Digital Signal Processing
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Lecture - 29
Random Signals

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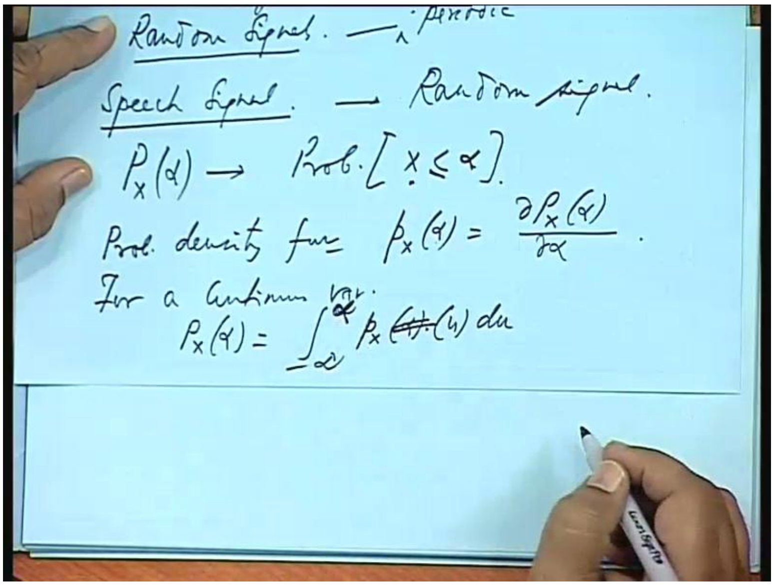

So far we have discussed about discrete systems, which were deterministic in nature; that means, with respect to time, with respect to a particular interval we know the value of the variable to be somewhat exact. Only in the last few classes, when you are talking about quantization error, rounding off error, we talked about noise, which were random in nature. So, random signals and defined in terms of the statistical properties, now they are not periodic, but the statistics will give you some properties, you cannot take Fourier transform or z transform of random signals.

But you can take Fourier transform that is in the frequency domain, you can describe some of the statistical properties like, auto correlation, co variance and so on. Because, these properties are again will show later on they can be also expressed in terms of the time variable or in terms of distinct variable n. So, suppose you have a speech signal, we speak a word we alter different sounds.

So, they are all random in nature, when you say a say b there is a particular signal which is basically a sample of, so many I mean, so many sounds that we make a, b, c, d and b may be pronounced by different people in different ways. So, there are, so many possible sounds, so many possible signals out of which say when I at a particular sound b, then it is one of those ensemble of those signals, which is the sound b.

Even in the same person when he utters the same sound at two different instant of time, suppose I have got cold, today I may have a little nozzle sound. So, the signal will change, so it is random in nature, there are so many instances of randomness in signals, that it is a very important study to be made. We are not going to the details of random signals, just basic statistical properties we are going to discuss, so let me first take up.

So, speech signal is a random signal, let me defined some of the simple quantities P x alpha, we define as the probability, that the variable x is less than or equal to alpha P x alpha we define as a probability of the variable x, x is the random variable, x is less than equal to alpha. We, define actually the probability in terms of a density function, which is small p you write alpha, which is variation of this probability with alpha any question.

Why should it be a partial derivative, it is implicit it may be a variable of time also, though you are not mentioning it here in a situation, when this varies with other variables like time, then it should be partial derivative. So, for a continuous variable x we can write P x alpha as minus infinity to plus infinity small p x I will write u du, this is alpha, this is minus infinity to alpha.

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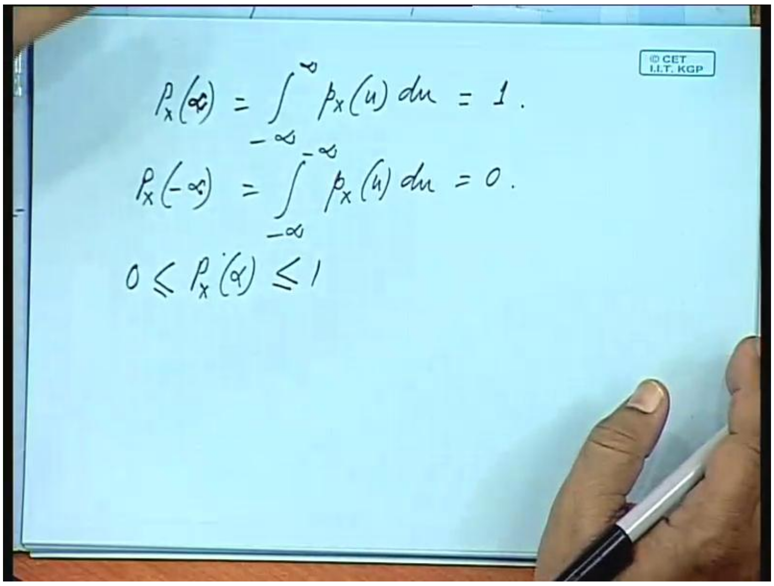

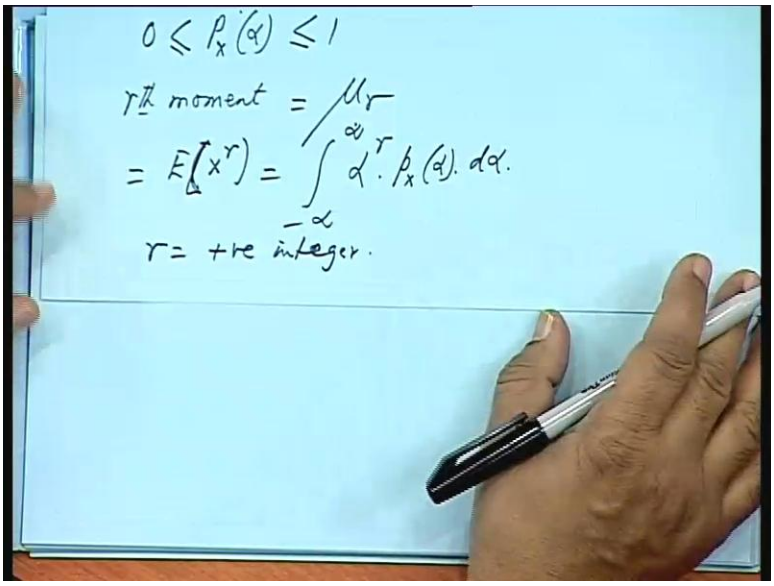

The value of P x at infinity will be minus infinity to plus infinity P x u d u that is the maximum probability, that you can have. And that is equal to 1 and P x minus infinity will be from minus infinity to minus infinity, so it will be 0. So, p x alpha will be varying between 0 and 1 as alpha changes from minus infinity to plus infinity.

Next we define the r'th moment, the r'th moment as mu r is equal to expected value of x to the power r let me put this bracket, x is the random variable to the power r. So, this will be defined as minus infinity to plus infinity covering the entire range alpha to the power r p x alpha d alpha this is the running variable alpha r is any positive integer. So, when you talk about random variables they are defined in terms of the statistics by that you mean all the moments.

