Digital Signal Processing

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Lecture - 30
Relationship Between Real And Imaginary Parts Of DTFT

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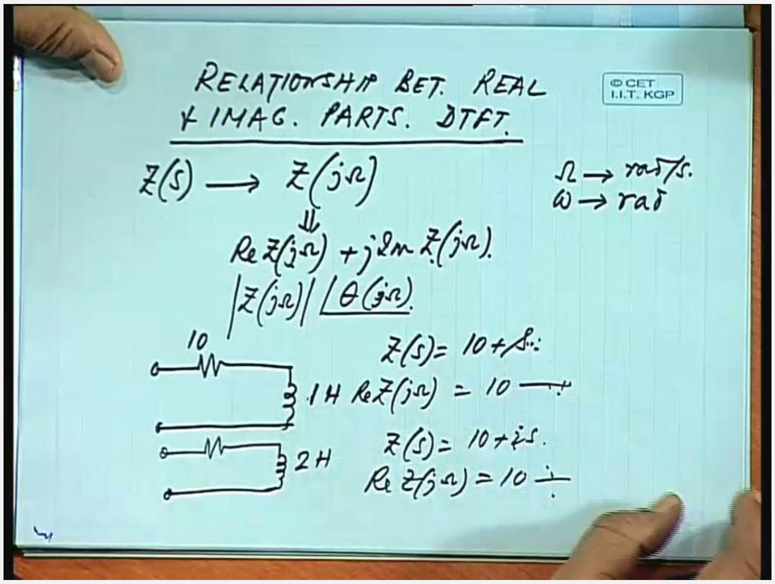

Good evening friends. Today we shall be discussing about Relation between real and imaginary parts of DTFT that is Discrete Time Fourier Transform. Now, in your earlier courses on DSP and signals and systems you must have studied something about discrete time Fourier transform of sequences discrete sequences. So, in this subject advance DSP will start off with this particular topic. Before we go to the subject proper I will just site a few examples in the continuous domain that you have studied. Probably for those of you who have done design of analog circuits with passive elements, they must be conversed with this I will just brush through some of the basic elements of those functions.

Now suppose, z s is a network function corresponding frequency domain response is z j omega, we shall be using capital omega for analog frequencies and small omega for discrete frequencies, that is small omega will have radiant's as units and omega will be the normal one radiant per second that you are conversion with. Now, this can be written as real part and an imaginary part or we can write in the polar form in magnitude and phase. So, to describe z s we need the information both real and imaginary parts. Now, is it possible to obtain the network function if you are given only part of it either the real part or the imaginary part.

For example, I have a resistance and an inductance say resistance of 10 ohms and an inductance of 1 Henry, what will be z s, it will be 10 plus s, what will be its real part, is only 10. Now, if I have another circuit with an inductance 2 Henry, then z is the real part, z s is 10 plus 2 s and real part of z j omega is still 10, so from here in these 2 cases you are having the same real part, but the network functions are different so there is an ambiguity.

So we can design, we can evaluate those functions which will be having just the minimum number of elements as reactive element we call it minimum reactance function for the realized network will be possible; that means we find out a network function I have taken a very simple series element it could have been much more complicated. So given the real part, you can realize the imaginary part where the imaginary part realized will be having minimum reactance value, anything you add any more reactance the real part does not change.

Conversely, if you are given the imaginary part then you keep on adding more resistance but the imaginary part will not change, so we can design only we can evaluate only a minimum resistance impedance function, an impedance function which has minimum resistance. I will just give you an example of how to evaluate this, a simple method then will go over to other areas for example.

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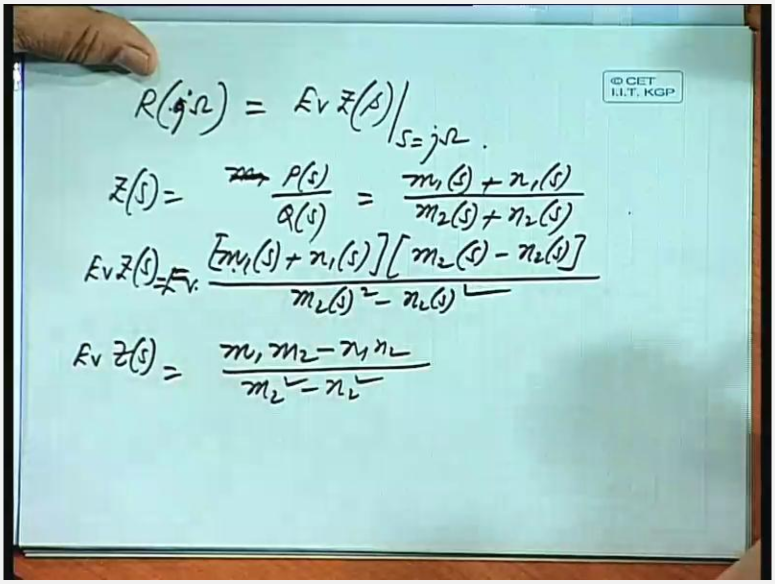

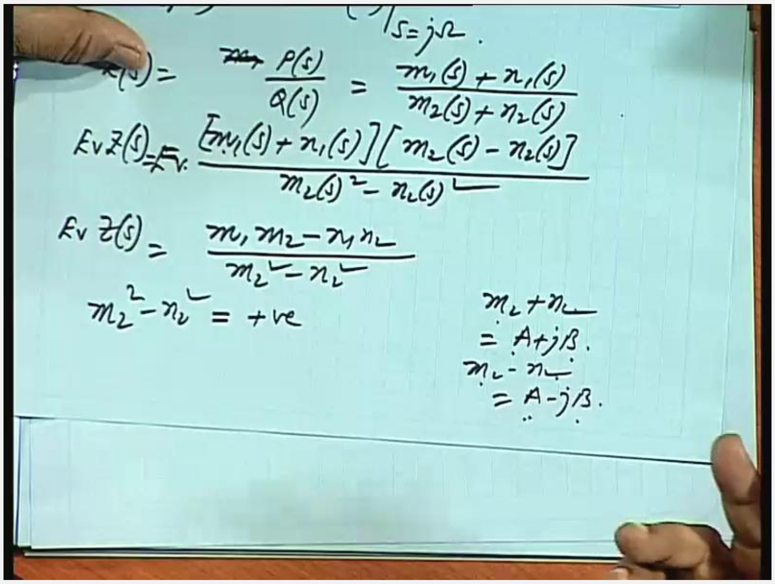

The real part will correspond to an even z s at s equal to j omega, the even part corresponds to the real part and the odd part corresponds to the imaginary part is it not. So, even z s if you can evaluate, suppose z s is written as some polynomial P s by Q s, I can write P s in terms of an even part and then odd part, similarly even part and then odd part. So, even z s will be even part of this, which can be written as m 1 plus n 1 s, if I multiply by m 2 minus n 2, then denominator will be m 2 s squared minus n 2 s squared. So, even z s will be, even part of this function this is z s, z s the denominator has been converted in to an even function.

See, m 2 square minus n 2 square is an even function, because this is odd function square it is an even function, so the denominator is an even function. So, the even part therefore even z s that is even part of this can be written as m 1 m 2. If you permit me I will not write argument s any more minus n 1 n 2 divided by m 2 square minus n 2 square and m 1 n 2 n 1 m 2 that will correspond to the odd part m 2 square minus n 2 square for any value of s equal to j omega will always remain positive. Why m 2 square minus n 2 square is always positive, why m 2 plus n 2 if I put s equal to j omega will be say something like A plus j B.

