Digital Signal Processing

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Lecture - 22
FFT & Computer Aided
Designs of Filters

We shall be discussing today fast Fourier transform. Basically, it is the computation of discrete Fourier transform in a, in an algorithm where the computational pardon will be reduced as we discuss last time. This fast Fourier transform can be implemented for design of a an FIR filter.

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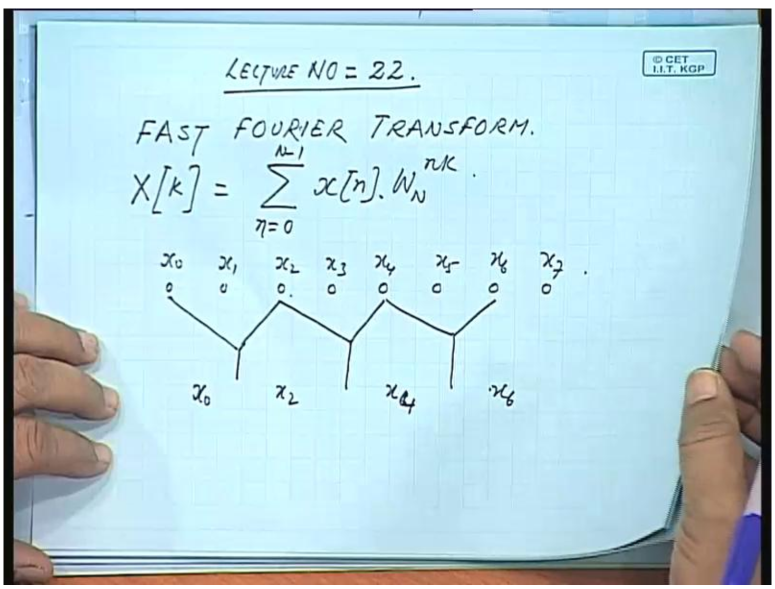

So, we will see how that can be done sorry, how that can be run at the end. So, let us see the discrete Fourier transform; the expression for discrete Fourier transform and then how this faster algorithm can be implemented, we have the expression for X k as x n, W N, n k okay. n varying from 0 to n minus 1, all right. Now, let us have say the points like this, let us take eight points; 1, 2, 3, 4, 5, 6, 7, 8. Let us pick up the even terms and odd terms and group them together. So, this is x 0, x 1, x 2, x 3, x 4, x 5, x 6, x 7. If I bring these, the order will be x 0, x 2, x 6 sorry, x 4 and x 6. This will be the even terms.

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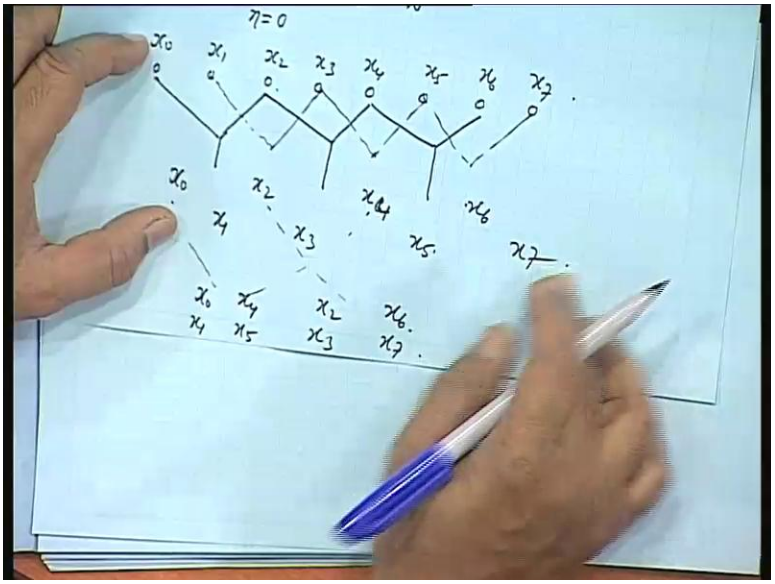

If you pick up the odd terms, so I will get x 1, x 3, x 5, x 7 all right. We have made two groups again out of these four points. If you pick up alternate terms then the ordering will be x 0, x 4, x 2 and x 6, all right. x 0 see in this sequence; x 0 then alternate will be x 4, similarly next one will be x 2 and x 6. Similarly, from the odd sequence if we again segregate in to alternate terms x 1, x 5, x 3, x 7 this should be the terms.

Now after breaking it in to two groups of two each, two elements each we cannot go down further. So, this will be our smallest element. Now, let us see how this helps, this kind of breaking down helps once again.

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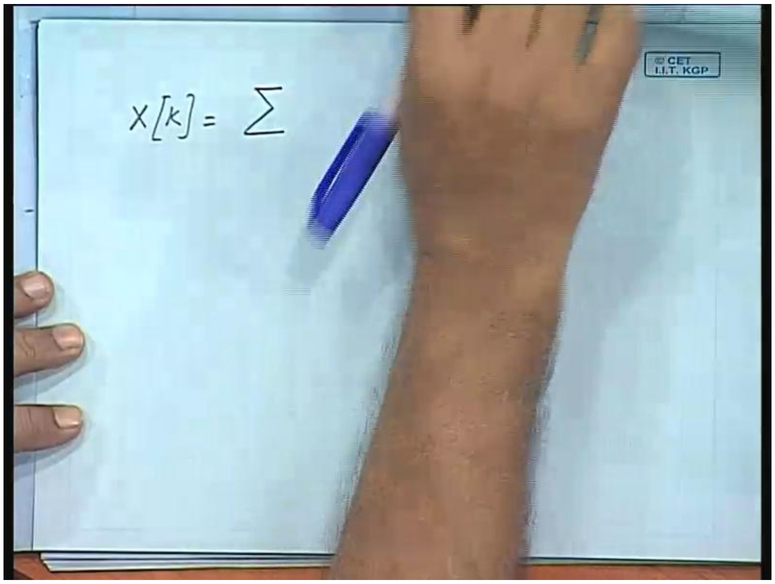