So, the 0 th order statistics would be r is equal to 0, if you put alpha equal to 0 it gives me only the total probability. If I take r is equal to 1, so that will be the mean that is if p x is equal to 1 p x is constant then it is a normal arithmetic mean, so it is basically p x gives you the weight age.

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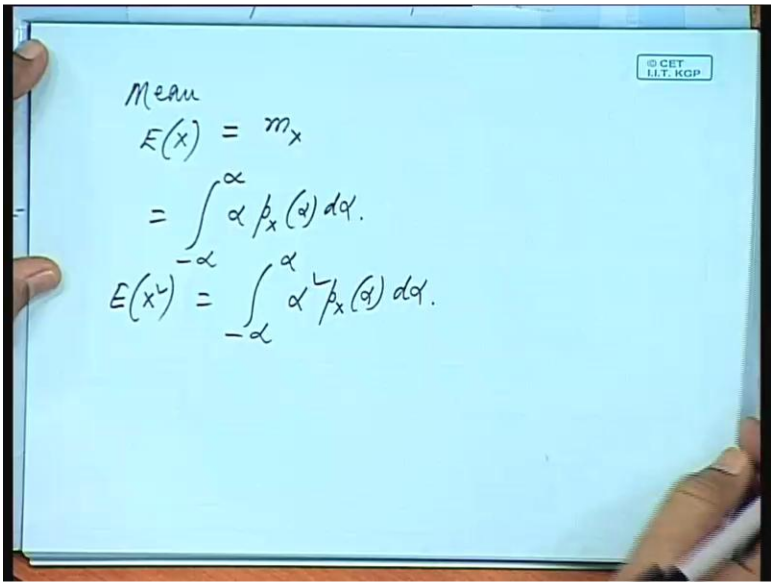

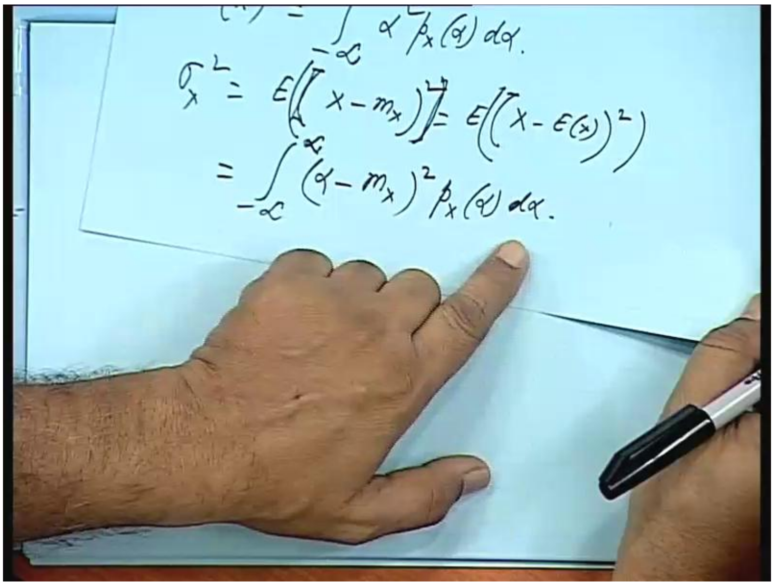

So, weighted mean that is known as the mean, so mean we write as expected value of x, so there are three quantities, the first moment, second moment and mean of the squared quantity squared variable, these are the three quantities, which define more or less the statistics of a random variable. We can go for higher order statistics also higher order moments, but for most of the practical situations we deal with the mean variance and the mean squared variable. So, this we write as m x is equal to minus infinity to plus infinity alpha p x alpha d alpha mean of the squared.

And we define variance as expected value of x minus m x that is expected value of x minus expected value of x. So, square of this I will put another bracket, so square this and what will be this minus infinity to plus infinity. Say if we take the running variable alpha m x square p x alpha d alpha, I have a very bad habit writing infinity and alpha more or less that the same symbol this is infinity and this is alpha, the limits are minus infinity to plus infinity.

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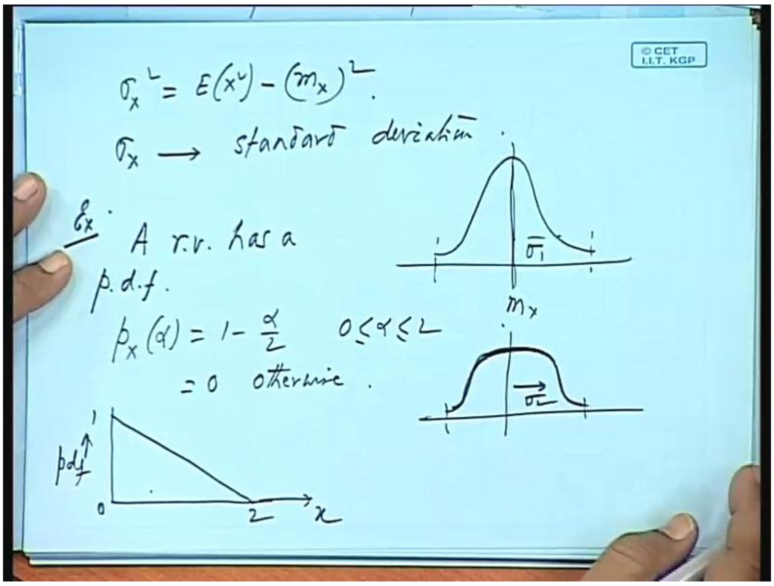

So, sigma x squared if you expand this will be at sigma x square as expected value of x square minus m x squared and sigma x we call standard deviation. Now, this is very important whenever there is a particular distribution given say distribution of marks, we talk in terms of the dispersion sigma x. How much is the variable changing, in which range it is changing and how is it distributed, you can have a the range similar.

Suppose, the performance of students in a particular class, there may be two subjects, where in one case the dispersion is high, that is the distribution of marks may be like this in this is a mean value around the mean value, in the other case. And suppose the range is only up to this, in the other case like this both of them are bell shaped having same range, same mean value, but the distribution is different. So, here sigma is more this sigma 2, this more than sigma 1 from the nature of the characteristics, you can find out.

Suppose, a random variable will take a simple example, a random variable has a probability density function given as p x alpha equal to 1 minus alpha by 2 for 0 alpha 2 and equal to 0 otherwise, what is the nature of this function, it will be a straight line like this. So, this is the random variable x and this is the PDF. Now let us compute the three quantities, the first moment, second moment and the variance.