Then what will be m 2 minus n 2 will be a minus j B, it is the odd part only which will be having a change in sign the even part omega square omega 4 etcetera if you put omega equal to minus omega then also they will remain positive. So, it is only the odd part which will be changed which will be having a different sign, so if m 2 plus n 2 gives you A plus j B. Then, m 2 minus n 2 will give you A minus j B, so m 2 plus n 2 in to m 2 minus n 2 is a square plus b square that is always positive is a magnitude square. So, we have to check whether this quantity is positive or not, then only the real part the even part will be always positive.

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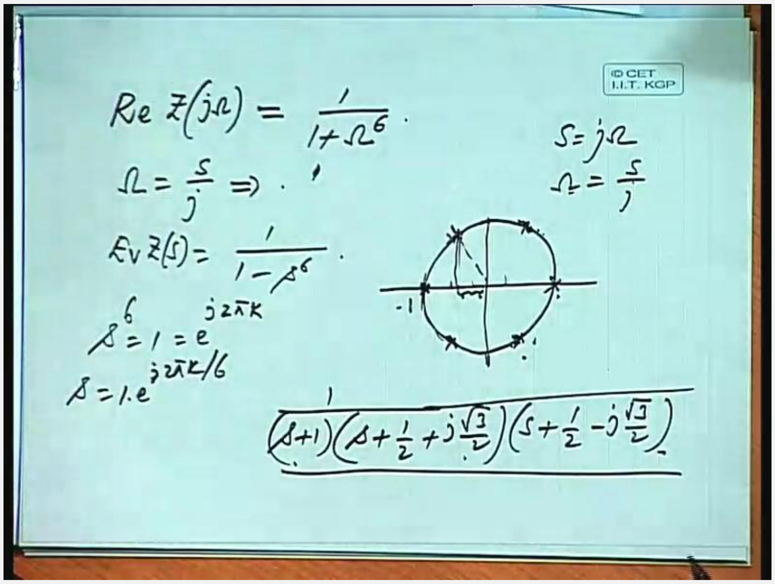

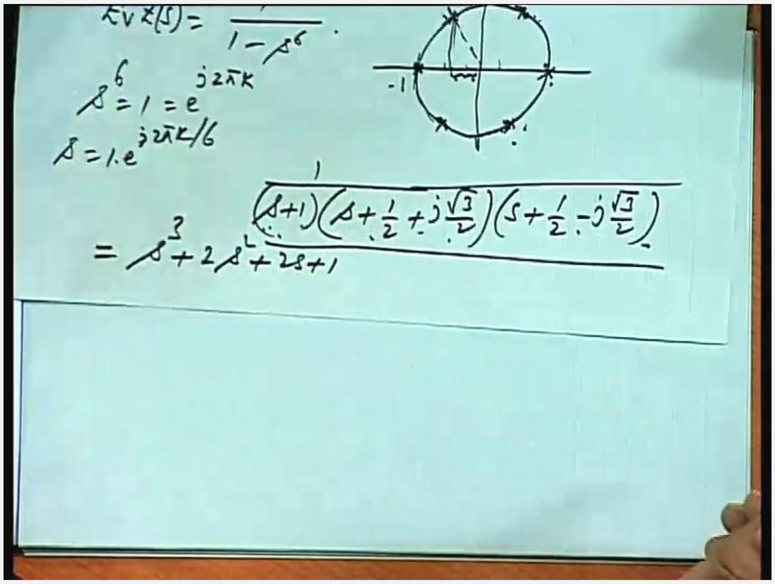

So, if you are given a real function, a real part of z j omega if that is given, say let us take an example it will be clear is given as 1 plus omega to the power 6. That means if I put omega is equal to s by j, s is equal to j omega the inverse of situation will be omega equal to s by j. So, from the given function in the frequency domain if I want to get back to s domain, I will just replace omega by s by j.

So what does it give me, that gives me the Even z s equal to 1 by 1 minus s to the power 6. If I substitute this omega is equal to s by j so s by j to the power 6, that gives you minus s to the power 6 1 minus s to the power 6, what are the roots, s to the power 6 is equal to 1. So, s is 1 means e to the power j 2 pi k, so this will be e to the power j 2 pi k by 6 in to 1. So, you keep on taking different values of k will be on a unit circle k is equal to 0, k equal to 160 degrees, k equal to 2, k equal to 3, k equal to 4, k equal to 5 and 6, 1 2 3 4 5 6 there are six roots.

Now for a stable system, if z s is a realizable network, all the roots of the denominator should lie in the left half plane. I know this z s that we have got that is m 2 plus n 2 and m 2 minus n 2 if m 2 plus n 2 is to correspond to set of stable roots, then that must correspond to these three roots and m 2 minus n 2 will be giving you the mirror images. So, once you have decided once you have calculated all the roots take away only the left half plane roots, right half plane roots are just the mirror images.

So, taking the left half plane roots you conform the polynomial, what are the roots, what will be the polynomial, 1 by the denominator polynomial will be 1 is corresponding to minus 1, so s plus 1 is a factor. The other one will be this is 120 degrees, so this is 60 degrees so cos 60 is half this is minus 0. 5 plus j 0 root 3 by 2 j root 3 by 2 minus j root 3 by 2. So, it will be s plus half plus j root by 3 by 2 in to s plus half minus j root 3 by 2, arising out of these three roots the corresponding polynomial will be this.

So, this is the denominator which means m 2 plus n 2 is it not we have started off with the roots of m 2 square minus n 2 square there are six roots. We know m 2 plus n 2 will give me these three roots m 2 minus n 2 will give me these three roots, because it is a stable system. So, the roots correspond to this must be these three and these are the polynomials.

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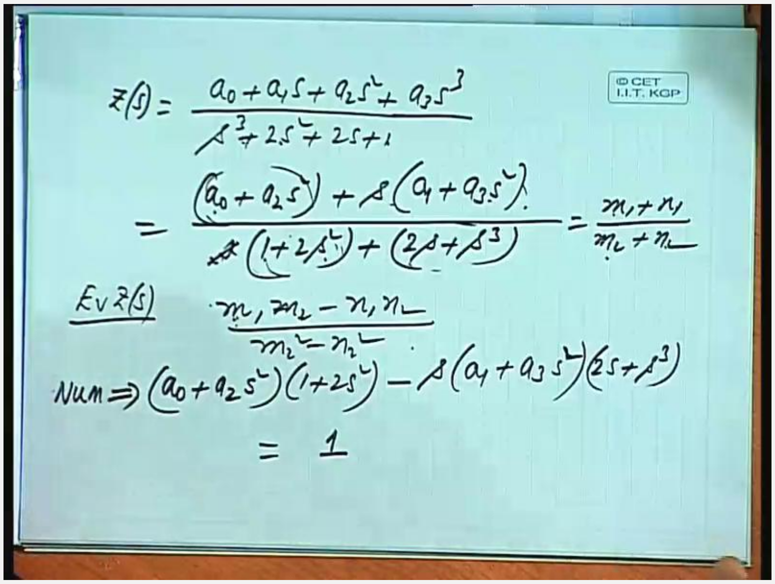

