Digital Signal Processing
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Lecture - 27
Effects of Quantization

Good morning friends, we are discussing about Quantization of quantization noise and the last class we discussed the noise model.

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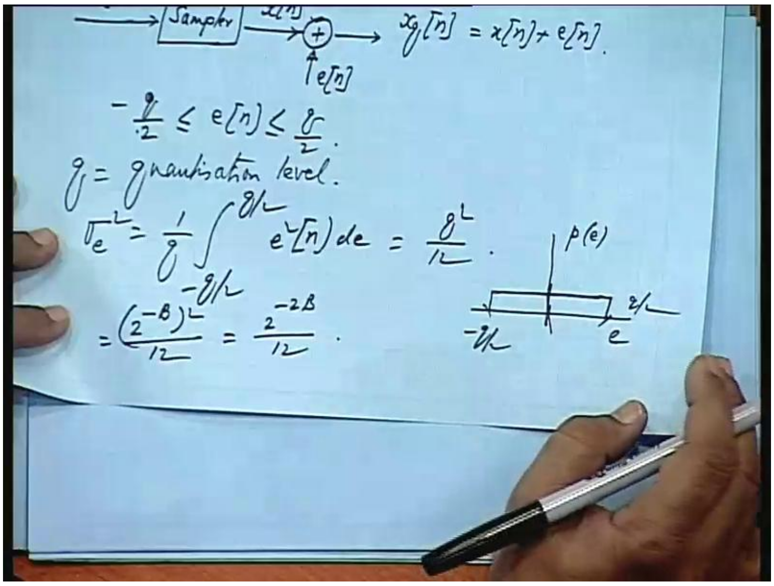

X t as continuous time signal passes through sampler, you get the sampled output x n, then this quantized. So, we added an error term, we get the quantized output x q n, which is x n plus e n, we also discussed about the limits of this error term, that should be minus q by 2 to plus q by 2. This one q is the quantization level, so in case of a random noise, this we are assuming to be random in nature, if it is round off. That means, you are truncating it, if you are truncating it then it will be the noise will be one sided, but if you are taking a value, so that means between minus q by 2 to plus q by 2, it will be quantization. Then, the variance is given by this is equal to q squared by 12, if you assume a really random noise, then it will be having uniform probability distribution. This is plus q by 2 this is minus q by 2 and that was 2 to the power minus b square by 12, so 2 to the power minus 2 b by 12, this is in case of rounding by the standard method.

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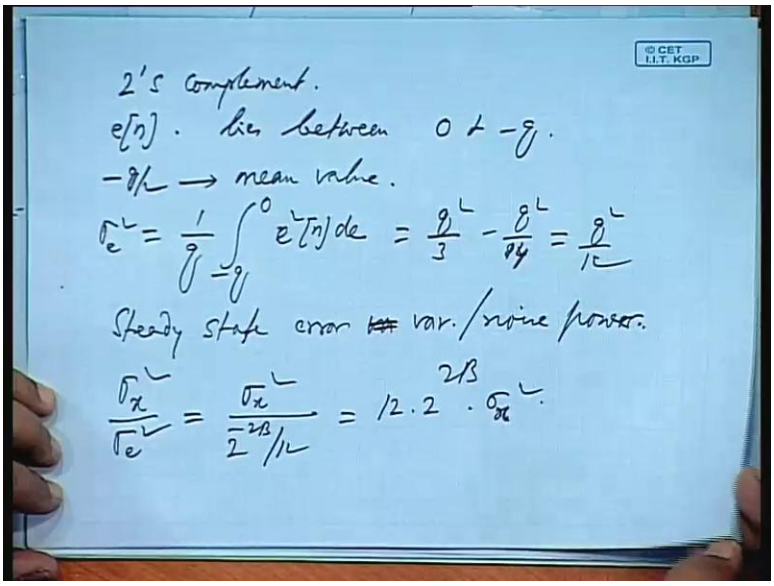

In 2's complement, if you have 2's complement, then e n lies between 0 and minus q having mean value minus q by 2, then again variance will remain same. So, that will give me q square by 3 minus q square by 4, 4 that gives me q square by 12, once again the same expression. This is the steady state error variance or noise power basically the variance represents a noise power, sigma x squared by sigma e square will be therefore, sigma x square by 2 to the power minus 2 b by 12, that is 12 into 2 to the power 2 b into sigma x square.

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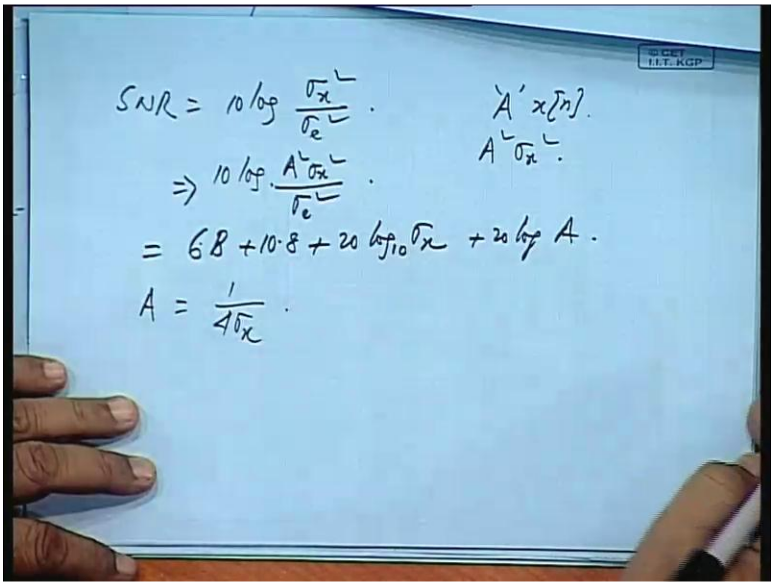