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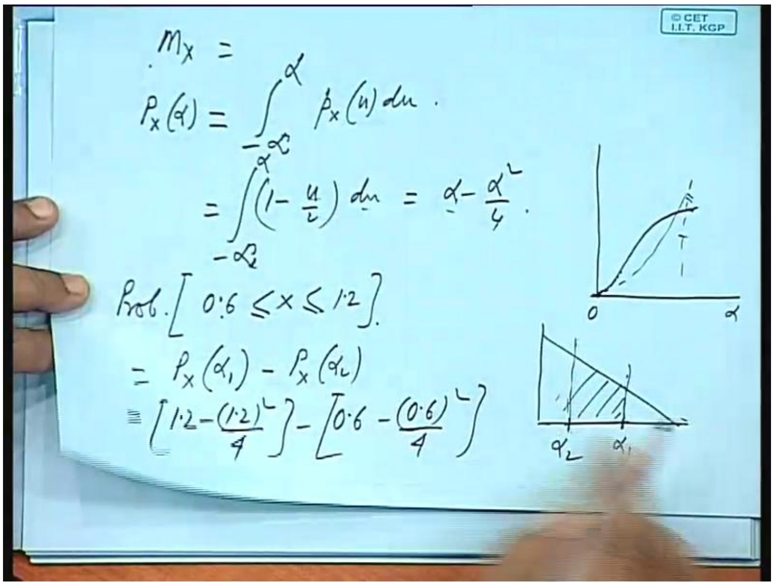

What will be the first moment could you check that is m x, what is the probability P x alpha to alpha, what is the dependence of this with respect to x that is this is the density function given. ((Refer Time: 16:07)) So, what is the probability, probability means area under the curve from minus infinity to that one, from minus infinity to this 0 it is all 0. So, you have to take only in this area and at x what is that capital P x it will be only this area, so can I write in terms of that variable x.

So, we are writing this as alpha, so it will be p x say d u which will be 1 minus u by 2 d u integrated from minus infinity to plus alpha is it all right. So, it will be alpha minus alpha squared by 4 is that I am just taking u as the running variable, so alpha minus alpha squared by 4. So, 0 alpha, so at alpha equal to 0 it is 0, it is the area is increasing and may be somewhat like this I do not know, that you have to compute it will be like this or may be like this, you have to check up you have to plot this.

What will be the probability that this lies between say 0.6 and say 1.2 what is the probability, this is the density function given what is the probability that x lies between 0.6 and 1.2. So, just I can write P x alpha 1 minus P x alpha 2, where this is alpha 2, this is alpha 1, so P x alpha 1 is this area, P x alpha 2 is this area, so the probability that it lies between 0.6 and 1.2 is this I mean area under this. So, it will be 1.2 minus 1.2 squared by 4 minus 0.6 minus 0.6 squared by 4.

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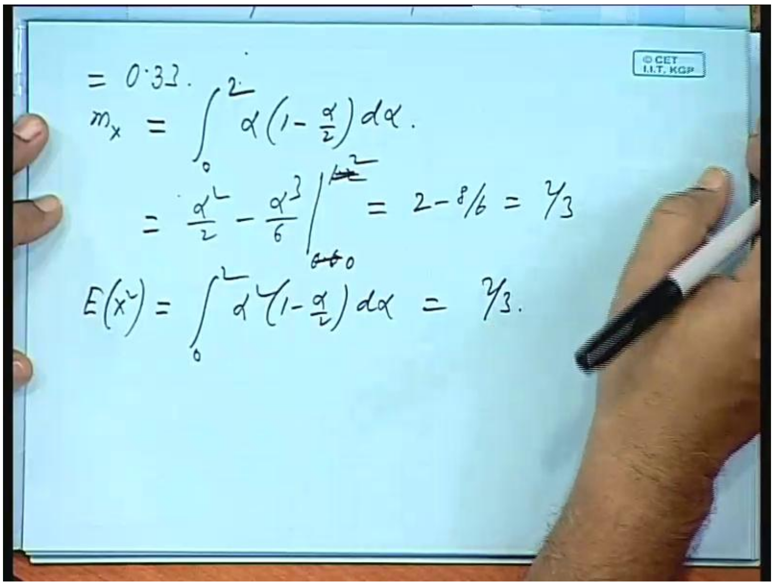

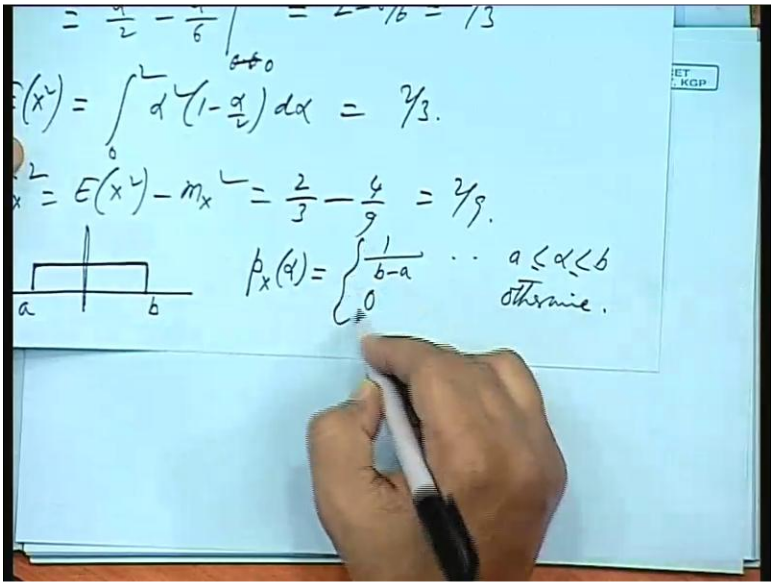

So, that gives me correct me if it is all right m x what is the mean value of this distribution. So, a 0 to 2 minus infinity to plus infinity will be now reduce 0 to 2 alpha into 1 minus alpha by 2 d alpha is that all right, the variable and this is the distribution. So, it is alpha squared by 2 minus alpha cube by 6 0 to 2 we are not computing the earlier, so 2 minus 8 by 6 that gives me 2 by 3, what is the expected value of the squared variable same, so that also comes out as 2 by 3.

Now, let us compute the variance, so it will be 2 by 3 minus 2 by 3 squared 4 by 9, so it is 2 by 9 quite often we come across a probability density function, which is uniform. If you remember, we are seeing the quantization error, so it may be say from a to b range may be specified. So, the density function can be written as 1 by b minus a when it is lying between a and b and equal to 0 otherwise.

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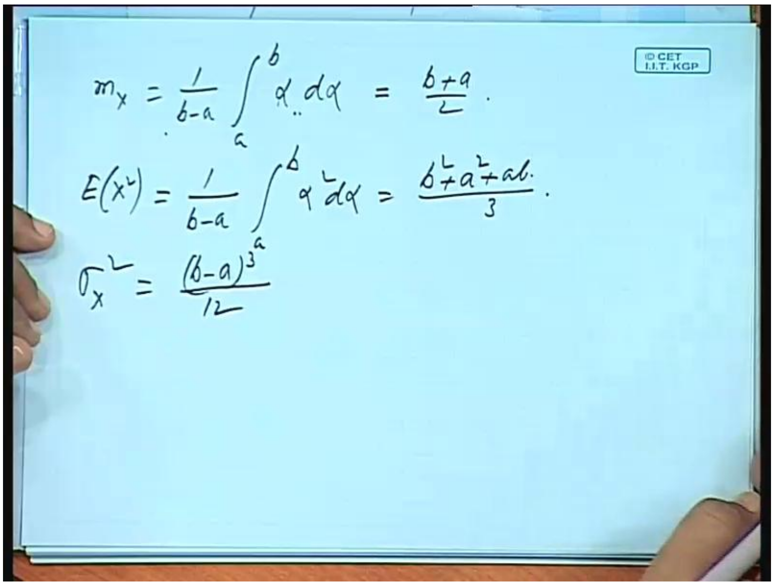