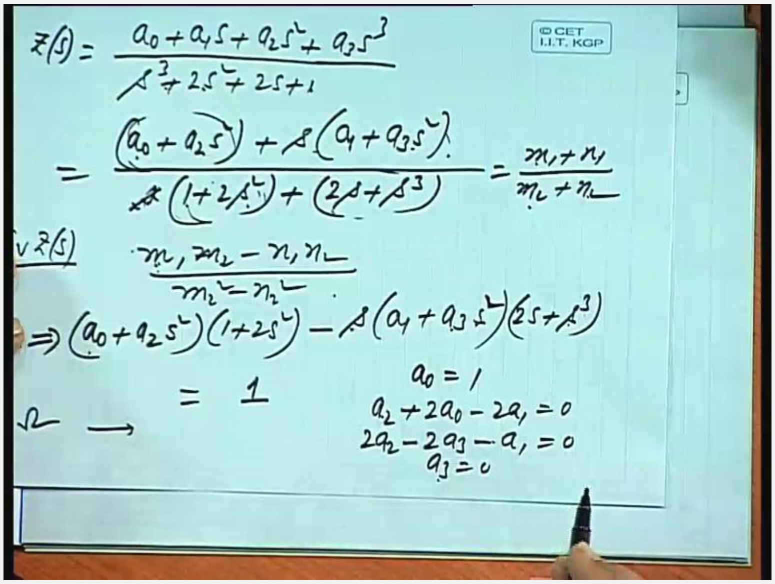

If that be so, then let us go back to our original function z s therefore can be written as this can be written as, if you simplify this it will be s cubed plus twice s square plus twice s plus 1. You just multiply and then multiply by s plus 1, it gives you this, so this is the denominator polynomial. So, our z s is some a 0 plus a 1 s plus a 2 s square plus a 3 s cube divided by s cube plus twice s square plus twice s plus 1, because we have been able to identify the denominator m 2 plus n 2 from the given function.

So, if you segregate again the real and the even and odd part, this is a 0 plus a 2 s square plus s in to a 1 plus a 3 s square divided by again 1 plus 2 s square plus 2 s plus s cube. So this is m 1, this is m 2, this is m 1, this is n 1, this is m 2, this is n 2. So, I have identify m 1 m 2 n 1 n 2, I have got this coefficients are not yet known. Now you try to find out, what is the numerator of Even z s, that should be m 1 m 2 minus n 1 n 2 this is the Even z s, the numerator part this is already known the numerator part, what was given, as 1 numerator corresponds to 1, so try to match the numerator.

Now, what is m 1, this one a 0 plus a 2 s square m 1 in to m 2 1 plus twice s square minus n 1 s in to a 1 plus a 3 s square in to n 2 twice s plus s cube, that should match with that numerator given that is 1 is it not? This is the numerator m 1 m 2 minus n 1 n 2 and that should be equal to 1. So, now by matching the coefficients of s square is equal to j omega by matching the coefficients of omega, you can get all the coefficients.

If you can write it a 0 in to 1, so a 0 comes out to be 1, all other coefficients of s square s 4 etcetera will be 0, what will be the coefficient of s square for example, and s to the power 4, we have up to s to the power 6. So, there will be three equations s to the power 2 s to the power 4 s to the power 6, three more equations all the coefficients are equated to 0. So, the equations are you can check up twice a naught minus twice a 1 is equal to 0, then twice a 2 minus twice a 3 minus a 1 is equal to 0 and a 3 is equal to 0. See a 3 in to s square in to s into s cube, so a 3 to the power a in to s to the power 6 and there is no, s to the power 6 term. So, a 3 is 0 substitute it there and a 1 you will substitute, a 0 you substitute here, so you get two equations in a 2 and a 1 solve them you get all the coefficients.

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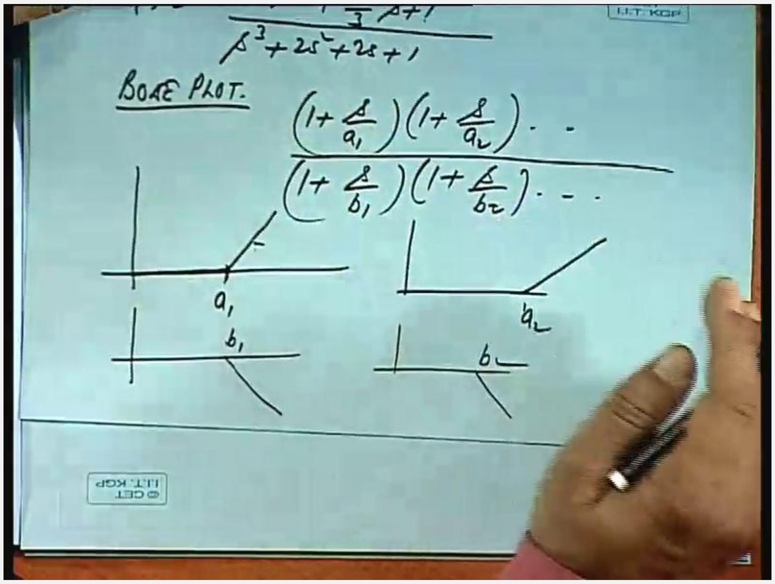

So, I will write finally the z s as 2 third s square plus 4 by 3 s plus 1 divided by s cubed plus twice s squared plus 2 s plus 1. This is as very simple technique of evaluating the network function you given say the real part. You have also seen in frequency plots say when you are given the Bode Plot of a system, suppose a network function is given in this form 1 plus s by a 1 in to 1 plus s by a 2, and so on. And divided by 1 plus s by b 1 in to 1 plus s by b 2, and so on. How do make a Bode Plot, you take individual components all the numerator functions will have the appearance like this 20 db per decade this is at a 1 this is at a 2, and so on. And the denominator functions will be having 20 db per decade fall at b 1 b 2, and so on.

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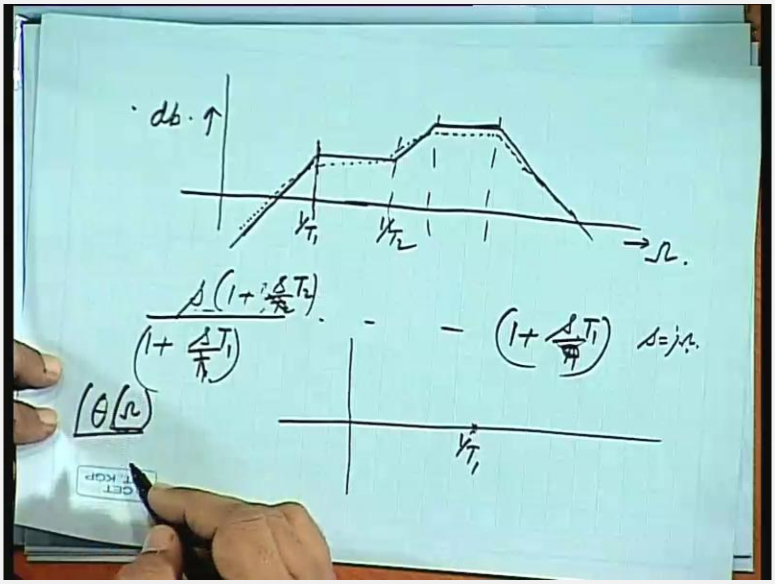

Now, if there is a mix up and if they are reasonably separated, then the total the asymptotic plots if you make will be it may be given to you the asymptotes may be say may be like this, the gain plot db gain that is 20 log of g this is frequency plot. Now, since these are asymptotic plots the exact plots may be somewhat like somewhat like this. We are not bothered about the actual plot, the actual plot has been decomposed in to approximate straight lines with 20 db or 0 db or minus 20 db slopes or multiples of 20 db.

So, with these clearly separated frequencies we can see there is a lies here, this corresponds to a term like s in the numerator, then there is a break 20 db per decade fall has been added here. So, corresponding to this frequency say 1 by T 1 there is a fall here, so that will be appearing here as 1 by s plus T 1 or b 1 whatever you call it. Similarly here, there is a rise to 1 by T 2 will appear as 1 plus s plus T 2 and so on this is how you realize from a Bode Plot. From the sketch you can realize the network function or the gain function system transfer function.