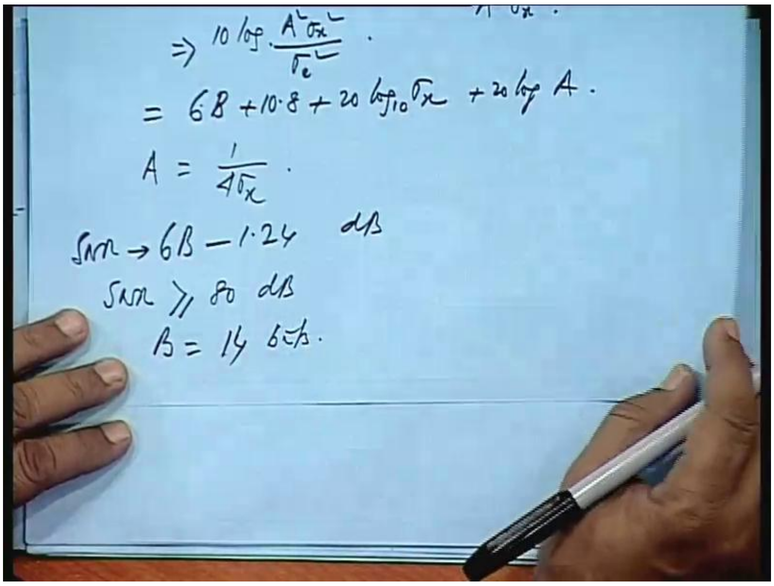

You also saw that SNR signal to noise ratio in terms of logarithmic ratio is 10 log sigma x squared by sigma e square, if you write this, if It is amplified by a factor A, that is A into x n is the output then variance will be a square sigma x square. So, this sigma x square will be replace by a square sigma x square, if there is an amplification by A. So, that gives me that approximately 6 B plus 10.8 plus 20 log of sigma x plus 20 log of A, if A is 1 by 4 sigma x.

So, that practically more than 99 percent of the cases, it will be within that limit 4 sigma x, if it is a normal distribution, then we obtained this SNR as 6 B minus 1.24 d B. If, I want to get an SNR of more than 80 d B for SNR greater than 80 d B, if this is specified how many bits will be required approximately 6 into 14 is that 84 minus 1.24, so many bits.

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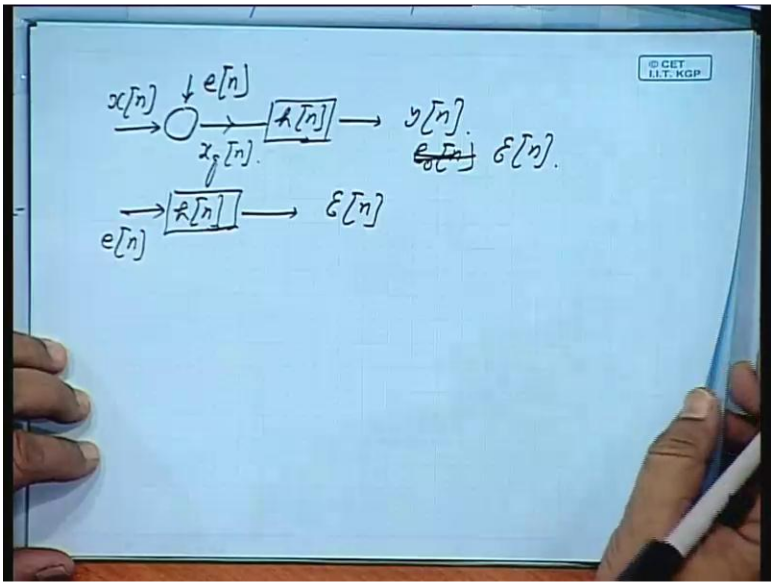

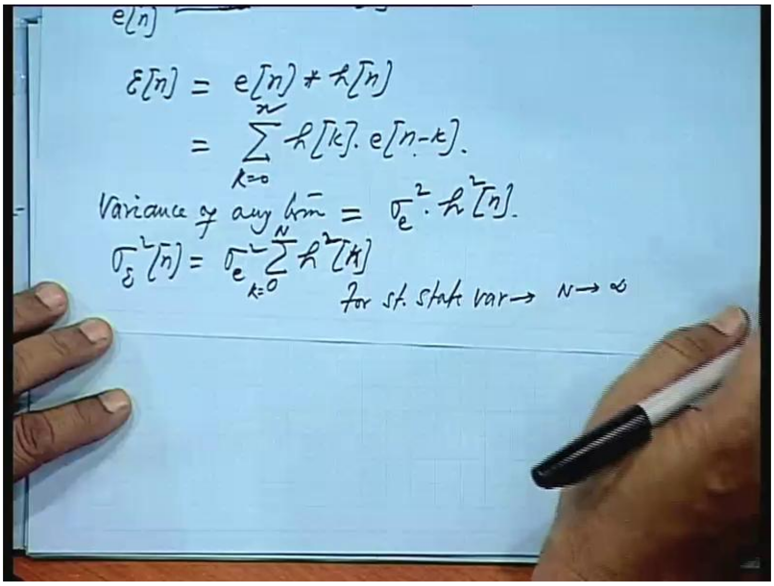

Now, let us see some other simple models, what will be the transfer function of the error model, the noise model, what will be the transfer function, the transfer function depends on the structure that you choose. Let us see how the errors coming, suppose this is the input, this is the h n, this is y n, this is the quantization error. So, what is the transfer function for the error, what is the output error by input error, how do you find out the transfer function of a multi variable input, suspend all the inputs, take one input at a time, as you apply super position theorem, you consider one source at a time. So, what will be the error, say e at output, if I call it, e output by e input or e n, so the transfer function will be again h n.

So, e n, h n will term this as Epsilon n, because e 0 will use for some other notations later on, so I will call it e n that is this is the error at the output stage. So output error can be written as e n convolution h n, h means h k, e n minus k, if it is a finite sequence. So, what will be the variance of any term of the above sum, what will be variance of any of the terms, if this is a random one, if signal e is random, it is quite in nature, 0 mean random process.

So, what will be the variance of any of the terms, it will be sigma e square into h square that particular term. So, will be sigma e square, summation h square n, n varying from, I will write h k, 0 to n, k equal to 0 to N. If you give an infinite sequence then 0 to infinity, so for steady state variance will put N tending to infinity.