I write X k as if I

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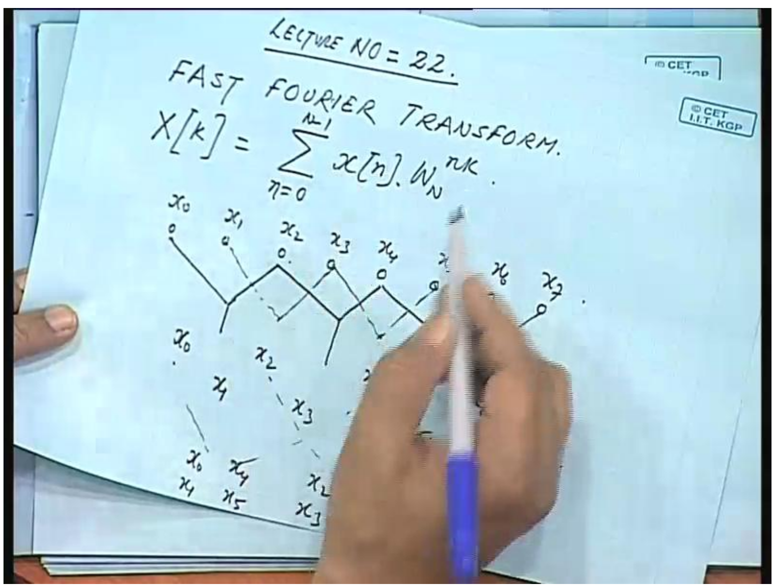

Now break it up in to even and odd terms.

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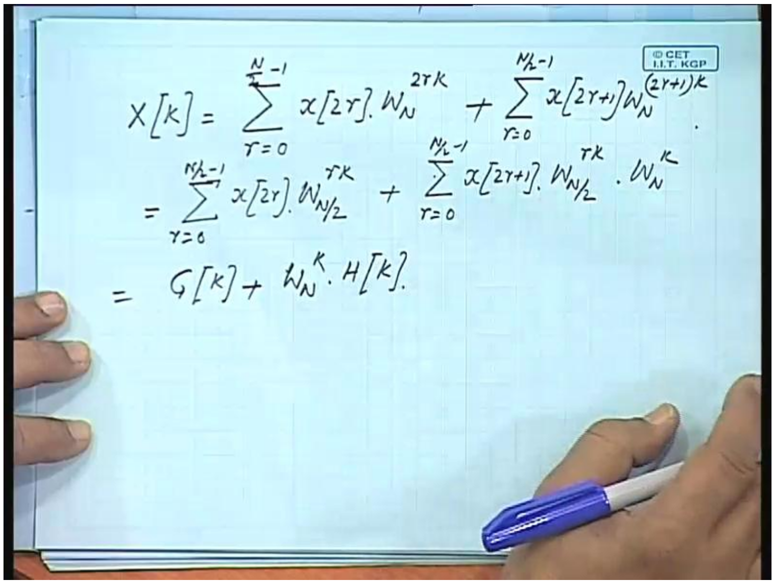

It will be say, x 2 r W N 2 r k plus x 2 r plus one odd terms, W N 2 r plus 1 in to k, do you all agree where r varies from 0 to N by 2 minus 1, okay. So, this can be written as x 2 r W N. Now, this two can be brought down here plus, x 2 r plus 1 W N by 2 r k in to W N k; again r varying from 0 to N by 2 minus 1, r varying by 0 to N by 2 minus 1, okay.

Now, this I write as G k plus this is similar to G k, all right. And multiplied by W N k, I call it H k, okay. So, it is basically half the number of points, if we have an eight point sequence then this is basically four point sequence, computation here also it is four point sequence. If we show it diagrammatically, it will look like this.

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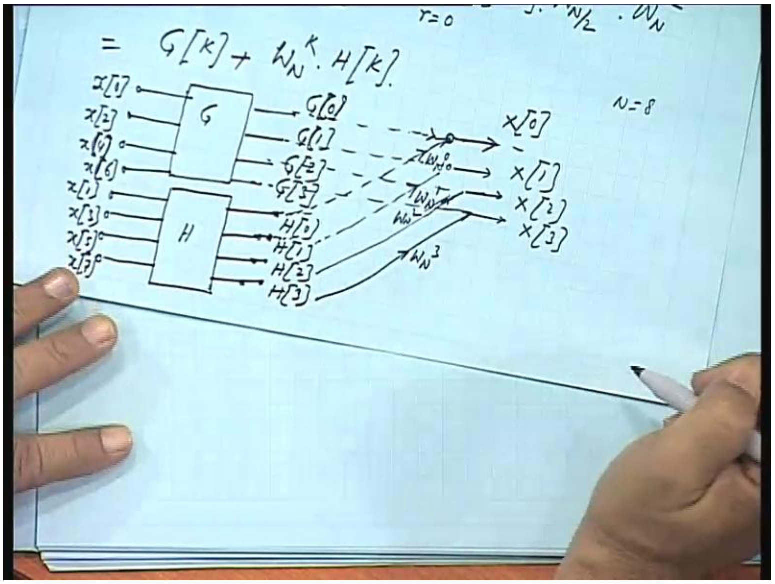

I have given x 0 then x 2, x 4 and x 6, these are the four points that will generate G, okay. And I get output as G 0, G 1 let me use the same bracket G 2 and G 3, okay. And similarly here H which is having inputs like x 1, x 3, x 5, and x 7, I will get H 0, H 1, H 2, H 3, all right. Now, I am combining G and H G and H in this mode, that is G plus H multiplied by this quantity will give me corresponding x scale.

So, let us do that. So, it will looks like this; G plus H when k is equal to 0, W N 0 is 1. So, H 0 plus G 0, G 0 plus H 0, this is W N 0. So, this is multiplied by W N 0 and in the signal flow graph you use this as addition okay when they are going inward. So, this will be X 0. Similarly, H 1, what should be X 1 like? It G 1 plus W N 1 in to H 1, so this has to be multiplied by W N 1 okay, N is eight in this example. We have taken four point sequence in each of these total, it is an eight point sequence.