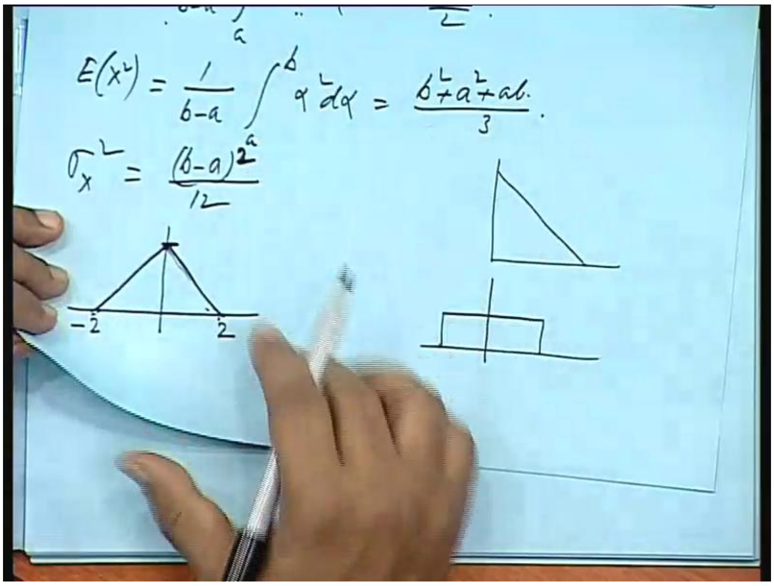

What would be the average or the mean it will be 1 by b minus a a to b alpha d alpha is this all right, see this is the probability density function, which is constant. So, alpha into 1 by b minus alpha b minus a into d alpha, so that 1 by b minus a I have taken out, that is b plus a by 2 alpha square by 2 and if you take the limits. Then common factor b minus a goes out, what is expected value of x squared again 1 by b minus a a to b alpha squared d alpha, so it will be b cube minus a cube by 3, so b minus a will get cancelled, so it will be b square plus a squared plus a b by 3.

So, it will be sigma x square, so you have computed for two typical distributions. One was a distribution like this, the other one is a distribution like this, compute all these three for a probability distribution, say minus 2 to plus 2 the area will be equal to 1. So, correspondingly you calculate the height 1 by 2, 1 by 2 half base into altitude that will be the area, so base is 4 half into 4 into height, so height will be 1 by 2. So, once you know this you know this we can find out the equation of the line and from there.

Student: ((Refer Time: 25:09))

This one, yes thank you very much all right, so this one you know, so you can find out the equation of the lines. And hence you can compute all the three quantities, the integrations for different a moments, try this is as an assignment.

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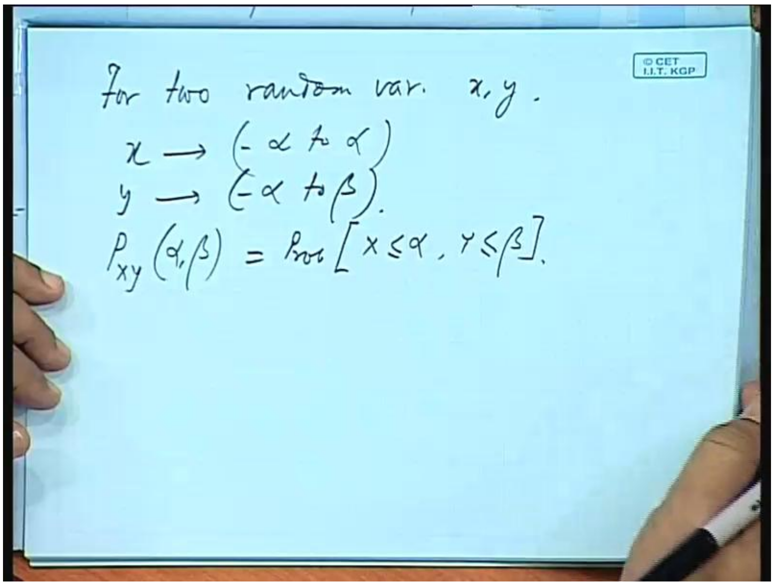

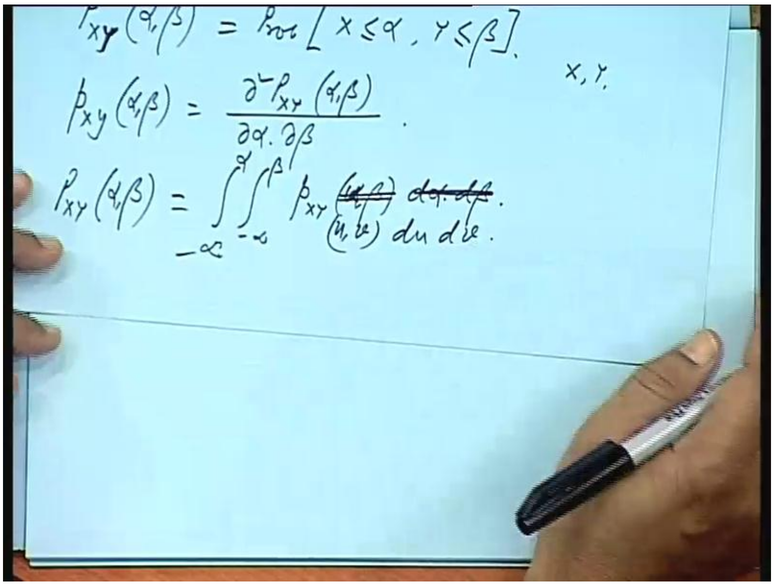

For two random variables x and y, we study their joint statistical property, in terms of the joint probability density function. There of practical interest quite often we find signals, which are either correlated or at least which are now whose properties are to be computed in a joint manner. So, suppose x is the variable which takes the values from minus infinity to alpha and y is a variable, which takes the value between minus infinity and beta, then probability x y alpha, beta is the probability that x is less than equal to alpha and y is less than equal to beta. In many books you will find random variables are written as capital X and capital Y.

So, hence forth I will be using capital X and capital Y and the joint probability density function is defined as the partial derivative with respect to alpha and beta. So, the probability p x y will be basically minus infinity to alpha minus infinity to beta, alpha beta we should not have the running variable same as the limits. So, let me put it as u and v and this is d u d v.