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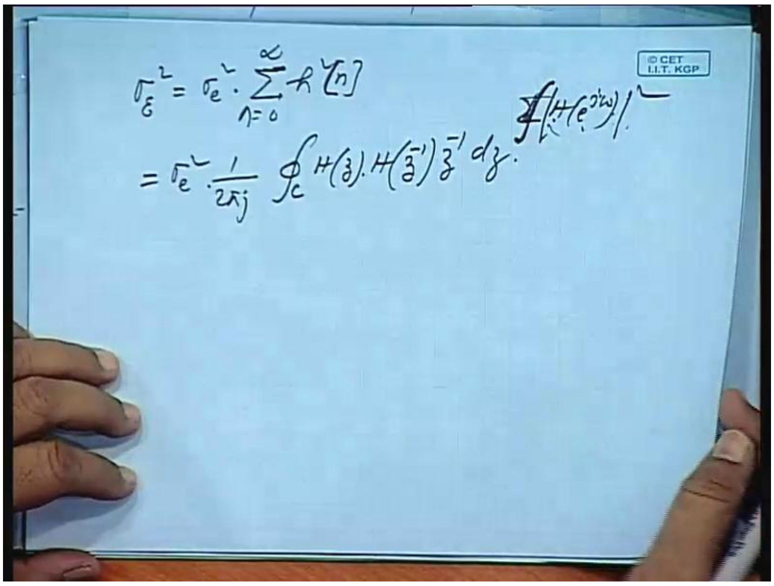

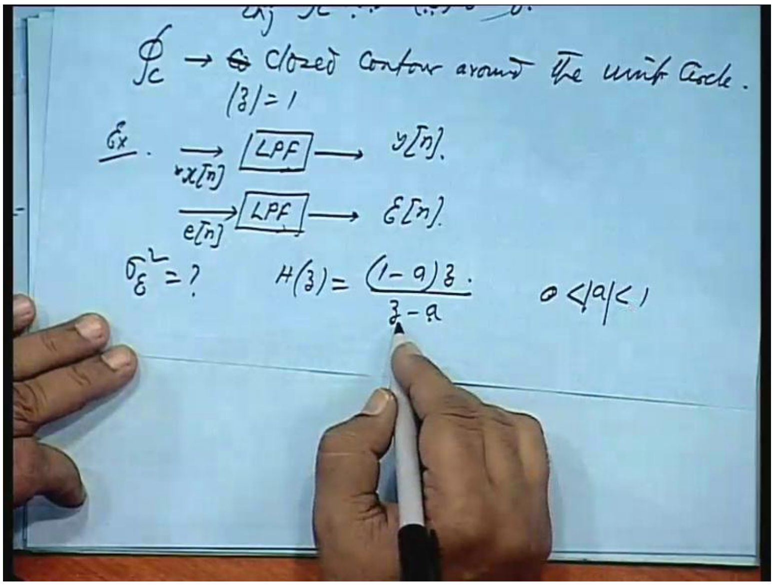

For sigma Epsilon squared equal to sigma e squared summation h square n, n varying from 0 to infinity and what is h square by pursuers theorem, it is the basically energy content. In that sequence h n, so it can be expressed in Z domain. It is basically sigma H e to the power j omega magnitude square also or integration this and if I integrate basically, this is H e to the power j omega into H e to the power minus j omega, if you take the complex conjugate product will be the magnitude square.

So, this can be written as this is again from (( )) theorem, this is the closed path, close contour that we take along the unit circle, around the unit circle. Summation of h square n will be H z, H z inverse z to the power minus 1, d z integration over this close path into

1 by 2 phi j. And this is what residue of H z, H z inverse within the unit circle, if there is any pole inside.

So, let us take up an example unit circle means z equal to 1, so let us take an example this is x n, this is a low pass filter and this is y n. Hence, it will be en for the error signal, also the same low pass filter same output noise e n, so you are ask to calculate sigma Epsilon square, if H z is given as 1 minus a into z by z minus a magnitude a is between 0 and 1. That means a is within the unit circle, so if you take the residue of this, basically this integration will be the residue of the pole at a, z is equal to a is a pole and a is within the unit circle. So, basically you have to calculate the residue add that.

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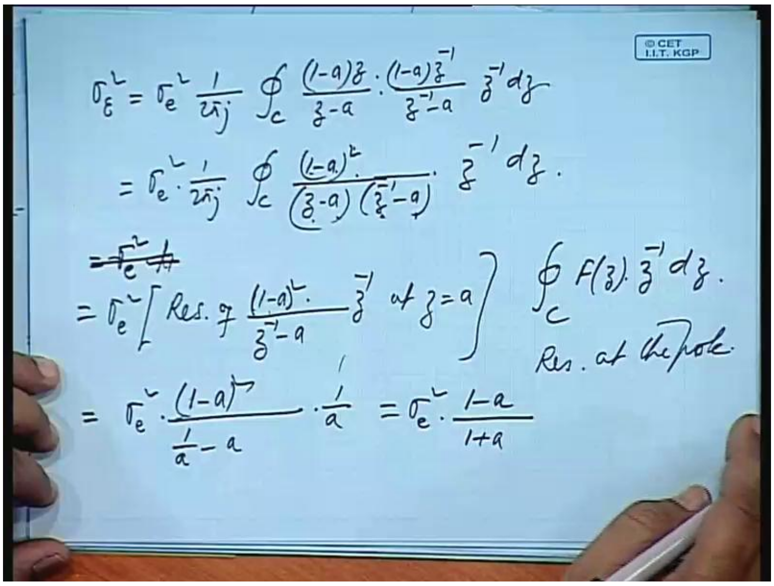