So, similarly H 2 has to be multiplied by W N 2 okay and then added with G 2, that will give me X 2, H 3 is to be multiplied by W N 3 then added with G 3; that will give me X 3 its fine, what about X 4, X 5, X 6, X 7? Now you see, W N

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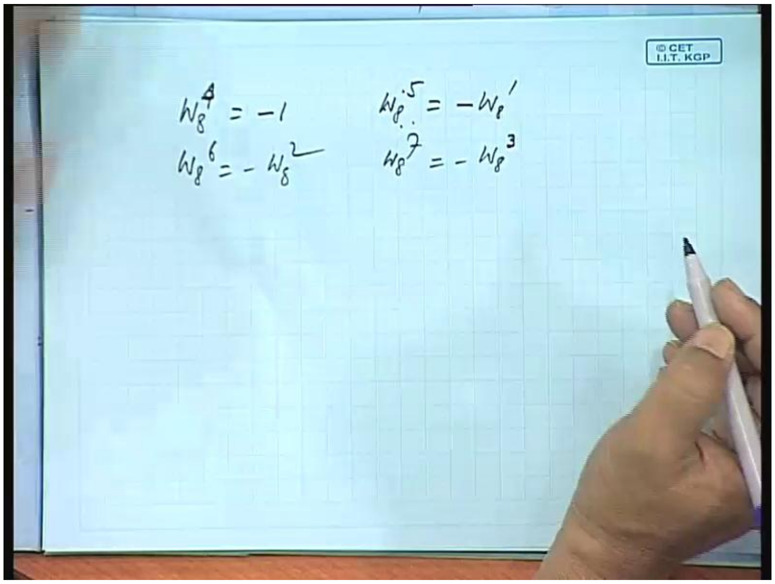

Say W 8, N is 8 in our case .W 8, 5 W 8, 4 how much is it? Minus 1? W 8, 5 minus W 9, 1 all right. I can write this as W 8, 4 in to W 8, 1. So, all of them after 4 will be negative of the previous values, so W 8, 6 is minus W 8, 2 and W 8, 7 is minus of W 8, 3 is that all right?

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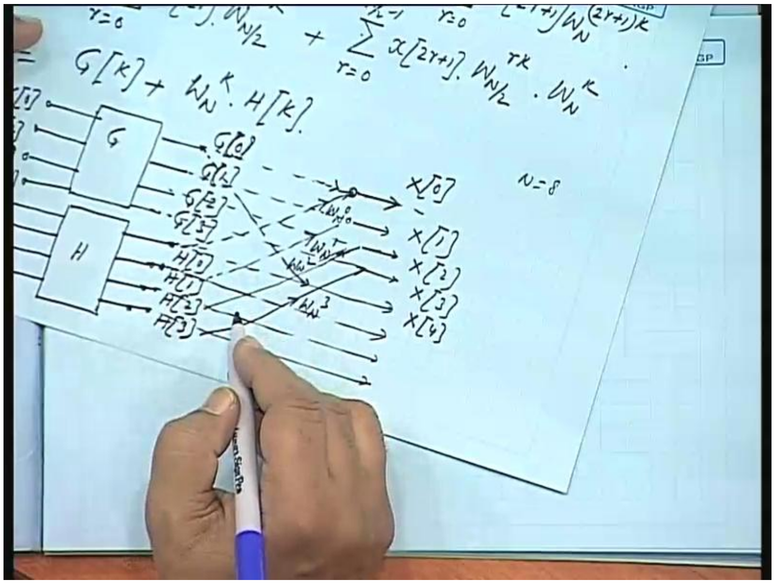

Therefore I can have the same combinations mind you, here here for computation of k; there is nothing like k is not beyond 4, k cannot be computed beyond 4 but you see, here when I put different values of r, it is basically G, G 0 minus H 0, that gives me a negative sign, that gives me X 4, all right. Because, why because G k is also periodic. It is a four point sequence.

So, every four point after, every four points it will be periodic. So, G 5 is same as G 1, G 6 is same as G 2 and so on. So, it is only this multiplier which you have negative signs sorry, so here G 0. If I club with H 0, what should be the sign? X 4 will be G 0 minus X 0, okay.

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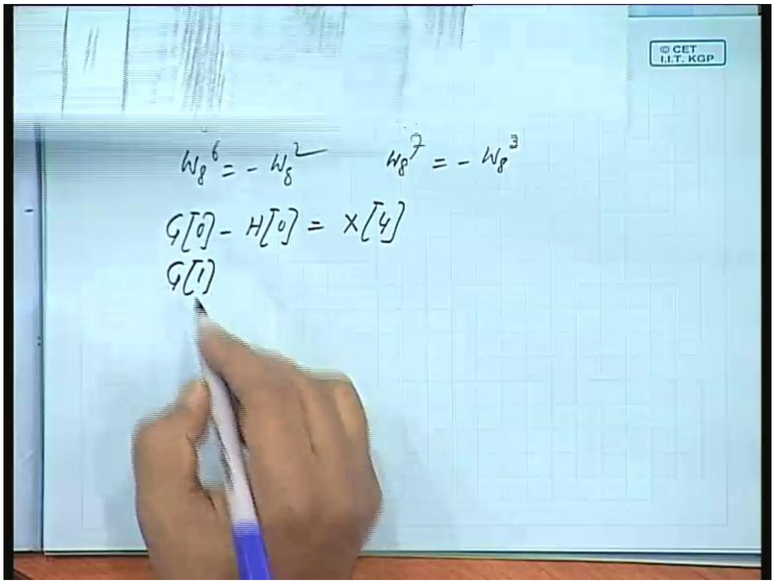

So, G 0 minus H 0 will be X 4. Similarly, X 5 will be G 1.

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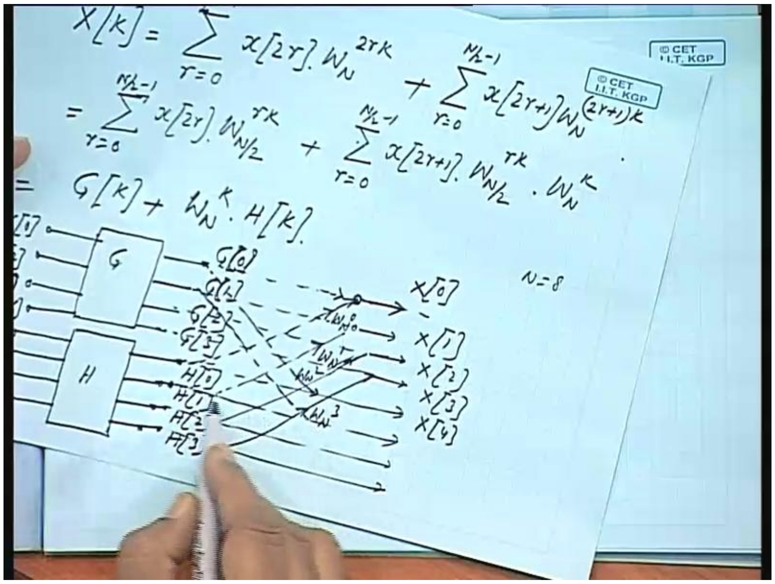