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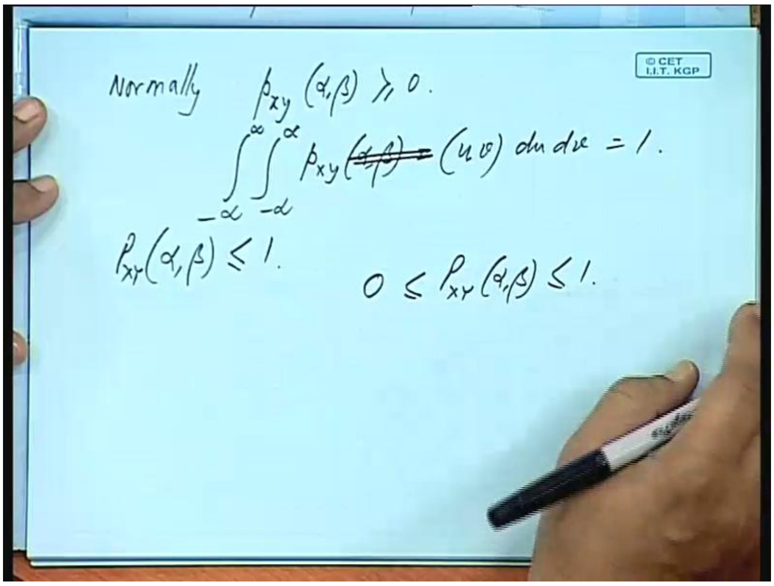

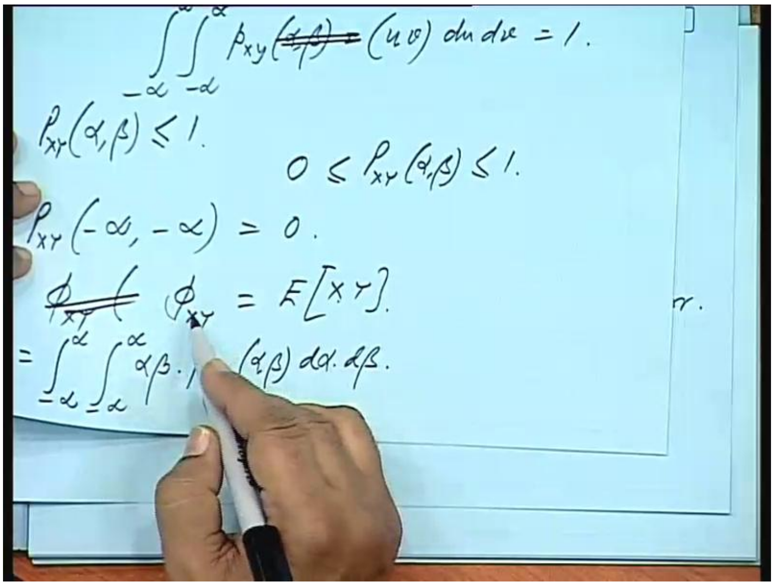

So, when you are considering the joint probability density function normally this distribution or this density and minus infinity to plus infinity for both the variables will be always equal to 1 again the same slip u and v should write d u d v equal to 1, anyway it was not alpha. So, I could have put alpha, beta p x y alpha, beta is always less than equal to 1, so the value of the total probability will between 0 and 1 minus infinity and minus infinity it will be 0.

Now, you define cross correlation between two variables x and y, cross correlation of x and y you define as expected value of x and y, which is between minus infinity to plus infinity alpha, beta p x y alpha, beta d alpha, d beta. That means, you take the product of x and y at all possible values of alpha, beta and then take their average expected value means basically average.

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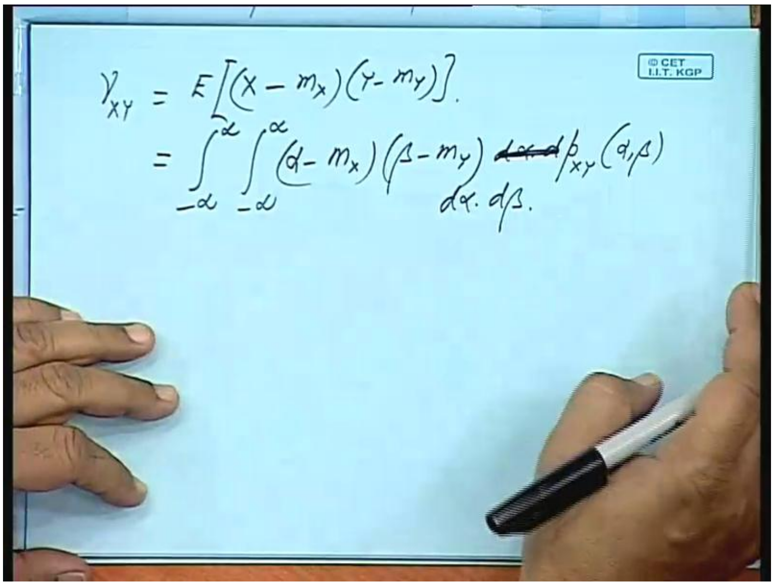

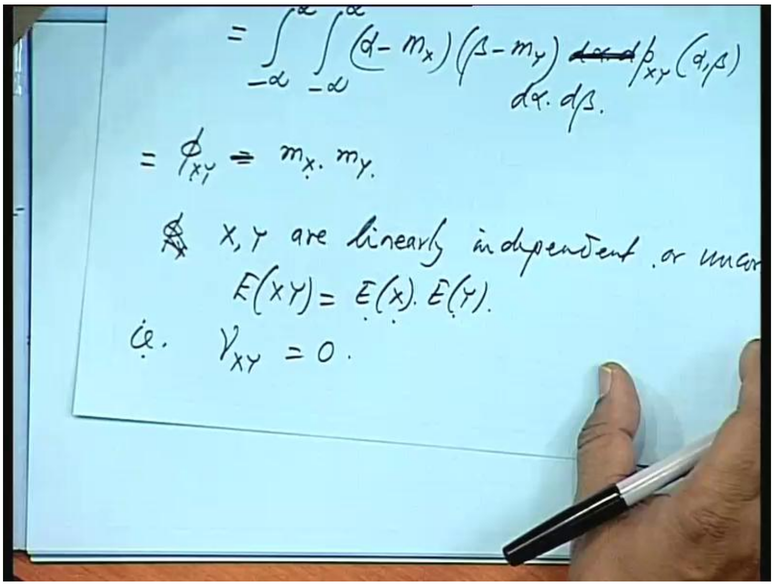

So, similarly, gamma x y is equal to expected value of x minus m x and y minus m y that is minus infinity to plus infinity alpha minus m x into beta minus m y d alpha into the probability p x y alpha beta d alpha d beta is that all right? Equal to phi x y minus m x m y I phi x y did we not define it yes, phi x y was expected value of x y. So, it will be like this, the variables x and y I linearly independent or uncorrelated if you find expected value of x y is equal to expected value of x in to expected value of y, what is expected value of x y is phi x y and what is expected value of x m x m y.

So, if phi x y is equal to m x into m y; that means, this will be 0 that is gamma x y is equal to 0. So, if gamma x y equal to 0, then if they are linearly independent or uncorrelated, quite often we try to find out if there is any correlation between two signals. Suppose, there is a random signal noise for example, a random noise when you try to find out the correlation of a random noise with any other signal it is always equal to 0; that means, they are two independent variables.

