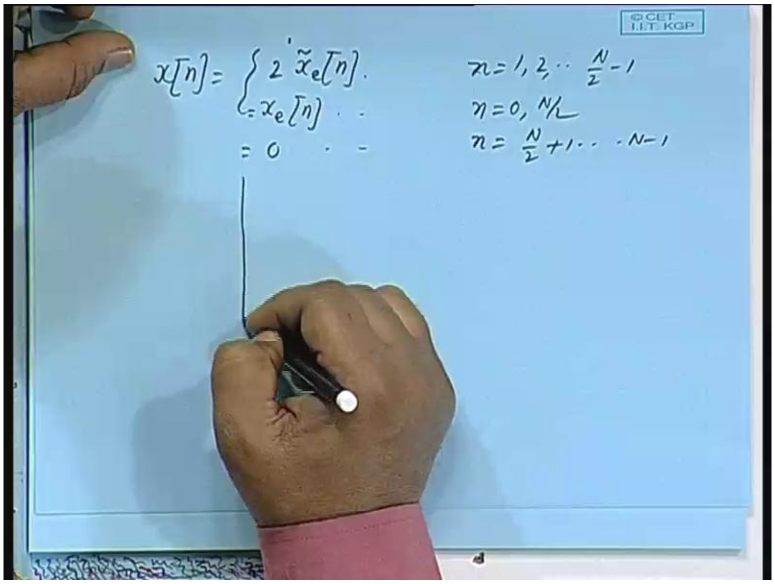

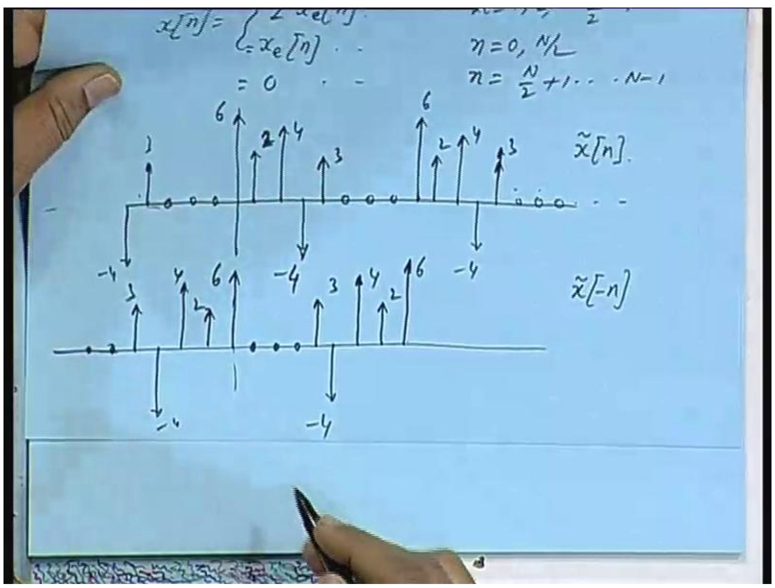

So, x n is 2 times x e n for n is equal to 1 2 up to n by 2 minus 1 and is equal to x e n at n is equal to 0 and n by 2 and equal to 0 at n is equal N by 2 plus 1 up to n minus 1 is that let us verify whether this is... Let us take some values of x n and x minus c n, et this be let me have a sequence like this 6 will compute this 2 4, they are not drawn to the scale that is why I am writing the values say minus 4 then 3, this is 2 then there will be 3 zeros I have taken a 5 points sequence, so this is x n tilde. This was x n, so I have padded with zeros and then I make it periodic so this is the periodic sequence and so on.

Similarly on this side, 0 0 0 and so on before 0 this is 6, so then this is 3 then this is minus 4 and so on it continues on this side. So, what will be x tilde minus n, because I will add the 2 and divide by 2, I will get the even part, so from x e n what will be x e minus n it will be 6. Then go from this side 0 then 0 then 0 is it not 0 0 0 then 3 then minus 4 correct me if there is any slip 4 then 2 then 6 and that repeats is that. Now, if you take the average that will give you x e n it repeats on this side also 6 2 4 minus 4 3 and so on zeros.

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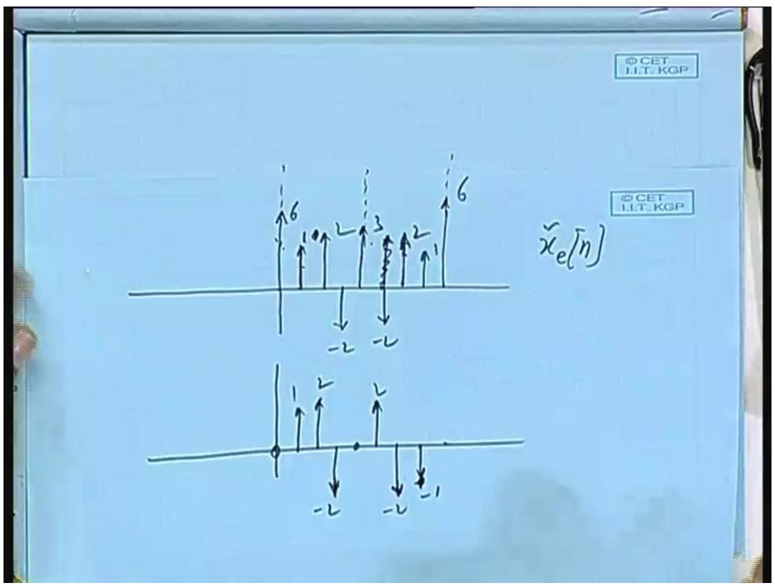