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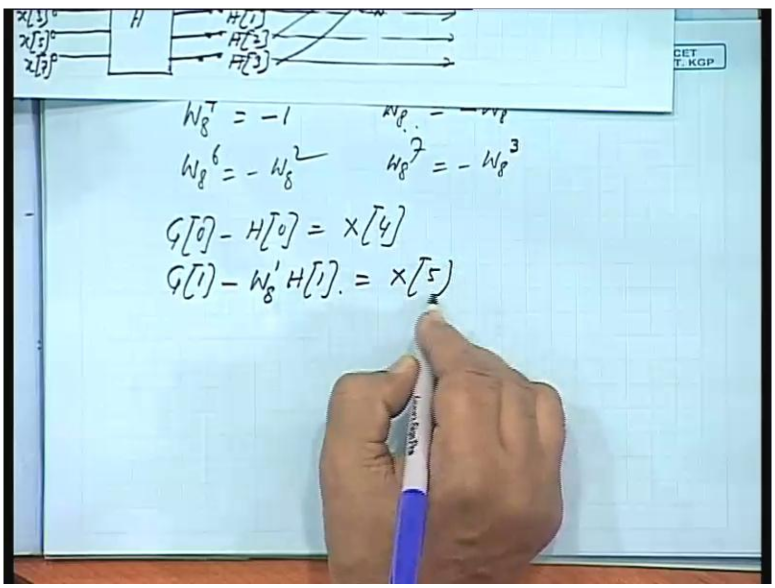

Minus minus W 9, 1 in to H 1, okay.

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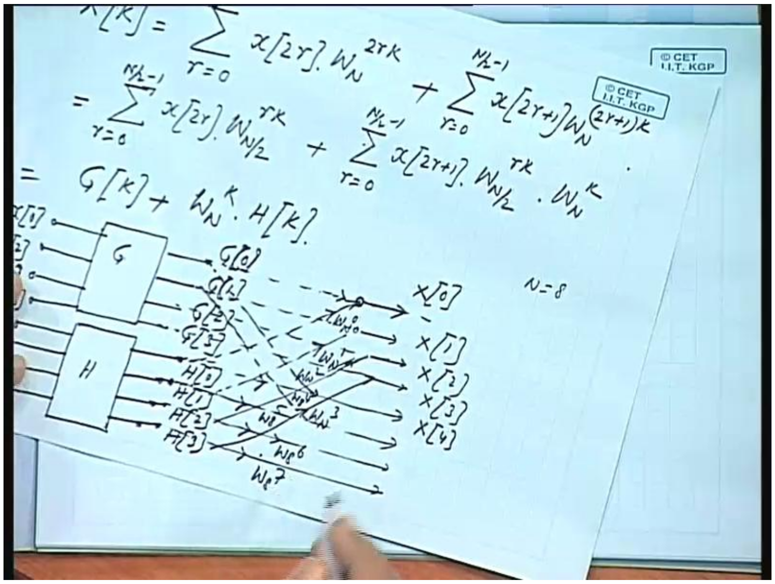

So when it is coming downward, there will be a negative sign all right. So, I will put a negative sign in this; that is to be multiplied with H the, when we are considering the bottom layer all right. H 0 multiplied by minus 1, H 1 multiplied by minus 1 in to W N 1 that is W N 5. So, this is nothing but W N, W 8, 4, W 8, 5. This is W 8, 6 and this is W 8, 7. So, this looks a little jumbled up, I will

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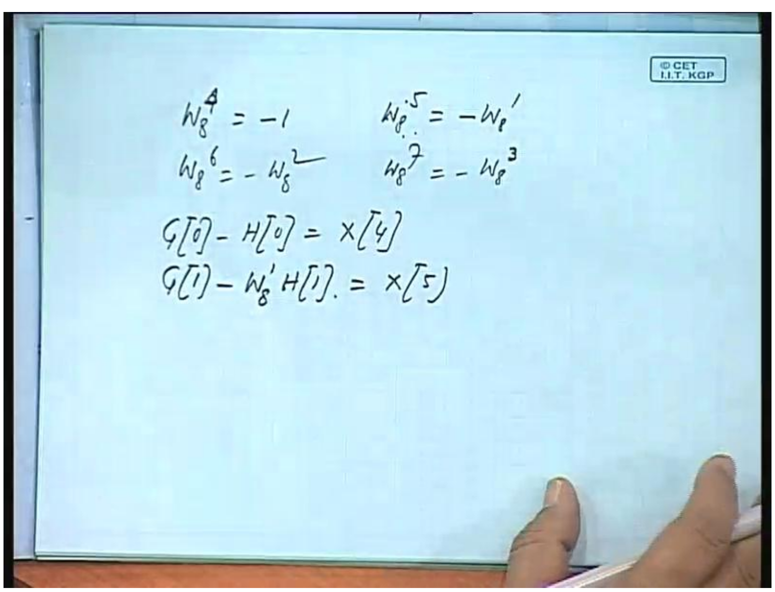

draw it a little more neatly, only this part. So, it will be looking like this.

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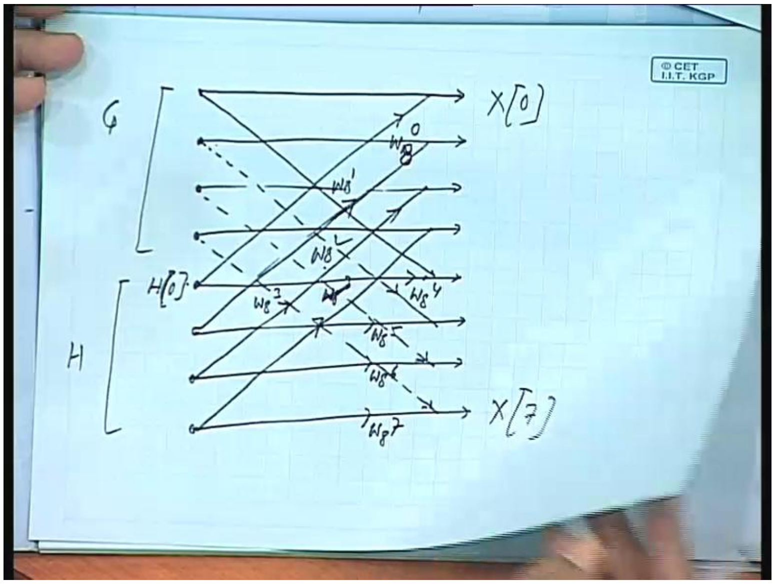

0, 1, 2, 3, 4. So, this is W N 0, W 8, 0. This is W 8, 1. Similarly, this is W 8, 2 and this is W 8, 3 and this one will be, this is H 0, this is. So, this is W 8, 3 I will write here. So, that it is not confuse, this is W 8, 4. This will be W 8, 5. This will be W 8, 6 and this will be W 8, 7 I think you can all complete this structure, okay.