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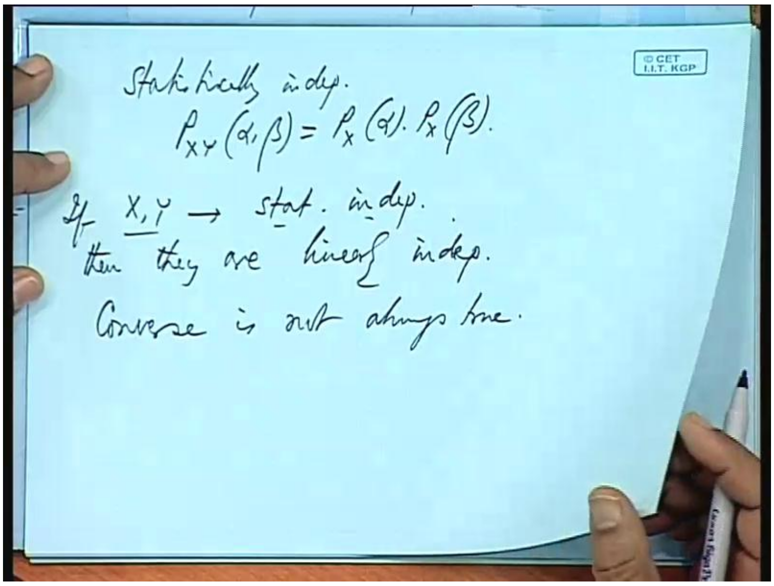

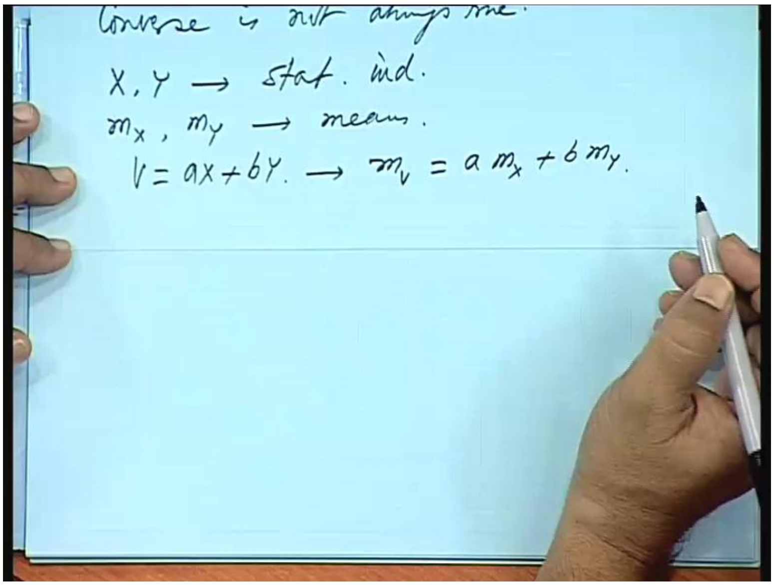

There are statistically independent if p x y alpha beta is equal to p x alpha in to p x beta, so it can been shown that if x and y are statistical independent, then they are also linearly independent. The reverse is not necessarily true, if x and y are statistically independent, then they are linear independent, converse is not always true I leave it as an exercise try to prove if not the second one at least the first one.

So, if they are statistical independent if x and y are statistical independent it is easier to derive the statistical characteristics. Suppose, x and y are statistical independent variables with m x and m y as their means, then the variable v is equal to a x plus b y, a x plus by is another random variable, it will have mean as say m v will be a times m x plus b times m y can you prove it.

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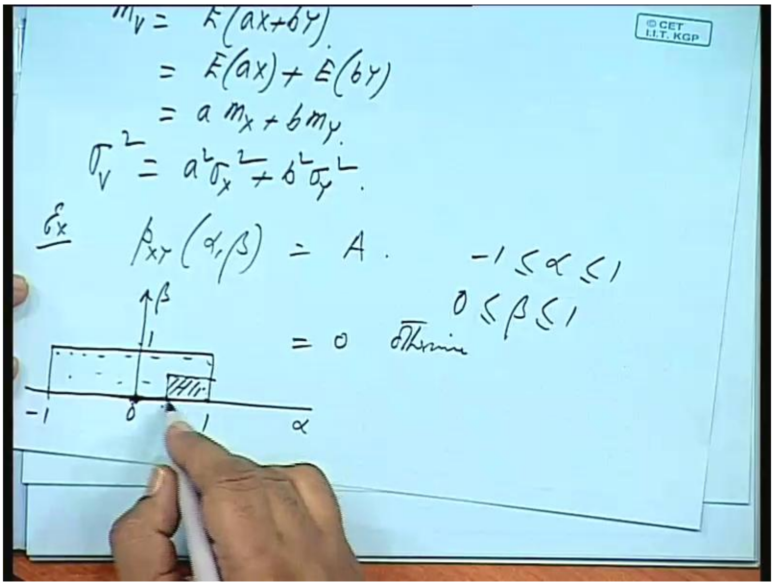

So, what will be mean of v.

Student: ((Refer Time: 37:41))

Expectation of a x plus b y since they are statistical independent, so expectation of, so a times m x plus b times m y, sigma v squared what will be the variance of the variable v is this precisely you calculate, when there is a mixture of say sinusoids, we talking in terms of the RMS value, RMS value. If when you compute it is the RMS value of the first component, square it multiply by the square of the weight age. Similarly, RMS value of the second quantity square it plus multiply by the weight age and so on.

And then take the under root of the entire sum we get the RMS value of the overall variable, when it is a mixture of say 2, 3 sinusoids you compute the RMS value in a similar fashion. Now, let us take a problem, suppose we have p x y as equal to A the range of alpha and beta, this given like this that is alpha if you write alpha it is between minus 1 and plus 1. And if I write beta, in the alpha beta plane I am just showing the variables it is between 0 and 1.

So, this is the range of alpha and beta, alpha and beta will be this set of points determine the and equal to 0 otherwise. So, the probability density function is if I take an alpha, beta plane and if the height represents the probability density function, then it is a it is like a rectangular cake know a just a brick. So, it is equal to a it is uniform for all the points of alpha and beta, the range is specified and it equal to 0 otherwise, say in this side it is all 0 what is the value of a.

Student: ((Refer Time: 41:07))

Very good, so what is a total area 2, 2 multiplied by a is the total volume and that is equal to one that is good.

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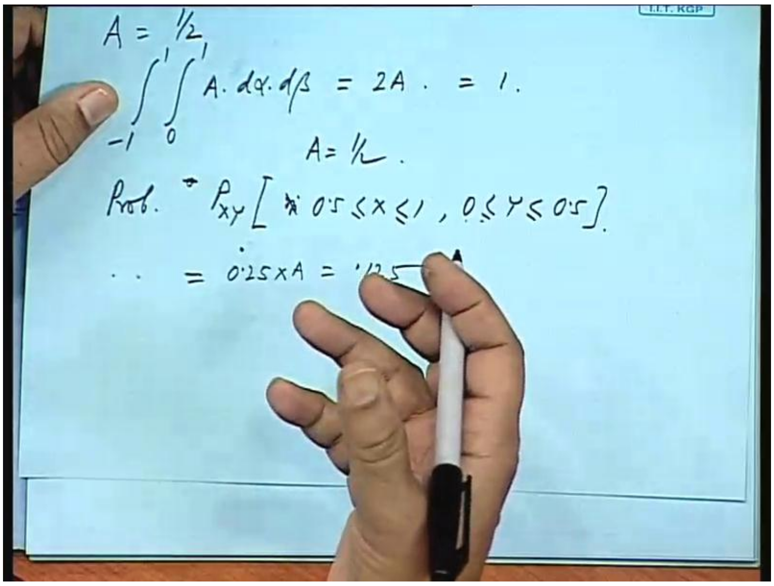

So, A is equal to half if you go by that integration relation it is between minus 1 and plus 1 and between 0 and 1 A d alpha d beta, the PDF is a and that is equal to 2 A and that should be equal to 1 total probability therefore, A is equal to half. So, compute the probability p x y, x lying between 0.5 and 1, y lying between 0 and 0.5, so what would be the value 0.5 and 1. ((Refer Time: 43:02)) So, once again A is between 0.5 and 1 and b is between 0 and 0.5 0 and 0.5, so is this small rectangular piece into A is not into A height is A. So, this is 0.5 and this is also 0.5 0 to 0.5 y is between 0 to 0.5, so 0.5 into 0.5 0.25 into A. So, the total probability will be 0.25 into A that is 0.125 that is the volume of the brick for a discrete signal.