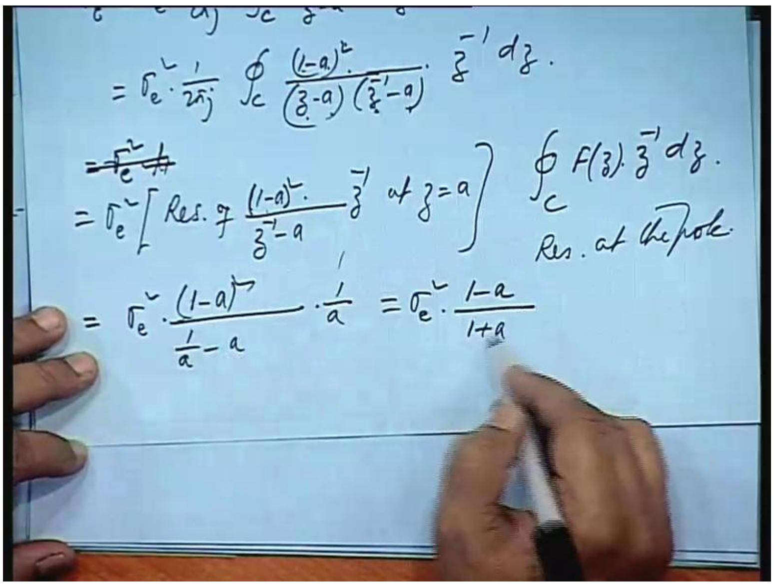

So, sigma E squared will be 1 by 2 phi j, 1 minus a into z by z minus a, you are taking H z, this h z inverse will be 1 minus a z inverse. By z inverse minus a into z inverse into d z, this is 1 minus a whole square, z into z inverse give me 1, by z minus a into z inverse minus a, z inverse d z. So, if you are having an integration some F z, z inverse d z, it is equal to the residue at the pole within the unit circle and what are the poles at z is equal to a, z is equal to 1 by a, 1 by a is outside the unit circle. So, you have to calculate the residue only for z is equal to a only for this pole. So equal to sigma e squared into this whole integration will be sigma e square into residue of this function multiplied by that pole removed is not, you multiply by that.

So, this is the function residue of 1 minus a square by z inverse minus a into z inverse, to be evaluated at z is equal to a, residue of this at z is equal to a, other residue is 0, because it is outside the unit circle, I am not writing that as 0. So, that is sigma e squared z is equal to a, so it becomes 1 minus, if I put z is equal to a, how much is this, 1 minus a square, 1 by a minus a into 1 by a.

So, sigma e squared, this will be 1 minus a square, 1 minus a square will get cancelled, 1 minus a whole square, so this will be 1 minus a by 1 plus a, does it come all right.

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It is, I have made a mistake here itself, it was 1 minus a whole square, so 1 minus a and 1 minus a will go, so 1 minus a by 1 plus a.

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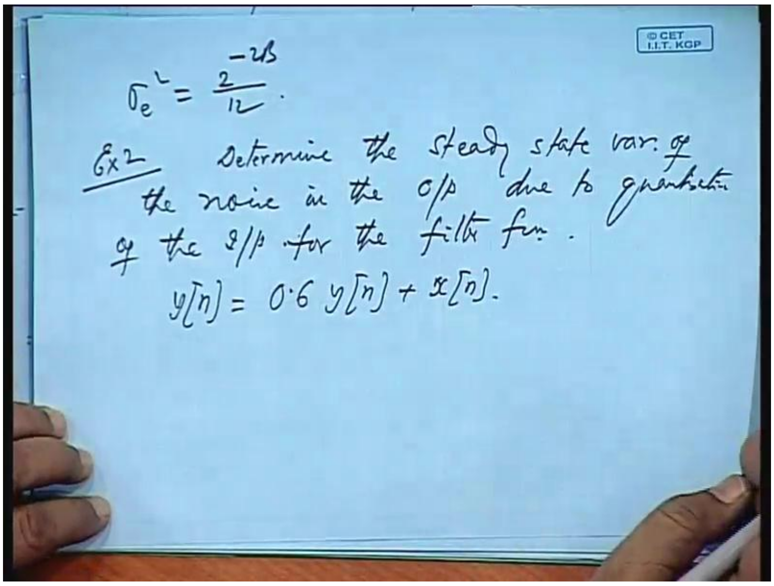

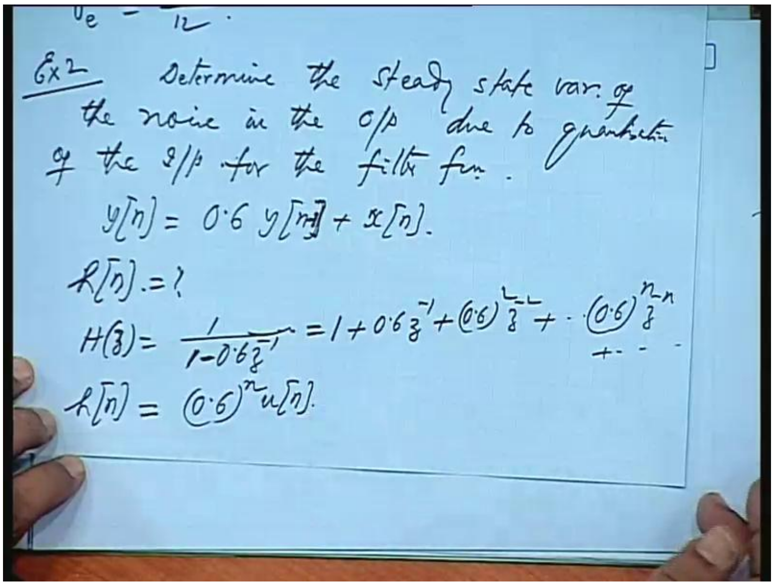

And sigma, e square is 2 to the power minus 2 b by 12, so this sigma e square, we can substitute that, so that will be the error variance the output sigma Epsilon e square. Another example, let us take determine the steady state variance of the noise in the output, due to quantization of the input. Further filter function, which is given by y n is equal to 0.6 y n plus x n, it is an autoregressive process is IIR. What is h n, how do you calculate h n, what is H z, if I bring it on this side, is it alright.

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Y n minus 01, obviously on this side it has to be 1, n minus 1, so this is H z, I can always write this as, it is the denominator having minus sign. So, it will be 1 plus 0.6 z inverse plus 0.6 squared z to the power minus 2, it is a G P series, basically which has been like this, 0.6 to the power n, z to the power minus n and so on up to infinity, so what will be corresponding h n.