Now, from the gain plot can you estimate the phase, once you have estimated these for each component you know if there is a factor say 1 plus s T by 1. Suppose it is in the numerator it can be in the denominator also, what is the phase contribution of this particular factor, at this frequency I should have written T 1 in to s dimensionally it should have been T 1 in to s T 2 in to

s. So that 1 by T 1 is the frequency, it should have been t 2 s T 1 s so 1 by t 1 1 by T 2, these are the corresponding frequencies.

Now, T 1 s, so at 1 by T 1, how much is the angle its 1 plus j 1 at a frequency s equal to j omega at omega equal to 1 by T 1 this becomes j. So, 1 plus j that is 45 degrees this is what you do in Bode Plot is it not? At that particular corner frequency you have 45 degrees 110th the corner frequency you have an angle of approximately 6 degrees and 10 times that you have about 84 degrees, and then some intermediate zeros you calculate.

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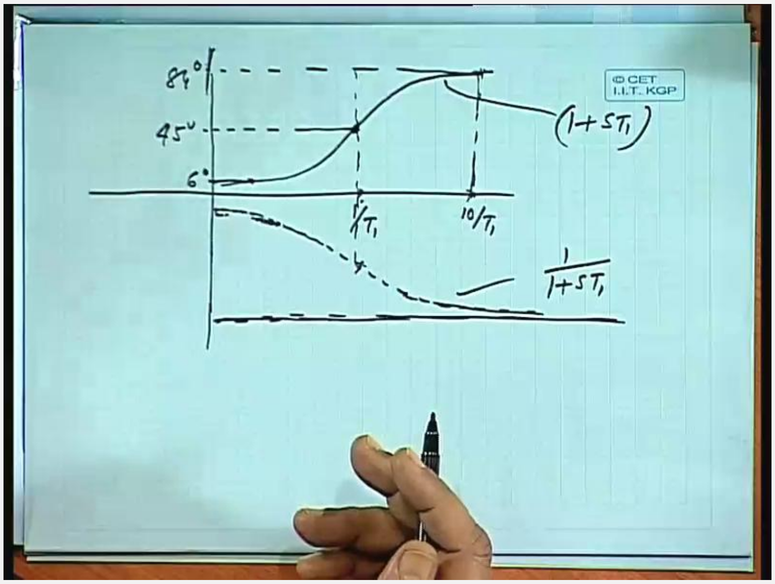

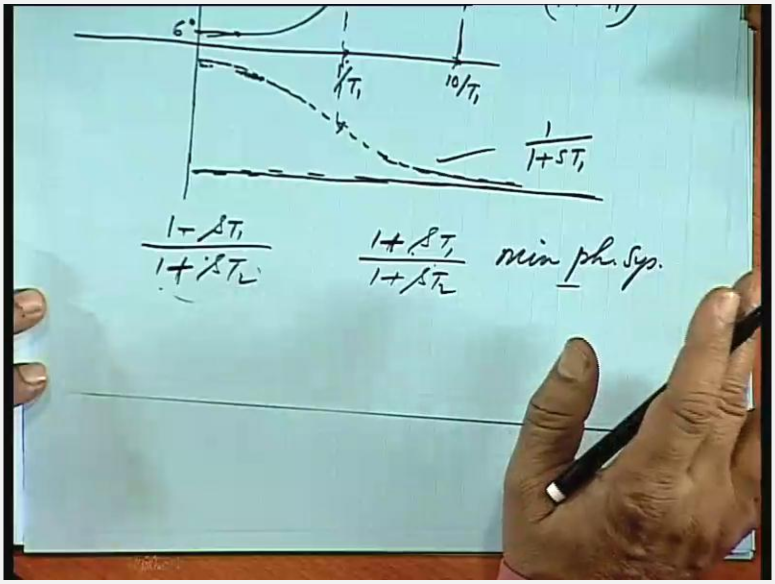

So you get a profile like this, either this is that frequency at which the change is taking place this corresponds to 45 degrees, 10 times that is 10 by t 1 it is about 84 degrees and 110th of that it is about 6 degrees. So, you get a profile like this and you can keep on using this as a stencil, you can keep on shifting it fix this point at 1 by T 1, 1 by T 2, 1 by T 3, you can reverse it. If it is in the denominator, the phase function is like this, distant image of this mirror image of this. So, this is corresponding to a factor 1 by 1 plus s T 1 and this corresponds to 1 plus s T 1 in the numerator.

So, once you are given a gain function you can resolve it in to approximate asymptotes, and then find out the frequencies and hence their nature, whether they will be appearing in the numerator or in the denominator. And accordingly, you can construct the contribution of phase for each component you can see the individual component of phase add them together you can realize the phase function. Or once you have realize the network function you just put s equal to j omega here s equal to j omega and then find out theta omega theta is a function of omega you can do either way. Now, based on this will now try to see whether it is possible to derive any relationship between the real part and the imaginary part of a complex quantity that is the frequency response discrete time frequency transform.

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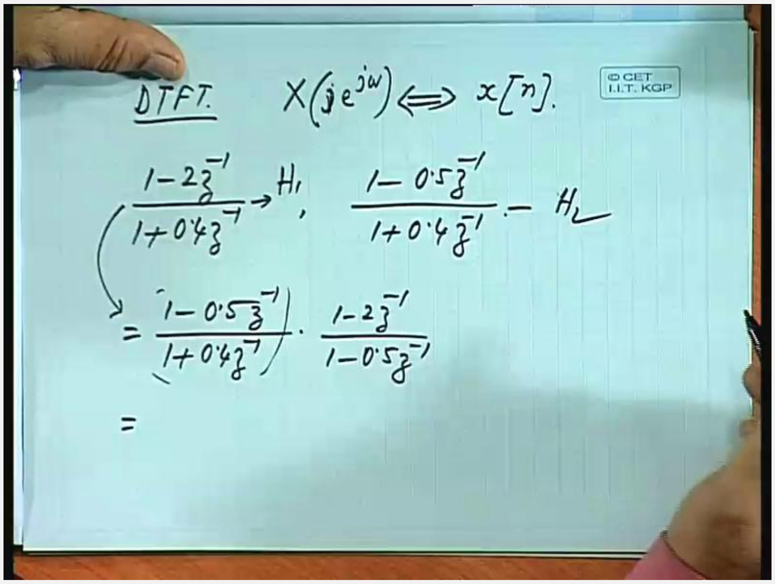

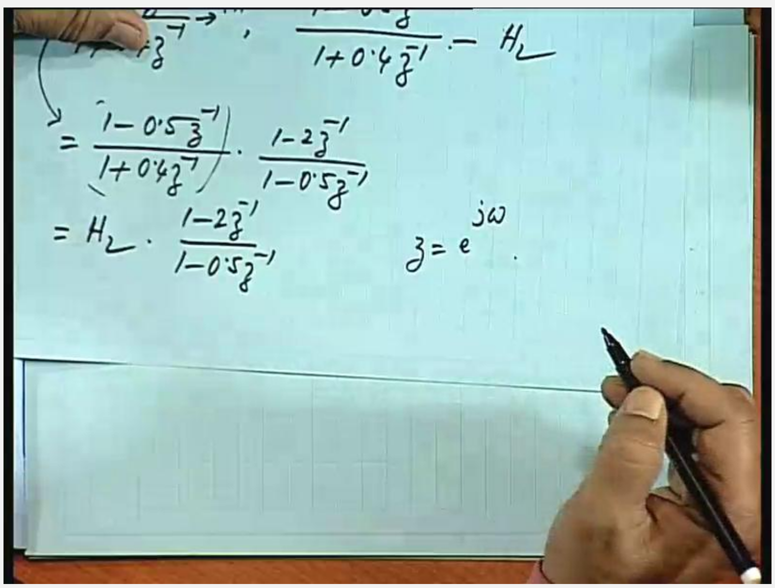

DTFT of a sequence normally, we write e to the power j omega of a real sequence x n. Now, this Bode Plot. Before we start here a 1 more points I would like to mention if you have a function like 1 minus s T 1 by 1 plus s T 2 and 1 plus s T 1 by 1 plus s T 2 will find the magnitude functions will be same for both of them is it not? So from the magnitude, we cannot really evaluate the phase.