So, take the average and then tell me, what will be x e n like, so x e n if you permit me to draw x e n 6 plus 6 by 2 so that will be 6, then here you see the advantage of padding 0s is 2 plus 0 by 2, so it will be halved all these values will be halved except at this point. The fifth point, if it is n by 4 whose index number is n by 4 that will be again 3 plus 3 is that, there is no midpoint for an even number sequence of say 8 points, there are 5 on this side 3 on the other side.

So, this will be half of 2 plus 0 that is 1, then this is half of 4 plus 0 so 2, this is 1 then minus 4 and 0 so minus 2, then 3 plus 3 by 2, so that is again 3 . Then on this side, this side is 4 plus 0 by 2 2 plus 0 by 2, so this side again it will be symmetric is that, what will be it, will be 4 am I 4 so 2 then 1 check if i have taken 2 plus 1 6 2 4 minus 4 3 minus 4 3 0 0 0 is that.

So, 3 plus 3 so immediately after that is that should be it should be minus 2 yes minus 2, and then it becomes plus 2 then plus 1, then again 6 so on. You can have just an image of on this side, I have not really drawn to the scale, I have that is why I have written the magnitudes.

You will find it is symmetric about the midpoint this also symmetric about the starting point 6. This will be x e n, that is why I wrote it is equal to x e n for 0 and N by 2 at this point and at this point the values of x n and x e n are identical 6 and 3 . And at other points it is just halved in magnitude. so twice of that even part gives me the original x n sequence at other points and it is 0 for N by 2 plus 1 to N minus 1.

Original sequence it is only for this part so even if x n x e n is extending up to this but it is twice of x e n only up to this point, because original sequence otherwise it is all 0 though x e n is having these values; that means if I add it with odd parts that should get cancelled and that should generate this, so you can write the odd part, how much is odd part odd part at this point 6 plus, how much will be equal to 6, it has to be 0 odd part the central value 0.

And, how much should be added to 1 to get 2 it will be just 1 again, because it is half of this, so up to this point, it will be same as x e n 1 2 and minus 2 and at this point, it is 3 is equal to 3. So, this value will be 0. And then at these points the resultant should be equal to 0 is it not, so whatever you are getting in the even part negative of that should appear in the odd part. So, it will be plus 2 minus 2 minus 1. So, you can sketch the odd part once you know the even part and x n the original x n is given. So, what 1 may ask so what you get this now, let us see how we compute the function x n.

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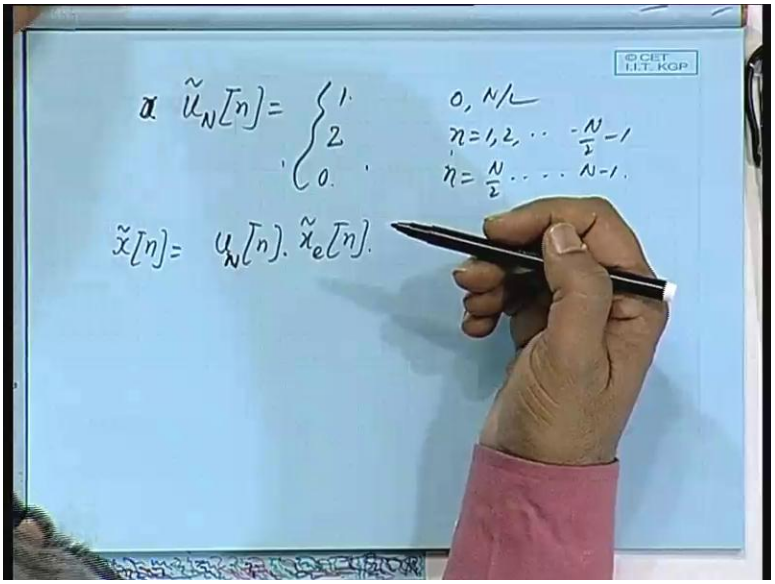