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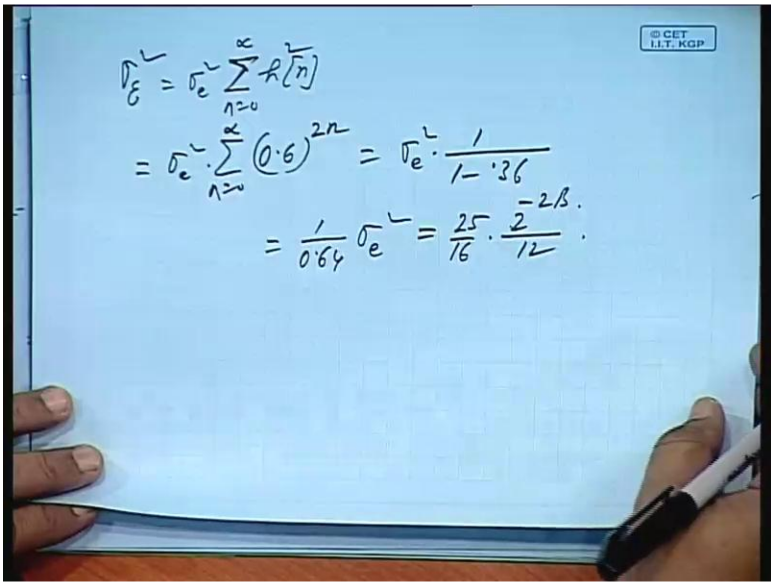

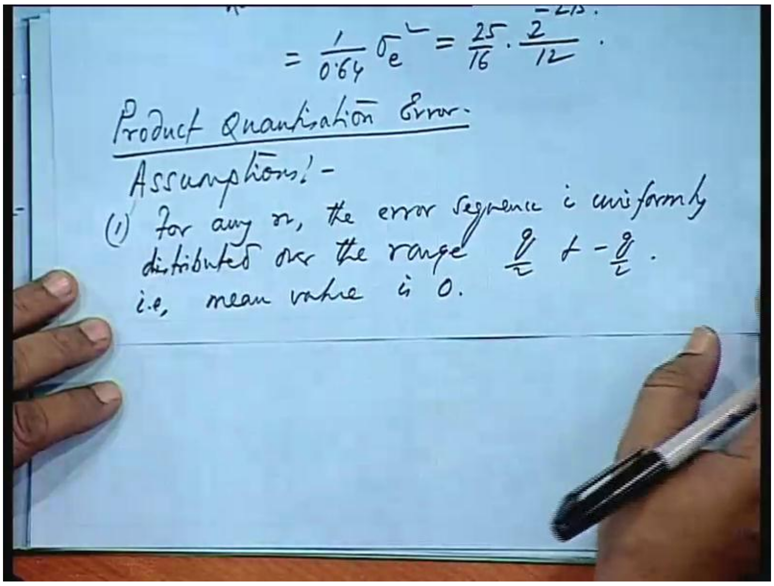

So, 0.6 to the power n, u n, so if h n is this, what will be sigma E squared, so sigma E squared summation, what is h n 0.6 to the power n. So, to the power 2 n, if I square it will be to the power 2 n, so sigma e square into n 10 into ranging from 0 to infinity, so what is the G P sum

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0.6 square, 0.36. So 0.36, so 1 by 0.64 into sigma e squared is equal to 25 by 16 and sigma e squared is 2 to 2 to the power minus 2 B by 12. Now, product quantization error, so far we are considering only input and output error, quantization at the input side. Correspondingly, what would the effect on the output, a product quantization error, this is the error, which you get while multiplying. So, there are certain assumptions very similar to the previous ones, they are for any n, the error sequence is uniformly distributed over the range q by 2 and minus q by 2 and that means; the mean value is 0, the error is uniformly distributed around that 0 values.

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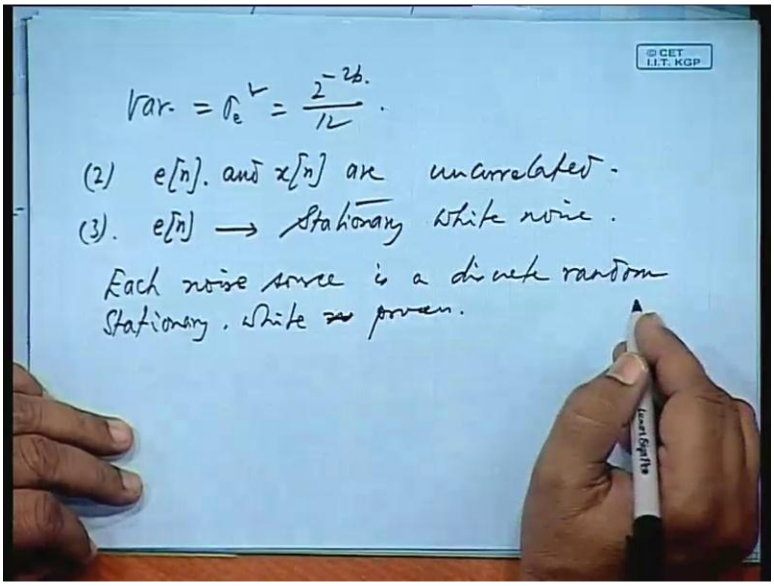