So, these are the G's and these are the H's, G 0, G 1, G 2, G 3, H 0, H 1, H 2, H 3. So, they are to be added like this. I will get X 0, X 1 up to X 7. See, these are four point butterflies, we call them butterflies by the same logic. G and H can also be computed from, say two point sequences. So, if we break it down further, it will be.

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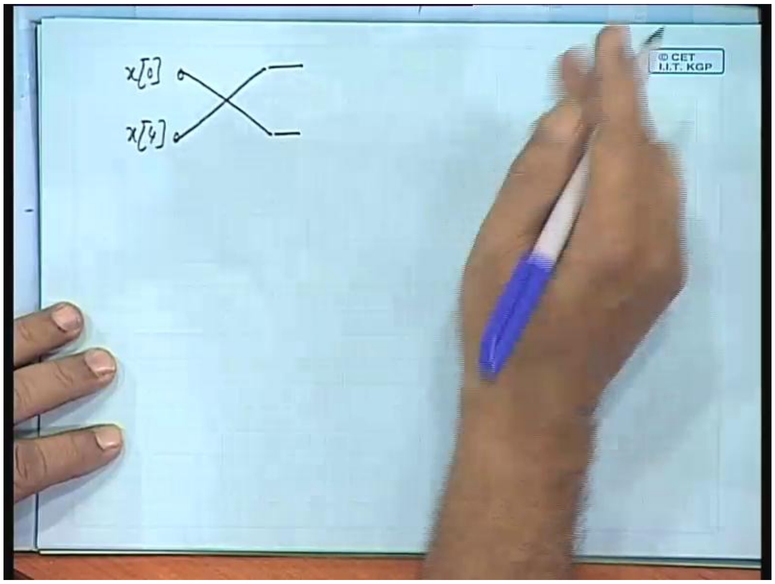

x 0 and just some time back we have seen has to be coupled with x 4 all right. And then what would be these values by the same logic as you have done earlier?

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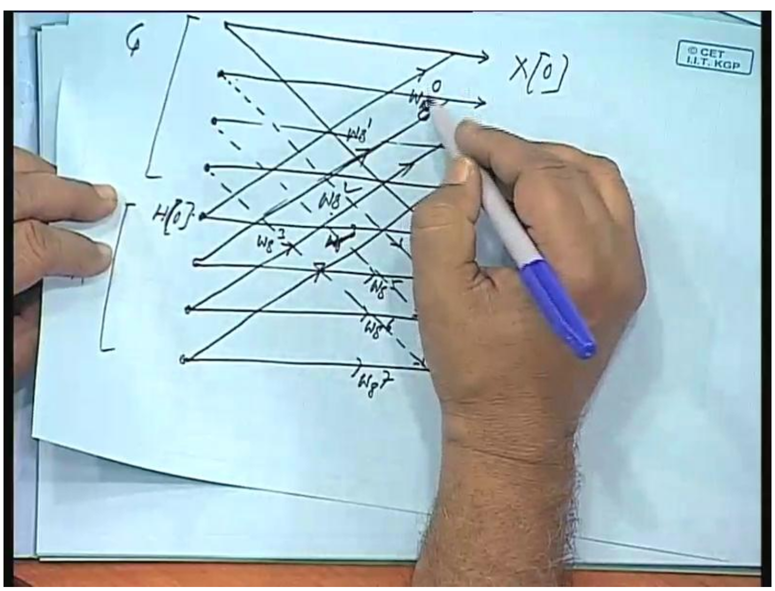

For an eight point sequence, you had W 8, 0, W 8, 1 and so on.

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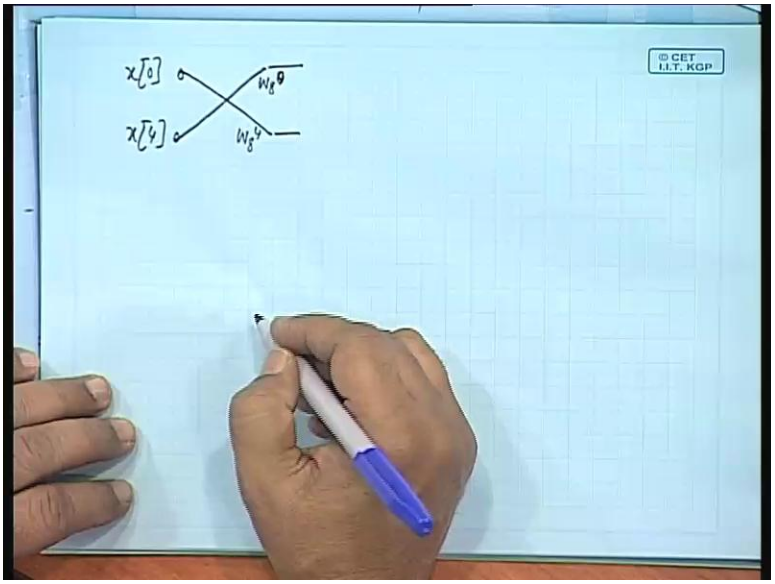

So, you will have in this case, W 8, 0, W 8, 4 okay sorry, W 8, 0. W 8, 4, because I am skipping just the previous step where you have two point sequences combined in the similar manner.

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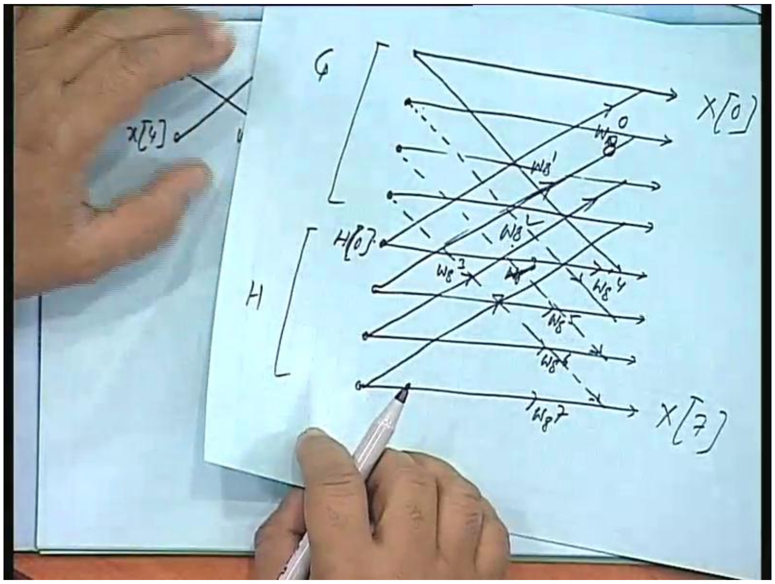