Unless there are certain restrictions imposed on this that is we would like to realize only those functions which have roots in the left half plane that is the zeros also should included in the left half plane. If that is show then only we can realize this it is not unique otherwise it can be either this or this. Even if it is stable, the denominator will be 1 by s T 2, I cannot put 1 minus s T 2, then it will become unstable, but even for a stable system it can be either minus or plus.

So, when it is all roots in the left half plane that is both poles and zeros are in the left plane we call it a minimum phase system. So, we can realize only minimum phase system like minimum resistance minimum reactance functions we are discussing earlier for network functions, similarly for a transfer function from bode plot we can realize only minimum phase system. Now, let us see in the discrete domain an identical nature of ambiguity, if I have a function 1 minus 2 z, inverse 1 plus 0.4 z inverse, and another function 1 minus 0.5 z inverse by 1 plus 0. 4 z inverse.

Now, this I can write as 1 minus 0. 5 z inverse by 1 plus 0. 4 z inverse in to 1 minus 2 z inverse by 1 minus 0. 5 z inverse. I have just multiplied by this factor 1 minus 0. 5 and divided also by the same factor. Now, what is this it is the second function is equal to I call it say H 1 and this as H 2 so this is nothing but H 2 in to 1 minus 2 z inverse by 1 minus 0. 5 z inverse. Now let us see, what is the frequency response of this, what is the discrete time Fourier transform of this. You put z is equal to e to the power j omega is it not?

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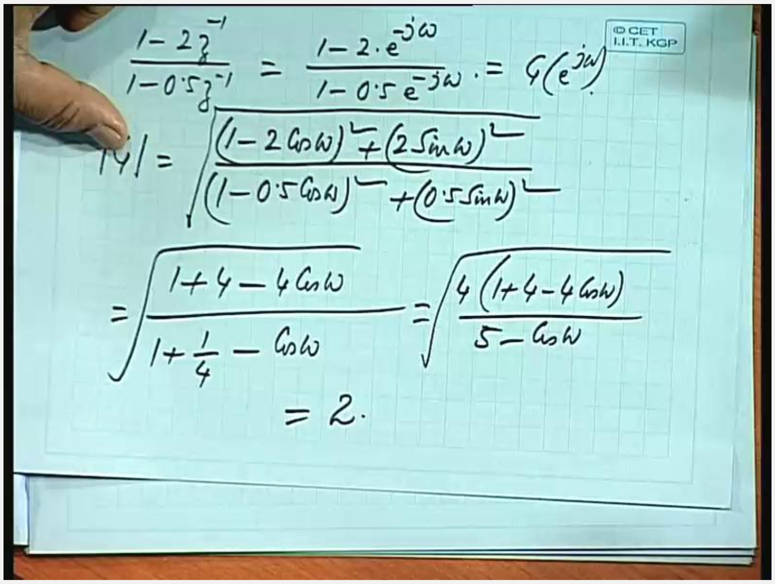

So let us see, 1 minus 2 z inverse by 1 minus 0. 5 z inverse equal to 1 minus 2 in to e to the power minus j omega by 1 minus 0. 5 in to e to the power minus j omega. Now, you would like to see what will be the magnitude of this, what is the magnitude of this function, let us call this as some G. So, we are interested in finding the magnitude of G will be 1 minus 2 cosine omega is that, squared plus 2 sin omega squared divided by 1 minus 0. 5 cosine omega squared plus 0. 5 sin omega squared correct me if I am wrong, whole thing under root is that.

Simplify, let us see this is 1 twice cosine omega squared that is 4 cos squared omega 4 sin squared omega will give me 4, minus twice a b, so 2 in to 4 cosine omega and here I get 1 plus 0. 5 squared that is 1 by 4 minus 2 in to 0. 5 in to cosine omega multiply the denominator by 4 numerator by 4. So, that gives me 1 plus 4 minus 4 cosine omega this whole thing square rooted divided by 4 plus 1 again 5 minus cosine omega. So, they will get cancelled, so this is only 2 it is a constant.

So, the first function is nothing but the second function multiplied by a constant, so for as the frequency response the gain is constant. So, when you see the gain plot when it is normalized then we cannot identified which one is what because both of them will have the same frequency response. So, this ambiguity once again can be dissolved only when we consider the roots for a discrete system, roots very similar to your continuous domain we consider the roots within the unit circle.

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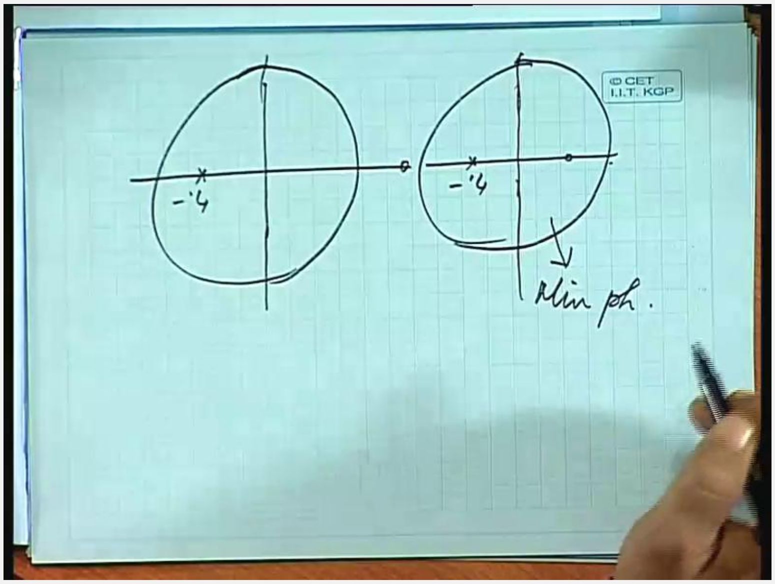

Now, for the first function, what are the roots, poles are identical poles are identical 0. 4 minus 0. 4 and minus 0. 4 in both the cases say this is the unit circle, it is not very much to the scale these other unit circle. Now, in one case in the first case, where is the 0 z is equal to 2, and here z is equal to 0. 5, so 0 is at 0. 5 in the other case it is here outside. So, when poles and zeros are included within the unit circle we call this a minimum phase system in the discrete domain. So, you can realize you will try to realize only those functions which will be of minimum phase nature, where is otherwise there can be any number of such functions giving you the same frequency characteristics for the gain function.