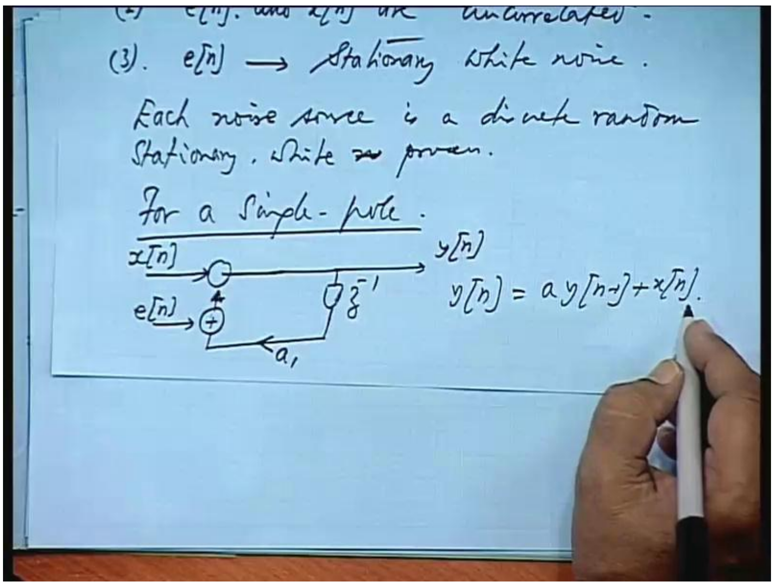

And variance will be 2 to the power minus 2 B by 12, where the same logic, then the error sequence has got no bearing with the input. So, error sequence e n and x n are uncorrelated, again error sequence is random and it is a white noise process. It is stationary, when do you call a sequence stationary, when the statistics that is the first moment, second moment, they are all constant.

Normally, if the first moment and second moment that is mean and variance a more or less constant, we call it a white sense stationary process and a strictly stationary process means when all other moments third order and forth order and so on. All other moments remain constant; it is a stationary white noise, so each noise source is a discrete stationary white random process.

For a single pole, let us consider, first a single pole structure and then a simple pole 0 structure and then a second order structure, what will be the effect of different structures, that you choose on the total noise statistics. For a single pole, we have a structure like this, so this quantity is multiplied by a 1 and then it is quantized. So, it is a product, which is quantized due to which there is an error, so our structure is once again, y n is a times y n minus 1 plus x n.

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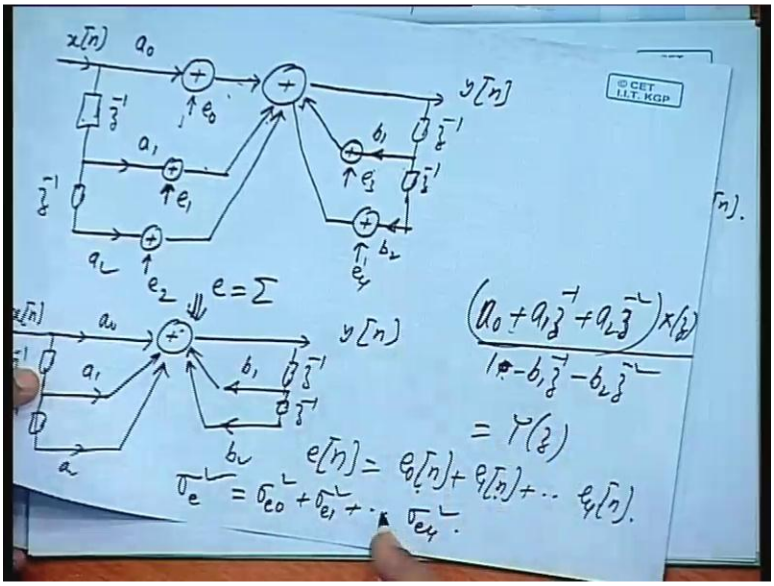

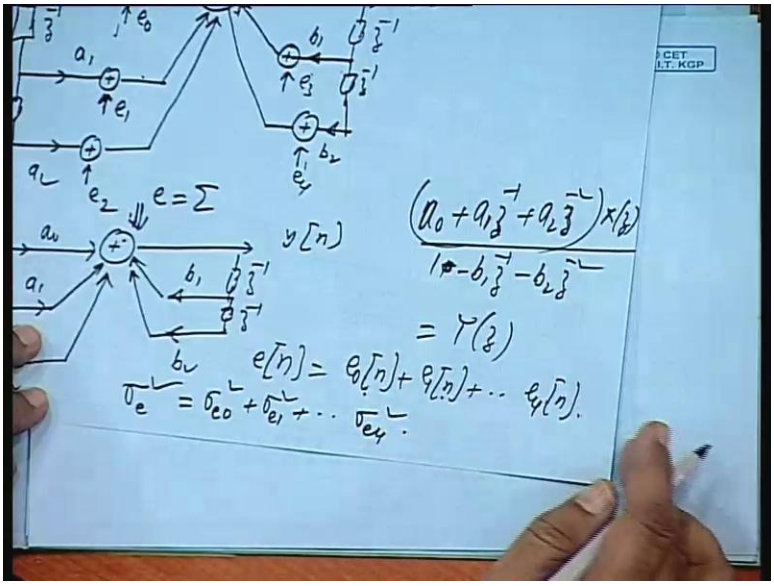

If, you have a quadratic structure, say x n both pole and 0, a 1, a 2, so there is an error here, error here, error here, so we have got e 0, e 1, e 2, e 3, e 4 errors coming at different poles. We are quantizing the products, and hence there will be errors at these places. Now what will be the error model, for the error, what will be the model, this is x n, z inverse a 1, again z inverse a 2. They are added and this total error e 0, e 1, e 2, e 3, e 4; they are all add it added here, this error goes directly contributes here.

So, as if all the errors are applied at this point, we are adding for this kind of a structure, what is this structure a 0, this is the forward structure plus a 1, z inverse plus a 2, z to the power minus 2. They are all multiplied by x z divided by 1 plus b 1, actually if I do not put any sign of this then it should be minus e 2 z to the power minus 2, this is equal to y z. So, for this kind of a bi quadratic structure, we can always put error at this point, the summation of all the errors together. So, e n is nothing but e 0 n plus e 1 n, e 0 n, e 1 n, e 2 n, e 3 n, e 4 n is equal to sigma e 0 squared plus sigma e 1 square, the variance of this error will be they are all independent random variables. So, the variance of this will be variance of each one of them added together, some of the variances.