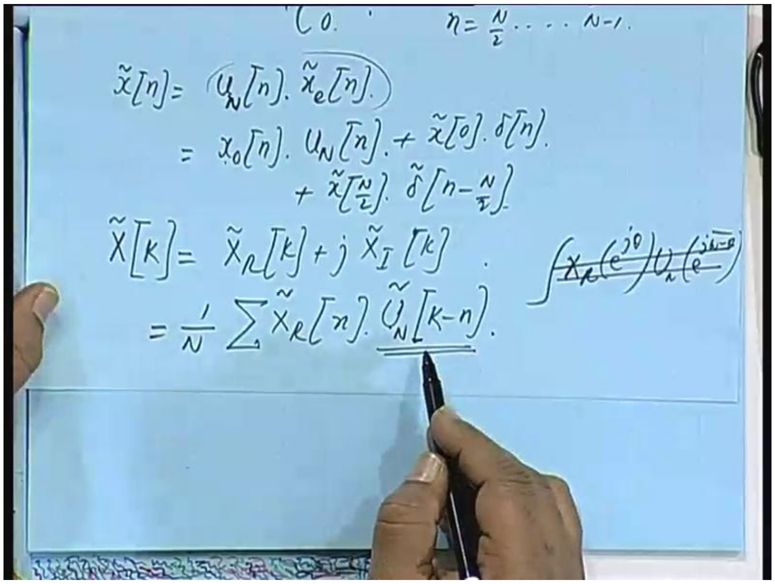

I define a function U N which is periodic which is equal to 1 for 0 and N by 2, for 0 and n by 2 it is multiplied by 1 the even part multiplied by 1. For these values it is multiplied by 2 and for these values it is multiplied by 0, so it is these sequences of multiplier 2 1 and 0 that I write in terms of function which is 1 for these values which is 2 for n is equal to 1 2 up to n by 2 minus 1 and which is equal to 0 for n is equal to N by 2 up to n minus 1.

Then, we can write x tilde n as U N n into x e n is it not x e n if I multiply by a function which is define like this I will get the product x n, term by term I multiply by 1 sometimes 2 sometimes 0 is that; that means or in terms of odd function, it is only at this point and that is at 6 and 3 this is 0 and N by 2 these values are missing in the odd part is it not when you subtract in the odd part that is missing.

So, I can also write this as x 0 n into U N we will write u and n plus x at 0 delta n plus x at N by 2 into delta n minus N by 2. Now, this delta means it is a periodic sequence of impulses at regular intervals, you have to multiply by that is the first value is to be given and then at regular intervals of 8 steps at 3 that is at n is equal to 4, then at 12 etcetera I have to also get this value of the original function that is not given I mean that has to be given. Then, only from the odd function I can recover x n odd function has certain values of the original sequence missing where as in the even function you just multiply by this.

So, I can straight away go for DFT or DFS as I told you DFS basically a part of DFS is DFT, so I can straight away take the discrete Fourier series X k there are 8 points. So, I will get 8 points sequence which will be repeated and X K which is X R K plus j times X I K, now this I can compute from here itself what is X K transform of this.

Now, what is this, 1 by N remember if you are having a periodic sequence, if you are having a periodic sequence and if you are having the products see for a periodic sequence, what is the DFT, it will be also period DFS that will also be a periodic sequence. You can compute that and then if it is coming in a product form we can straight away write X R N U this U is the DFT of this small u this time series K minus n in the frequency domain it will be convolved like this.

In the time domain they are product in the frequency domain they will be convolved and because it is discrete periodic function. So, in the frequency domain also you do not have to take the integral of X R e to the power j theta and U N e to the power j omega minus theta and so on. And for different values of omega you do not have to go in the continuous domain, you can straight away take those discrete points 8 points.

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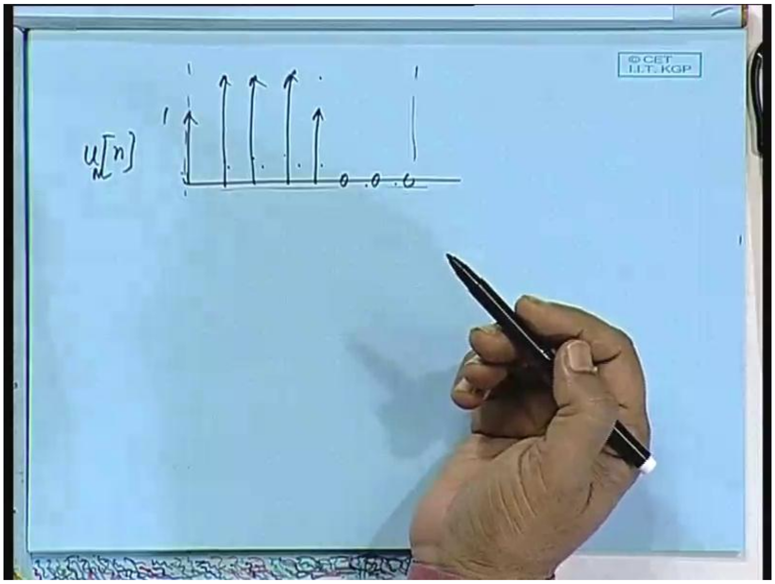

So, in the next class we will take up how to evaluate this; that means you are having a sequence 1 2 2 2, then 0 1 2 3 and then 4 that is 1 then 0 0 0 this is your U N of period n. What is the Fourier transform of this, so DFT how do you compute that, will take it up in the next class. So it is basically 8 points DFT's 1 2 2 2 2 0 0 0 we will stop here for today.

Thank you very much.