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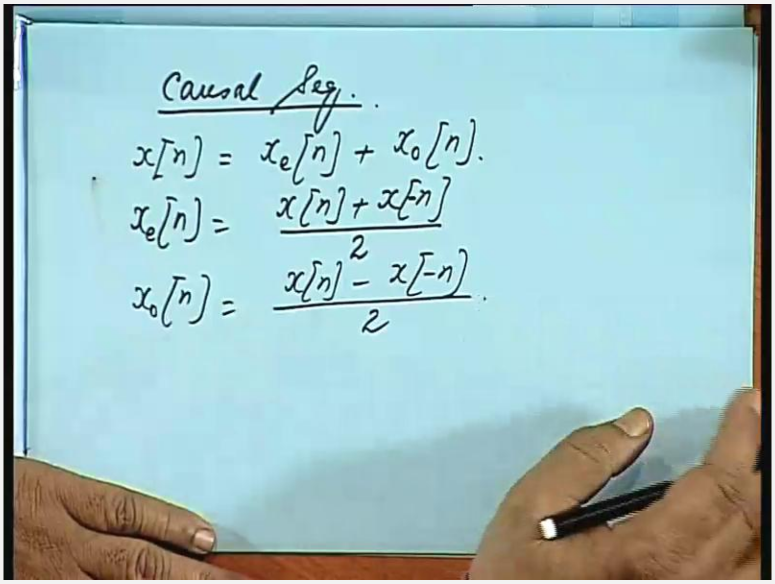

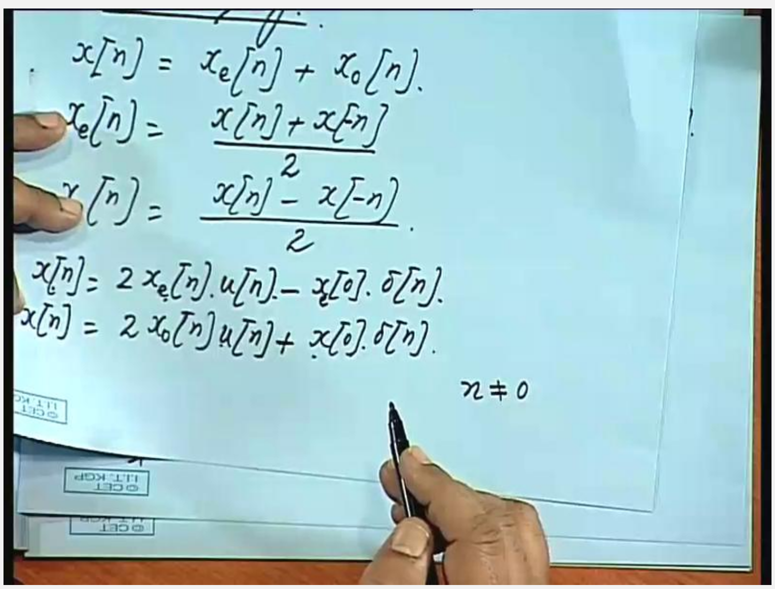

Now, will take a causal sequence x n any causal sequence, causal or non causal any sequence I can write as an even part and then odd part. Basically, they are even functions and odd functions, where the even part will be evaluated from the value of the function in the positive and negative region of n. Similarly, the odd part will be x n minus x minus n divided by 2, we are considering only real sequence at this moment. If it is causal then x minus n will be 0 causal sequence means it will be only on the right side of the x axis. So, from here can you I will uh write it later on let me draw some sequences.

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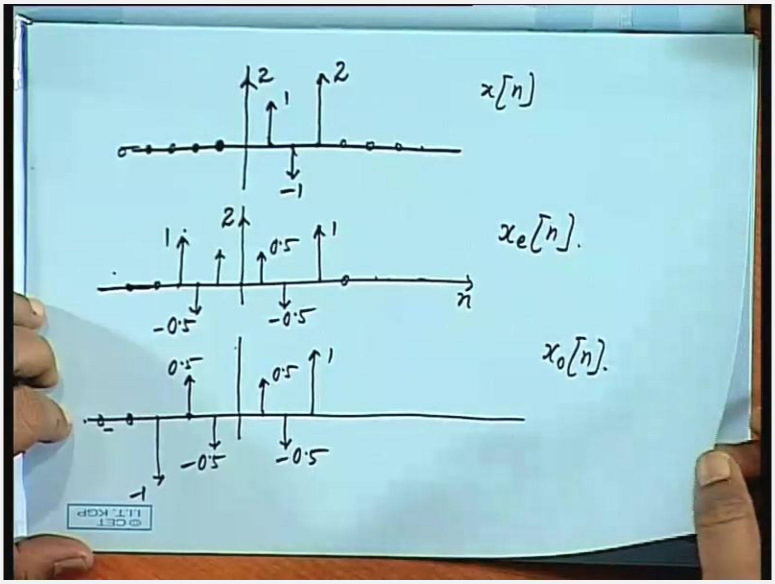

It will be then clear suppose I have a causal sequence like this let us take some values x 0 is a 2, then 1, then minus 1, then 2, just for example I am taking a sequence like this, rest are all 0, what would be the even part this is x n, what will be the even part, x n plus x minus n divided by 2. So, this is 0, so 1 plus 0 by 2 so here it will be half 0. 5, then here it will be minus 1 and 0 divided by 2 so minus 0. 5 and then this 1 was also 2, so this 1 also will be 1 rest are all 0. What about the central value, should it be 1 or 2 yes 2 because, there is nothing like plus 0 and minus 0.

Now from here, can you establish a relation let me complete the odd part also, then will come back, what will be the odd part like. Now, even part though we are defining in this side even part means there is a replica on the other side that is not necessarily to be shown every time. But actually it exists up to this points this is the even part, odd part similarly these are all 0 value. Odd part what will be the odd part, this minus 0 by 2, so 1 minus 0 by 2 it will be again 0. 5, then minus 1 minus 0 by 2, so minus 0.5 and then 1 and then what would be its replica on the other side, it will be double image, so you take image about the y axis.

Then, again about the x axis, so it will become minus 0.5 plus 0.5 and minus 1, rest are all 0, so this is the odd part. We shall keep this in front of us to write x n now in terms of x e n. Now, x n if I just multiply x e n by 2 I will get these values except the first one is it naught if I multiply this by 2 I will get rest of the values that is for not n not equal to 0. So, x n I can write as 2 times x e n in to u n if I multiply by u n; that means if I multiply 2 times u n the even part is having non zero values on the left hand side. Similarly, odd part though the original sequence is having 0 values on the left hand side.

So, unless I am multiplied by u n this will also be multiplied if i multiplied by 2 this function then everything will be doubled, so I have to multiply only this part by 2, so I have to also multiplied by u n. Then, what is left, this twice here it is 2 so whatever is the value here that has to be subtracted, because when I am multiplying by u n I have already taken it when I am multiplying by 2 I have taken 2 times this. So, one value has to be subtracted so minus x at 0 in to delta n is that.

Similarly, x n in terms of odd function also if I multiply by 2 and again further multiplied by u n I get all these terms except the first value. So, x n will be 2 times x odd n mind you x at 0 is same as x e at 0 is it not even part gives me the same value at x equal to 0, other values are halved but this value same so x at 0 is same as x e at 0. So, while subtracting it I can also subtract x e at 0; that means if x e is known the complete information is given to you for realizing x n, whereas for odd functions the first value is missing, because I do not have any no knowledge about the first value from the odd function.

So here, minus x 0 in to delta n, it should be plus, so whereas from the even part I can realize the original function. From the odd part I can realize only the non zero values of for non zero values of n, n not equal to 0 I can realize this, it will be only this much. But, for realizing it completely that is at the initial value I must know the initial value of the function.