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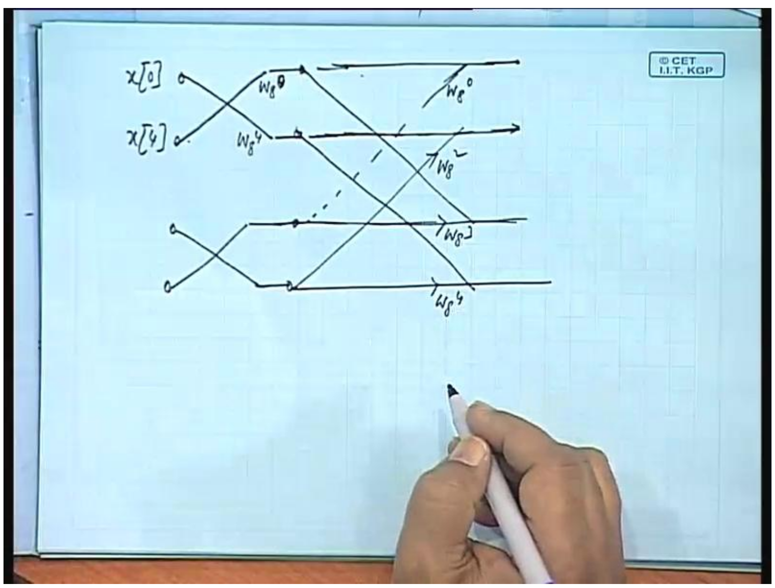

I just show one of them. See, these two pairs if they are to be combined to get the four point sequences; it is like this, similarly for the other group okay. So, this particular intermediate step, I am skipping so previous to that, if you go down then it will be W 8, 0, W 8, 4 here it will be W 8, 0 and W 8, 2 okay. So, this will be W 8, 3 and this will be W 8, 4 okay, is it not very similar to the previous one?

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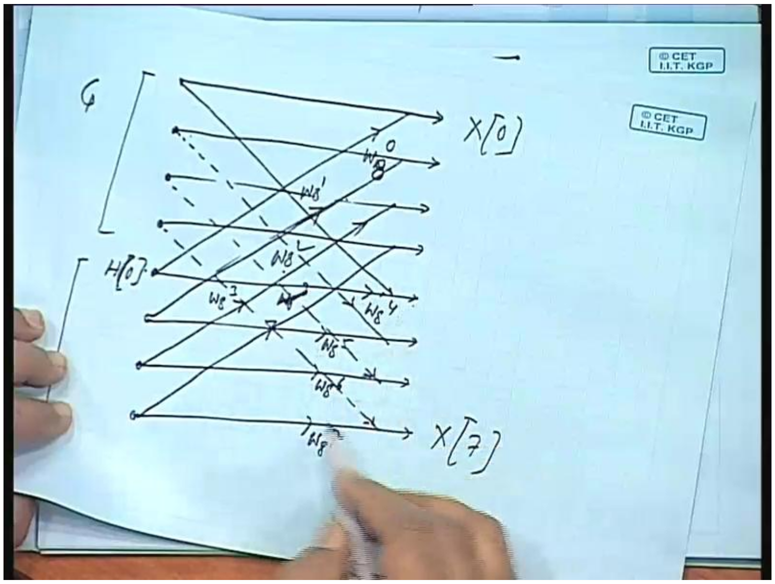

You have W 8, 0, W 8, 1, W 8, 2, W 8, 3 and the horizontal slope where W 8, 4, 5, 6, 7

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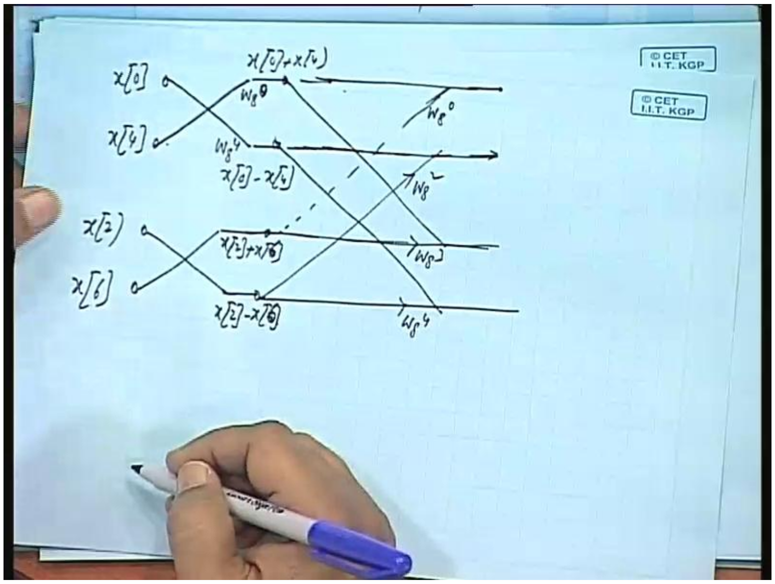

Similarly, here it will be W 8, 0, W 8, 2, W 8, 3, W 8, 4 and the horizontal one should be, the other ones and then this one will be x 2, x 6 and so on, I am skipping the other four all right. What will be the values of this? This will be x 0 plus x 4, W 8, 0 is 1. So, this will be x 0 plus x 4 and this one will be x 0 minus x 4.

So, when you come down to the lowest level will be just addition and subtraction of two numbers. Similarly, it will be x 2 plus x 4, x 2 minus x 4, okay. So it is this quantity, so which is to be again multiplied by respective values okay. Let us take an example, it is all right? x 0, x 4, x 2, x <a_side> ((x two plus)) (00:20:31) <a_side> x 6, yes? Thank you very much, okay and so on. So, let us take an example and then work out one problem, it will be clear okay. Let us take a sequence.