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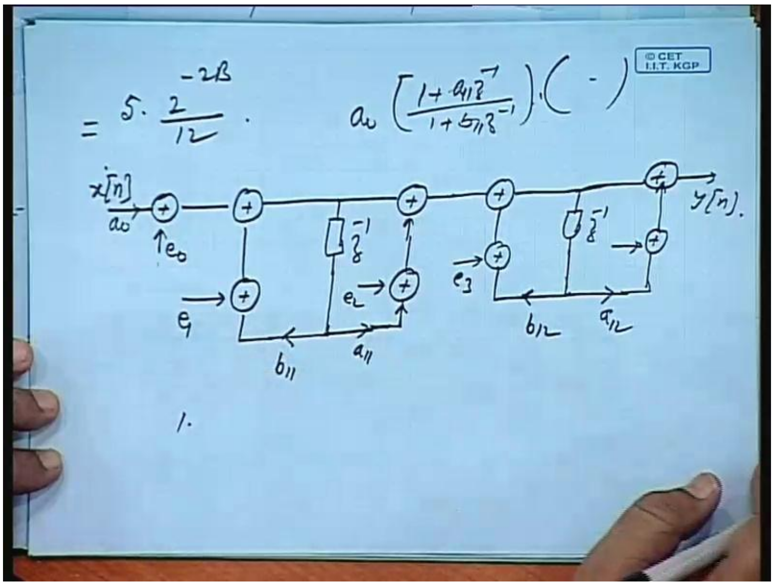

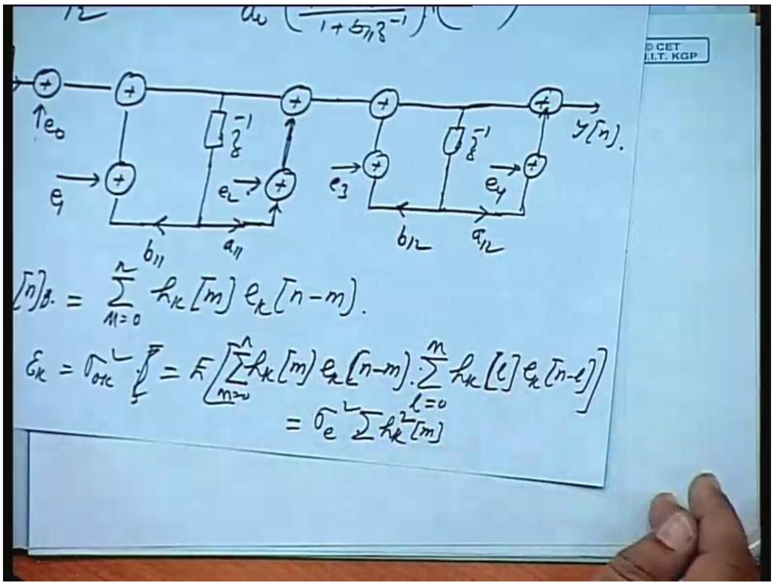

If, they are independent noises, then variance of each one of them you compute, add them together, that will be the total variance. And since, all of them are identical ranging between minus q by 2 to plus q by 2, so it will be 1, 2, 3, 4, 5, so 5 times 2 to the power minus 2 b by 12, this is the error variance. Now suppose the second order filter, we can resolve into two linear functions, both for numerator and denominator. Then we can have first order filters on both sides, we can put them in cascade, then what happens b 11 and a 11 are the corresponding coefficients, it is like this, it is basically, it has been put. The earlier structure, I could have put the z inverses on this side will reduce the number of delay elements. So, this is what has been done here, suppose we have structure like this, then what happens, this is first multiplied by a 0.

This factor, I have taken out one constant a 0 and then how, after that I have factored into 1 plus a 1 z inverse by 1 plus b 1 z inverse, something like this, similarly this one, it is a 11 and b 11, a 12 and b 12 into a constant a 0. So, because I have multiplied by a 0, so there is an error e 0, then this b 11, after multiplication by b 11 this variable, that is again quantized. So, this is the error e 1, similarly at this 0, e 2, so these are the product contribution.

So, at every loop, whenever you are feeding back, there will be an error associated with a product quantization. Now here any error E k will be dependent on that particular transfer function at the kth level, suppose we are considering e 2. Then what is the transfer function of e 2, against the output or error, here it is only this transfer function coming in, and this does not take into account this one.

This error is the forward one, so this is independent of this whereas, when we compute this error, the transfer function corresponding to this error, this loop is coming into picture. So, you have to evaluate the transfer function for corresponding error, so that will give you the corresponding response h k. So, h k m, e k n minus m, m varying from 0 to n and variance E k will be sigma 0 k square.

Expectation of h k m, e k n minus m, summation m equal to 0 to n into summation h k l, e k n minus l, l varying from 0 to m, which will be giving me sigma e squared into sigma h k m square. So, at the kth level, for the kth signal, if you calculate the variance, it will be sigma h k square into sigma e square. So, for each 1 of them e 1, e 2, e 3, e 4 and e 0, you calculate the corresponding h k's, this k means that particular transfer function, that particular h n series and then that will be giving me total variance.

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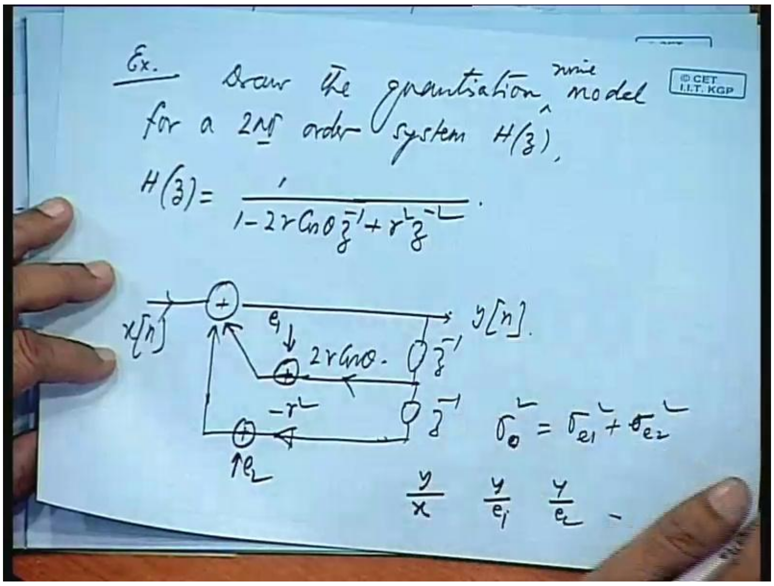

Let us take another example, draw the quantization noise model, for a second order system h z, given like this H z is 1 minus 2 r cos theta z inverse plus r square z to the power minus 2. So, what would be the structure, this is the one, that we discussed in our earlier parameter quantization remember, so z inverse 2 r cos theta. The other one is minus r square, z to the power minus 1.