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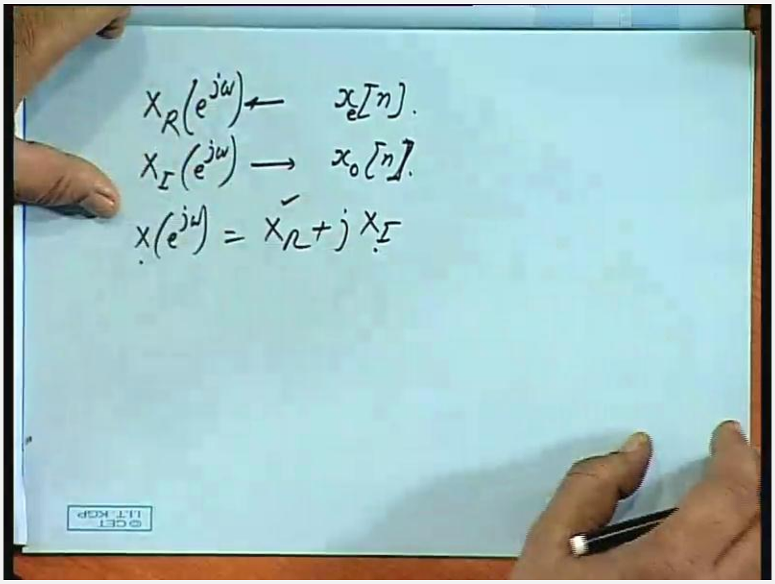

Now, will start from here the real part of a DTFT of any function any sequence x n, the real part corresponds to the even part of x n. Real part will be corresponding to the even part of x e n. You can see for yourself there are proofs given or you can prove it and the imaginary part corresponds to the odd part of the function. So, if you are given the real part of the DTFT that is the frequency response of the discrete system you are given only the real part this function, then how do you realize the odd part, but the total function total x, total x is x r plus j times xi this plus this. So, if you are given this, how do you realize this or how do you realize this.

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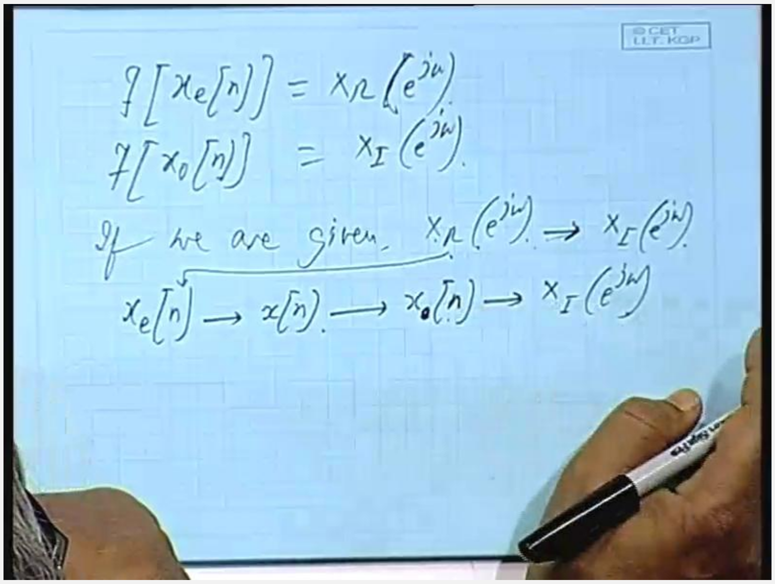

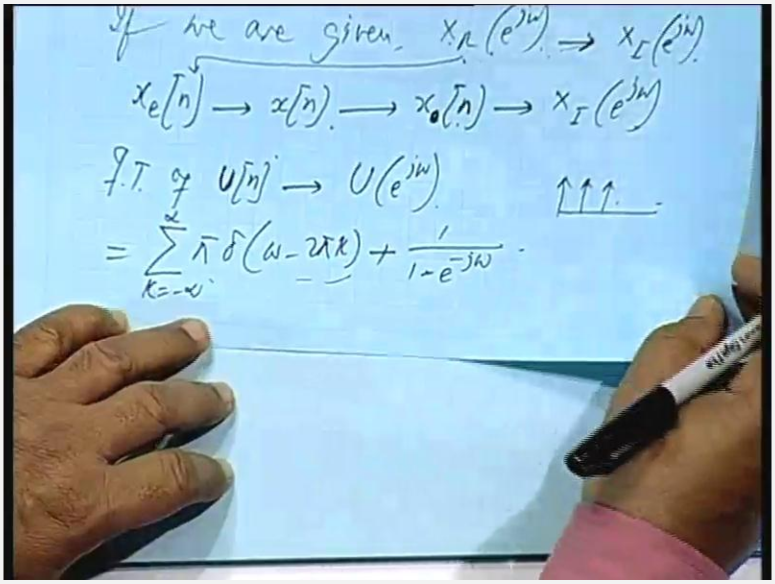

So, Fourier transform of the even part is x r e to the power j omega we have also observed the Fourier transform of odd part is the imaginary part of the Fourier transform of the sequence. So, if we are given the real part of the Fourier transform, how do we get the imaginary part, if we know if we can find out a mechanism of directly finding it from the real part thus the best. Now, the logic is the real part will give me in the time domain the even part and from the even part we can reconstruct x n and from x n we can obtain the odd part and from the odd part you can find out the imaginary part.

Now, is it necessary to go through all these exercises that is obtain from the real part the even component, then from the even component generate the original sequence x n, then from the original sequence generate the odd sequence and from the odd sequence again go back to the imaginary part. Let us see, whether we can skip this, now Fourier transform of U n, small u n we are writing is capital U e to the power j omega.

Now, it can be shown that this is equal to pi delta omega minus 2 pi k plus k varying from minus infinity to plus infinity plus 1 by 1 minus e to the power minus j omega mind you, if it is just a series of one’s it would have given me this result but since this is not an absolutely. So, the Fourier transform the normal Fourier transform is not possible, any way so that is why this additional term comes when we take the Fourier transform of only this series this is sum in the normal case.

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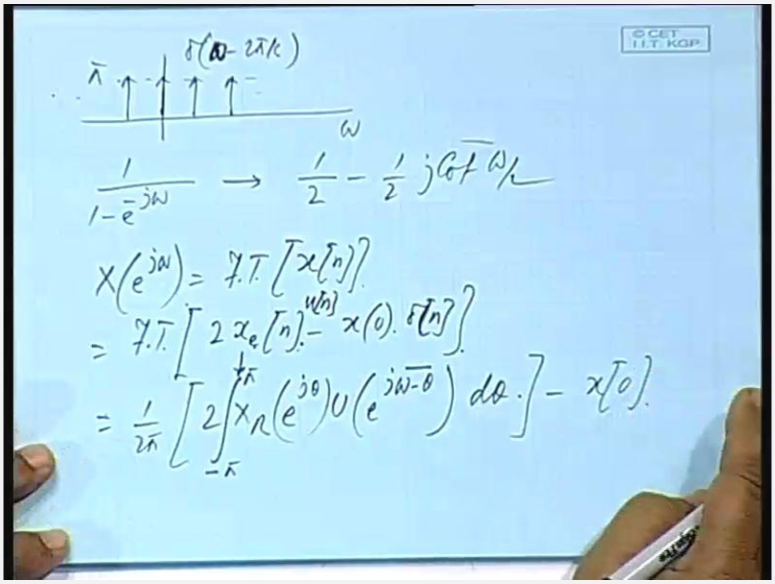

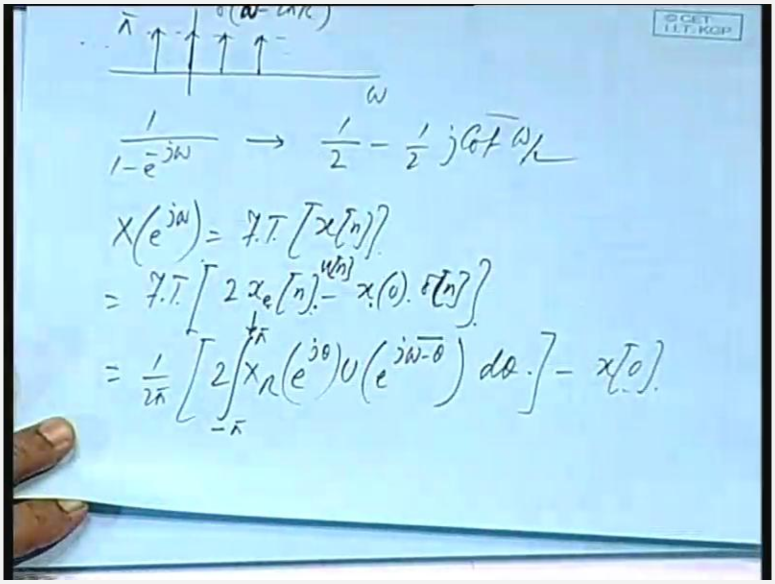

So, we get basically a sequence of impulses in the frequency domain impulses delta k minus 2 pi omega minus 2 pi in to k. This multiply by pi, so this magnitude is pi it is this infinite sequence which will come as an additional term apart from this function. Now, 1 by 1 minus e to the power minus j omega that gives me you can variously resolve it multiply both sides by e to the power j omega by 2 numerator and denominator you will get finally j cot tangent omega by 2. I leave it as an exercise you can do this therefore what would be r x e to the power j omega.