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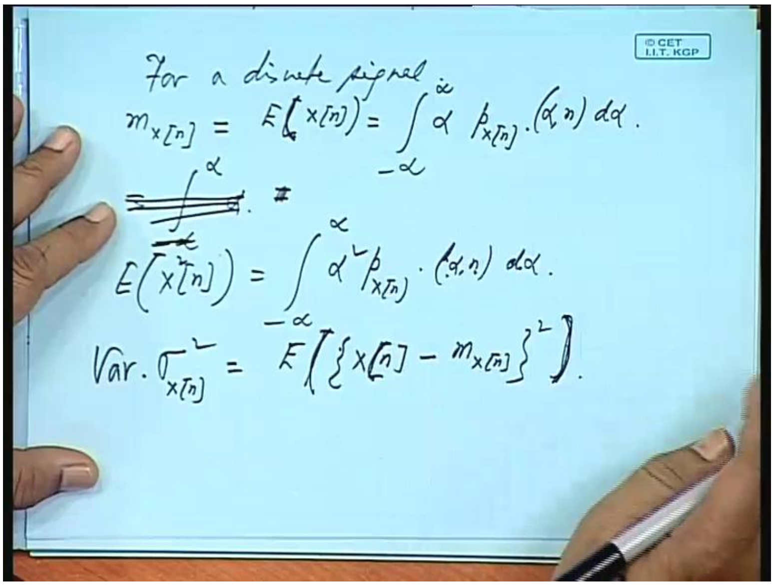

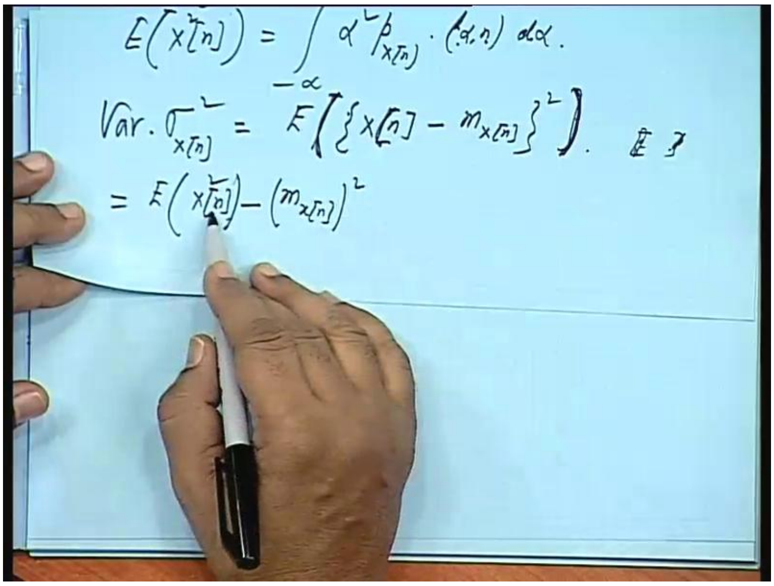

We write the mean as expected value of x n, let me put once again the first bracket, so that is minus infinity to plus infinity, in the discrete domain will put summation. So, expected value of x n how do you put that.

Student: ((Refer Time: 45:00))

Now, in the beginning itself I had mentioned about the speech signal, we are taking discrete domain of time, but we are considering at any moment there can be, so many possibilities at any moment there are, so many possible values. So, it is one of those values, what is the probability that at this moment the say sound will be like this, what are the possible values of the sound.

So, it is in the discrete domain of time is it all right we are not varying n at any instant of time n, what are the possible combinations, what are the possible values of x n. So, it will be minus infinity to plus infinity, this is the range of x at any instant of time alpha p x n alpha n d alpha. Actually, we are in the habit of taking mean with respect to time not the mean at a particular instant of all the possible values of that variable at a particular instant, if that mean is same as the other mean, then it is definitely a stationary process it is an ergodic process.

So, expected value of x n squared will be minus infinity to plus infinity alpha squared p x n alpha n d alpha, if you want to calculate for all possible times then only the summation comes, I am not submitting over n it is integrated over alpha. So, variance it will be expected value of x n minus m x n squared after using the third bracket, so often I am use to it. So, I have forgotten about the first bracket that is equal to expected value of x n squared minus m x n squared.

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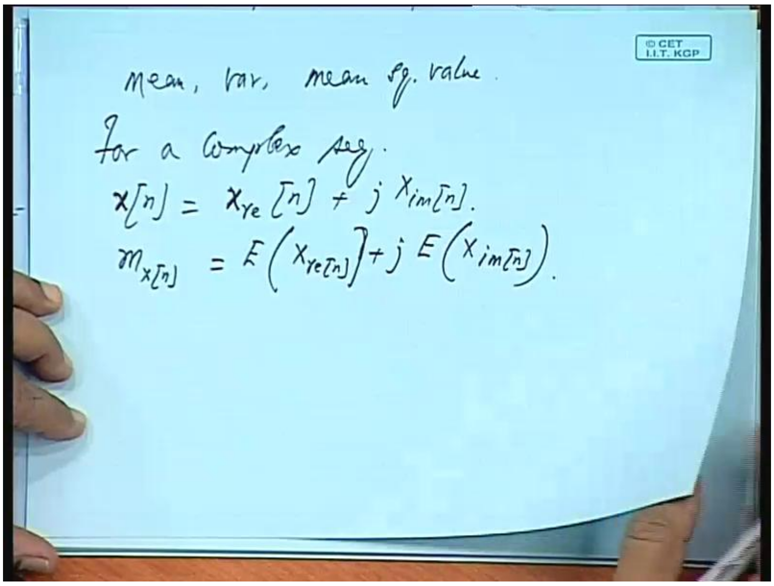