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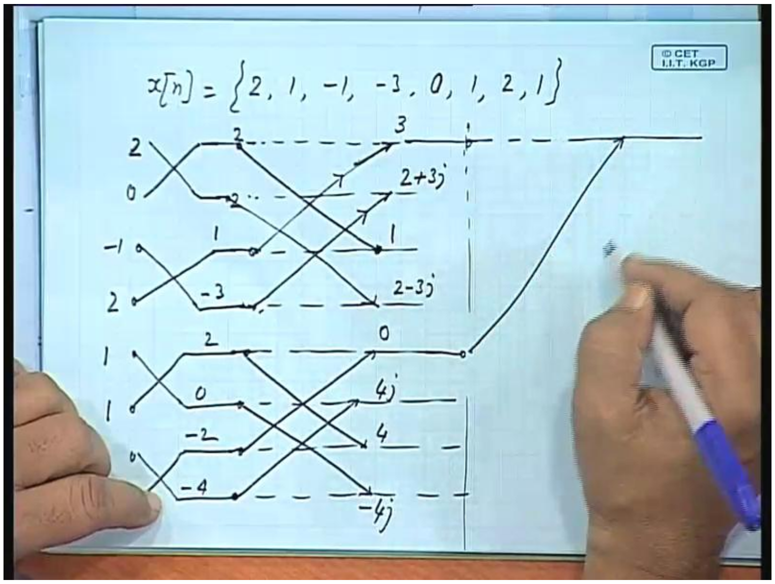

X n equal to, you suggest a sequence say; 2, 1 minus 1 okay, minus 3, 1, 1, 2 I am taking very simple values, 1, 2, 3, 4, 5, 6, 7 okay, in between I can put 1, 0 and 1 okay, this is a sequence. So, what will be the first stage of computation? It will be arranging the terms in this sequence. So, it will be 2. Next, it is 4, 0, 1, 2, 3, 4 a 0, all right have a butterfly. Then this will be 2 and this will also be 2. Next, minus 1 and 2.

So, this is this will be 2 minus 1, 1 minus 1, minus 2. So, this will be minus 3 okay. Next, what will be the other values? 1 x 1 and 1 and 5, is it all right? So, 1 is that okay and then minus 3, correct me if I am wrong minus 3 and 1. So, what will be these values? 1 plus 1, 2 and 0 minus 3 plus 1 minus 2 minus 3 minus 1 minus 4, correct me if I am wrong, is that all right? Next, we should combine these two.

So, let me draw it okay. Let us draw horizontal lines first, that will be easier to trace. So, these has to be paired with this, okay. And this has to be paired with this. This has to be paired with this, this has to be paired with this. What would be the multipliers? This plus this 2 plus 1, okay. So, 3 and then what should I do with this? This is is that all right? This is 2 plus one1, 3. And this one will be 2 minus 1, 1. And what about this? What should I multiply with, what should I multiply with? when we are taking two point sequences; W 4, 1 know should it not be W 4, 1, W 8, 2 you means; W 4, 1, so W 4, 1 is minus j okay. So, I will write this one as 2, this minus 3 in to minus j that becomes plus 3 j, is that all right? If this is 2 plus 3 j, then its counterpart will be 2 minus three j okay, once again you draw the lines.

So this one will be clubbed with this, this with this, this with this, and this. So, this one will be 2 minus 2, 0 and 2 minus minus 2. so 4, is that all right? And then 0, this is 4 multiplied by minus j all right. So, 0 minus of 4 in to minus j that becomes plus 4 j, is that all right. So, this one will be minus 4 j. In the next stage of computation, this one will be clubbed with this okay. So, this will be first one will be always with a cost at 1. So, 3 plus 0, 3

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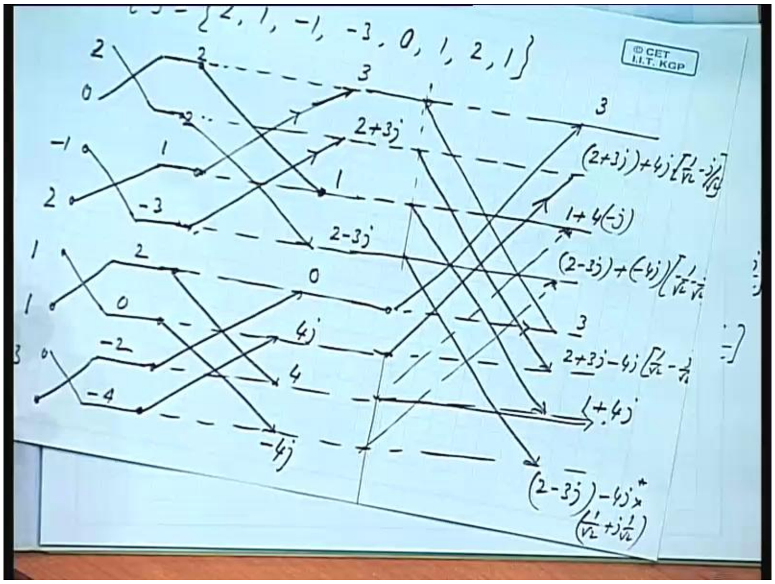

Next this one with, this one. Next, this one with this one and this one with this one. What will be this value? 2 plus 3 j in to 4 j in to how much? Now, it is W 8 to 1, is that all right? W 8 to 1 is minus 45 degrees. So, minus 0.7 or 7 sorry, the real part is positive. So, I will write 2 plus 3 j I read the computation to you plus 4 j in to 1 by root 2 minus j by root 2 okay. This one it will be 1 plus third? 1, 1 plus 4 in to how much? 4 in to minus j and this one will be 2 minus 3 j plus minus 4 j in to minus 135 degrees.

So, both minus so minus 1 by root 2 minus j by root 2; whatever be that you compute, similarly you can find out for these four, all right. So, these lines what should this be multiplied by these are basically W 0, W 8, 1, W 8, 2, W 8, 3 then this will be W 8, 4 means minus 1. So, it will be 3 minus 0. It is also three then this one, it will be 2 plus 3 j minus 4 j in to this same thing, is that all right?

1 by root 2 minus j by root a root 2, similarly this one, third one and fourth one; so this one will be 1 plus 4 in to, what will be this plus? J, is it not? And this one, last one will be 2 minus 3 j minus 4 j in to; it will be plus 45 degrees. So, I will write here 1 by root 2 plus j 1 by root 2, okay. You will find the symmetry in the conjugate symmetry of the values. This is 3, this is also 3. The first one will be always real for a real sequence and then this point onward, you will find this one will be complex conjugate of this. You see here, 2 plus 3 j 2 minus 3 j. This is plus 4 j,

this is minus 4 j, 1 by root 2 minus j by root 2; it is plus j by root 2, all the signs of the imaginary terms are just opposite here.