So let us write, this as the Fourier transform of x n which means Fourier transform of x n you know x e n in terms of x e n twice x e n minus x 0 delta n. So if I do that the 4ier transform of this, if I write then it will be 2 times x r now 2 times x e n in to u n this is to be multiplied by u n. Now, if they are in a product form in the time domain, so in the frequency domain will get a convolution of x r which will be u e to the power j omega minus theta d theta minus pi to plus pi, this is for the first part, this part cot tangent theta and minus x of 0, x of 0 will come.

Because, this x 0 in to delta n if I take a 4ier transform delta n that is 1, so that will be coming as a constant x 0 and if you see this is half j of cot omega by 2, which has given me this. And there is a term half separate half and also a term like this these two will generate, they will get cancelled, so they will generate nothing extra.

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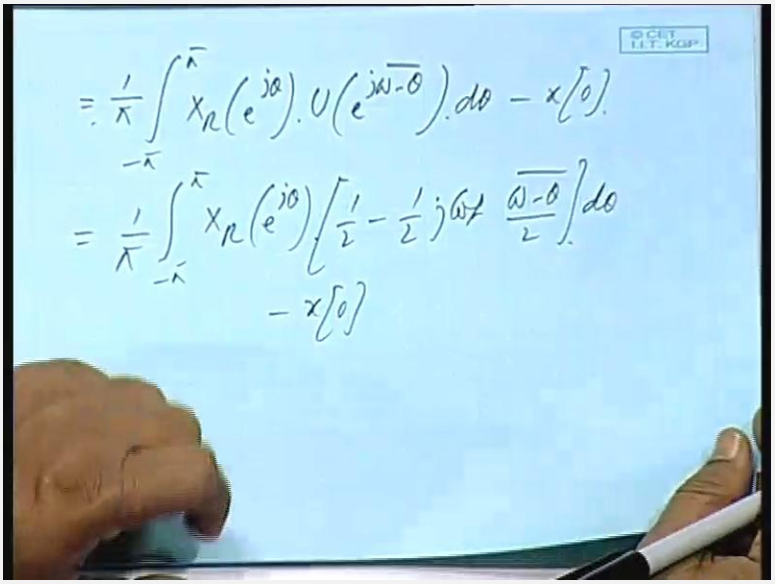

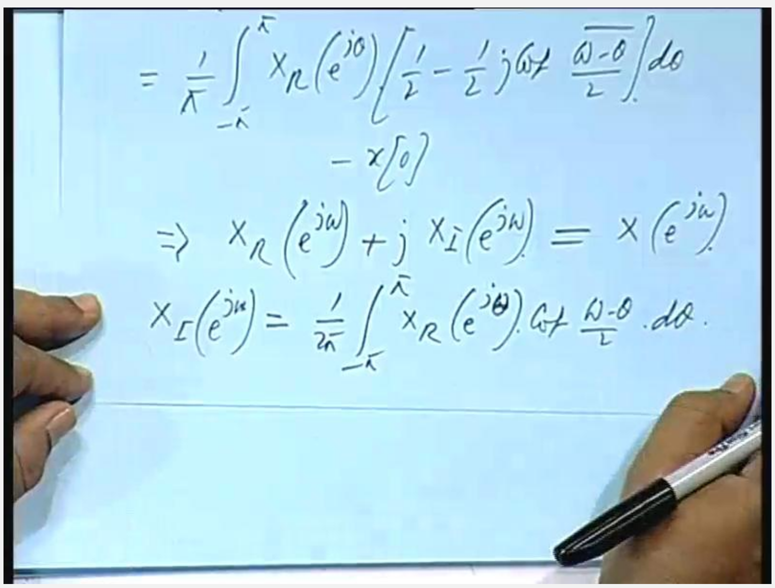

So, if I can write in the final form it will be 1 by pi minus pi to plus pi x r e to the power j theta u e to the power j omega minus theta d theta minus x 0. So, 2 pi and the two got cancelled so finally if I now substitute that values of this function which was giving you this I finally get e to the power j theta half minus half j cot tangent omega minus theta by 2 d theta minus x 0. So finally, the relationship is, so this is nothing but this one first part if it see this will be 1 by 2 pi X R e to the power j theta and integrated over this.

So, that will give me X R now you can simplify this you can see X R e to the power j omega plus j times X I e to the power j omega is X e to the power j omega is it not and X e to the power j omega is also this term now this is generating an x r term. So finally, what you get x imaginary part will come on this side, so X imaginary part j omega if you just equate you expand this and equate with this X R will get cancelled one X R will be inside. So, you will get X I in terms of X R as 1 by 2 pi minus pi to plus pi X R e to the power j theta cot tangent omega minus theta by 2 d theta.

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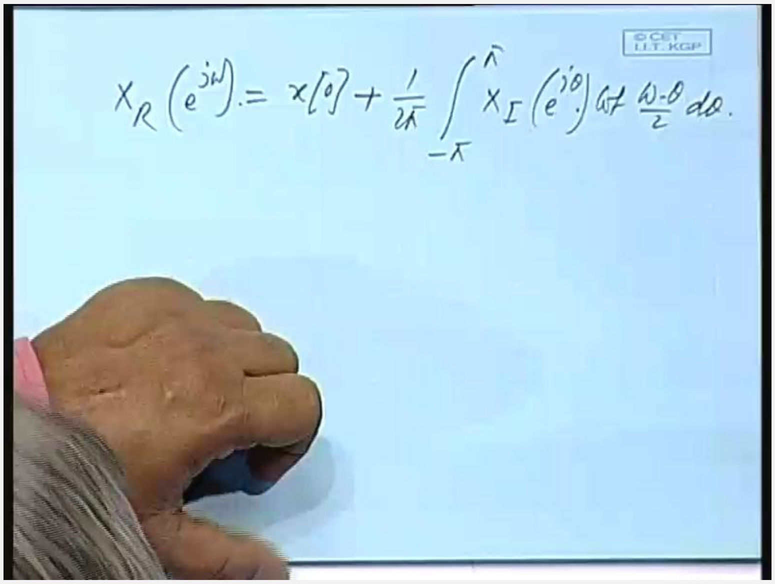

Similarly, from starting from the imaginary part that is starting from the odd function in the time domain if you take the Fourier transform. Then, going by the same logic we can get X R is equal to x 0 there is only one additional term to be written, when we go from the imaginary part X I e to the power j theta cot tangent omega minus theta by 2 d theta the expressions are almost identical except that. In case of imaginary part you have only X R term in case of real part there is an additional term except that there is nothing else.

So, thank you very much.

Wll be using it in the next class or the discrete domain relationship.