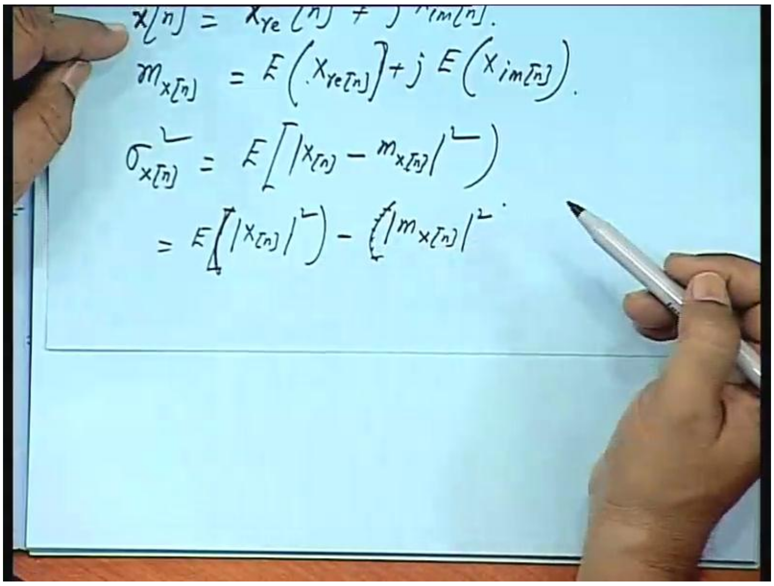

Now, mean variance and the mean squared value mean of the squared value, these are function of n is it not; that means, they can change with time. So, if a process does not have a constant value; that means, does not have a value which is independent of n, then it is not a stationary process, when it remains constant with respect to n, then it is a stationary process.

We call it strictly stationary process, if all other moments higher order moments are also remaining stationary; that means, they are constant they are not changing with n. So, say when we speak for example, in a speech signal when I make a sound for some time you will find it is having a particular average, particular variance. Then I change the word, I mean I change the sound I switch over to another sound, then again the mean changes.

So, it is a constantly changing mean, variance and other moments for a complex sequence we write x n x real n plus j times x imaginary line. So, mean of this will be expected value of x a real n plus j times expected value of x imaginary line, so you segregate the real and imaginary parts in terms of the means. Similarly, variance will find expected value of x n minus m x n and when you take magnitude square we will get expected value of x n magnitude square minus m x n magnitude square. Actually it is immaterial, you need not put a bracket you can expand and then see for yourself x, you put here and then see.

Sometimes we are interested in the statistical relation between two quantities in the time domain or; that means, at two different instance how are they related x at this moment, y at the other moment or x at this moment and x at the other moment. So, when we are talking about the two different instances x and y their cross correlations of different time span when they are with a same variable, then they are auto correlation.

So, these are two important quantities now depending on the nature of variation of these with respect to the time span the time range, if you can express them and if the statistical properties are definite, then we can take their Fourier transform. So, we can take transforms not of the original signal, but they are properties, original signals x and y it will be difficult to take transforms, but if you take their auto correlation or cross correlation, you can take their transforms. So, this is one very important property I do not think will be able to a take up any example, will take it up in tutorial classes in a little more detail about the transforms and other relations if time permits.

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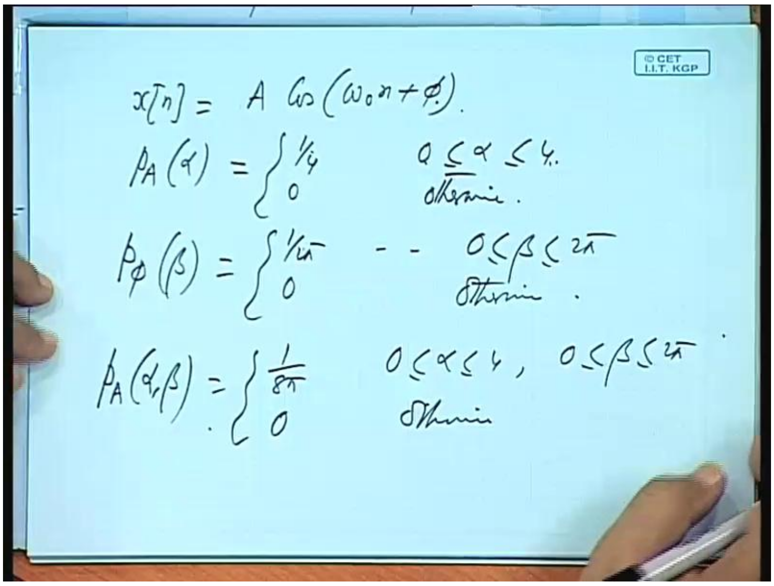

Let us take a simple example, A cosine omega naught n plus phi, the probability distribution of this amplitude, you say 1 by 4 and is equal to 0 otherwise. Similarly, for this phase phi we have 1 by 2 phi and 0, this is between beta is between 0 and 2 pi equal to 0 otherwise. That means, the phase difference can vary between 0 and 2 pi and the magnitude can vary between 0 and 4, so what would be the statistical properties of this.

So, p a that joint probability density function will be 1 by 4 in to 1 by 2 pi, so 1 by 8 pi in the range this 0 alpha 4, 0 beta 2 pi and equal to 0 otherwise it is a constant. So, you can calculate m x n x n x is this, so it is having dependence on two random variables a and phi.

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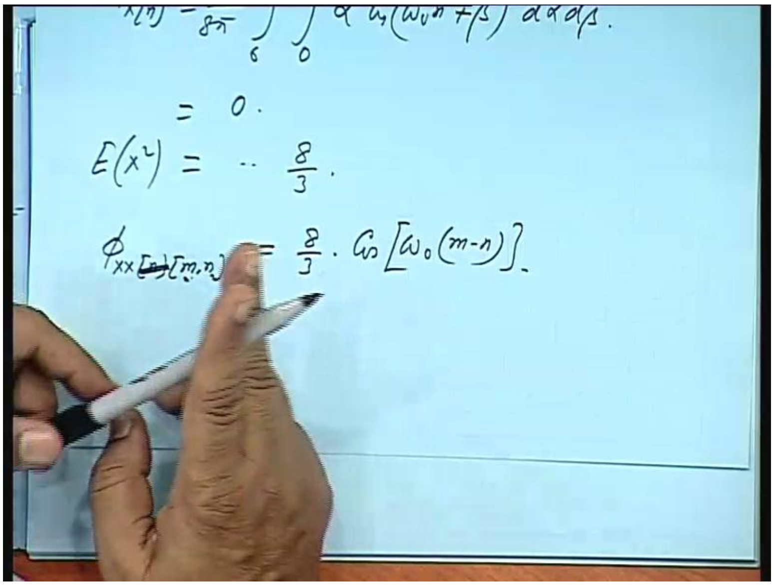

So, what is the average value of this what is the mean 1 by 8 pi 0 to 4, 0 to 2 pi then the variables a and phi will write as a running variable alpha and beta. So, alpha cosine omega naught n plus beta d alpha d beta, so if you complete this you I leave it as an exercise, you complete this integration it will be 0, two sinusoidal quantities separated by 2 pi. Similarly, you also compute expected value of x squared you work it out find out the value.

And then autocorrelation, if I take an autocorrelation of lag m and n there are two variables is it not a and phi magnitude and phase. So, there are two variables same may done, I am just giving the result check up whether you are getting this, so autocorrelation of two variables, means in both m and n you can take two dimensional transforms, autocorrelation of one dimension you can always take one dimensional transform. So, we will stop here for today, I will take up some problems in the tutorial class.

Thank you very much.