Now, in this model, since this is the product, we can put an error signal here, say e 1, similarly here e 2, e 1 and e 2 and e 1 and e 2 they are added here. Both of them are having similar transfer function, because they are finally, getting added here e 1 plus e 2, so what will be the variance, there are 2 independent noise. So, sigma if I call it sigma 0 square will be sigma e 1 square plus sigma e 2 square.

Both of them are observing the same transfer function H z, check for yourself, the noise is added here, so it is added with x n, so x n e 1 and e 2 they are all added together. So, you suspend any 1 of them, I mean keep any 1 of them and then drop others with y n, it will be the same relation. So, y by x, y by e 1, y by e 2 all of them will give you the same transfer function H z.

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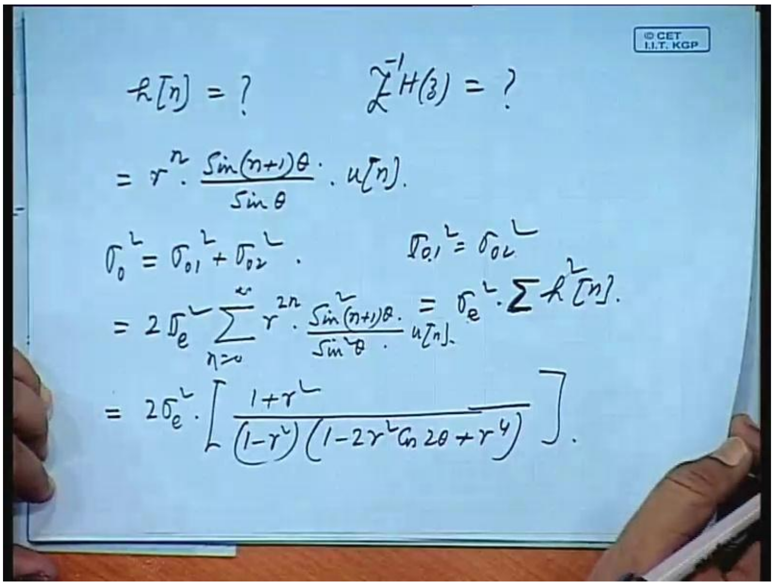

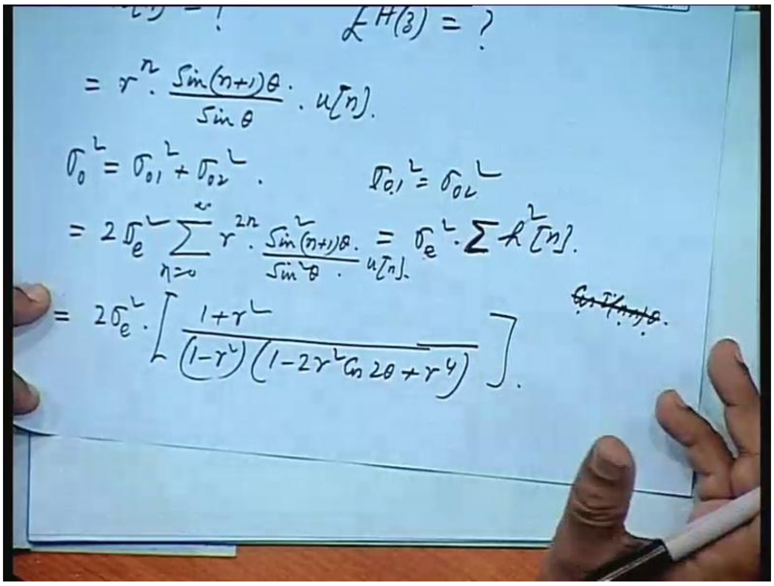

Now, before we do that, what would be h n, that is what is the inverse of H z inverse, so what is the inverse of this, I leave it to you as an exercise, I will give you only the answer, you verify for yourself. H z inverse will be r to the power n sin n plus 1 theta by sin theta u n, best thing would be to take this as a function, find out it is z transform, converts you prove. So, determine the z transform of this function, try this at home, you will find it will be coming as this one.

So, sigma 0 squared, which was sigma 0 1 squared plus sigma 0 2 squared and sigma 0 1 squared is same as sigma 0 2 squared, both are identical. They are having the same transfer functions, so this would be 2 into how much is this, both of them will be equal to sigma e square into sigma e square n and h n is this much. So, square of this, so it will be 2 into sigma e square into sigma h square n this sigma h square n, where h n is this much.

So, it will be sigma square of this is r 2 n sin squared n plus 1 theta by sin square theta into u n, n varying from 0 to infinity. So, this sum once again, I leave it as an exercise, you try will be 2 into sigma e squared, this sum check whether you are getting 1 plus r square be 1 minus r squared into 1 minus 2 r square cos 2 theta plus r to the power 4, r to the power 2 n sin squared n plus 1 theta. You can sin square theta, you can always write in terms of what cosine 2 into n plus 1 theta.

So, express that and then make a G P series, if you write in terms of cosine 2 theta, then write in the exponential form e to the power j theta plus e to the or minus j 2 theta divided by 2. That gives you cos 2 theta, so write in terms of e to the power j, twice n theta or twice n plus 1 theta.

So, you will get a G P series r to the power 2 n and cosine that will give you, e to the power j 2 n theta. So, take the sum you will be getting finally, this kind of a series and that will be a convergent series, so that will give you the output like this. So, in the next class, will take up coefficient quantization error a co coefficient quantization, I have already discussed a little bit more will take up and then will take some more problems, varieties of numerical problems. And will also, discuss about limit cycle oscillation, how oscillations may take place, because of the quantization.

Thank you very much.