So, these two will be complex conjugate. Similarly, this is 1 minus 4 j, this is 1 plus 4 j. As you go towards the center, the center one will be again a real; this is 2 plus 3 j 2 minus 3 j is minus f4 j in to this, okay. Check whether we get minus 4 j in to 1 by root 2 okay. This is plus 1 by root 2. So, this is also complex conjugate.

So by this, we have seen that if you can arrange the terms and we start with two point sequences and gradually develop, it we can get the output in order. So, what should be the sequence at the input? It is all shuffled. So, how do you do the shuffling? If N is very large, for an eight point sequence you can do it manually, for sixteen point sequence you can do it manually; when it is very large say two fifty-six or five hundred and twelve, then how do you {mak} (00:32:18) make those arrangements? So, this is in place computation that is before we start the computation, we first of all arrange these terms and these two will be replaced by these two values, okay.

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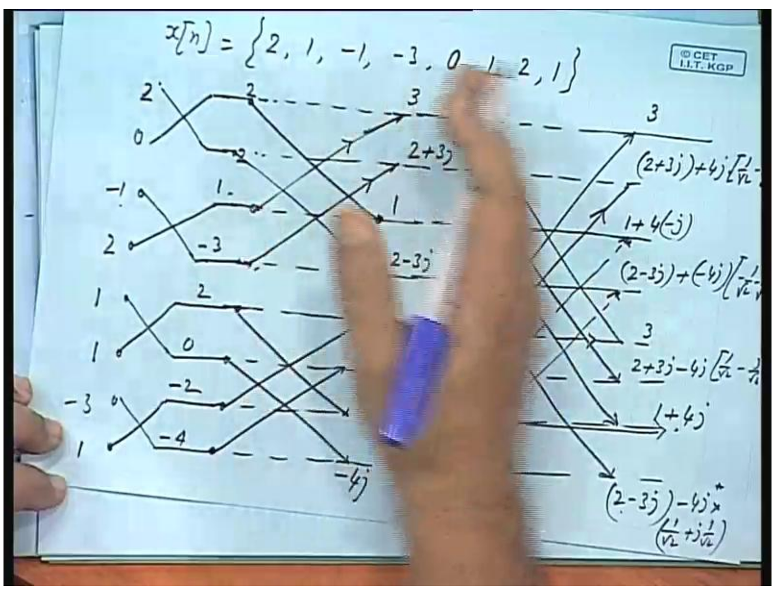

These two will be replaced by these two, then again when you take the combination of these two pairs; these four will be replacing these 4.

So, gradually you keep on replacing the positions by the newer values and at the end, you get the final product. So, how do you replace them? So, what to be the position of the sheets in the beginning?

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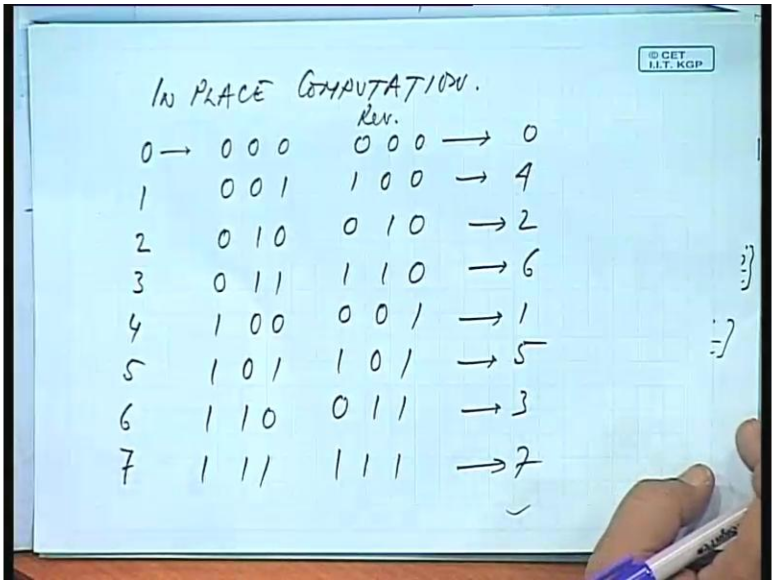

So in place computation, you will find for an eight point sequence. it is output is 0, 1, 2, 3, 4, 5, 6, 7. What is the binary expression for this? We are going for an eight point sequence, so there will be three digits 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 1, 1, 1, 0, 1, 1, 1. If you reverse them, reverse sequence this will be 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 1, 0,0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 1. Now, if you express them in decimal numbers; this will be 0, this will be 4, 2, 6, 1, 5, 3, 7.

So, basically this is the shuffle sequence. Say, this is known as bit reversal technique; so express them in binary digits then you reverse them and again, convert it back the decimal numbers, you get the reverse sequence. So, you shuffle the data in this sequence, you will get this. Now, what is the logic behind it?

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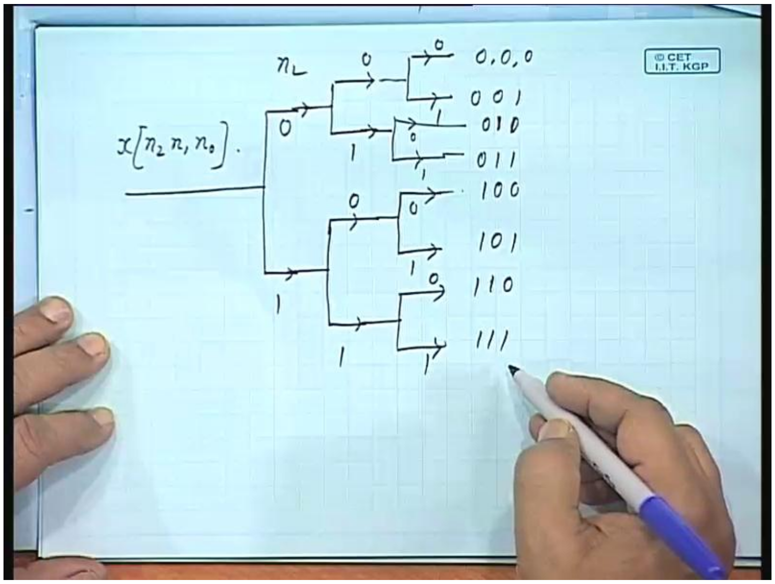

If you remember, our binary digits are say; three digits like this, what we have done is; so this is 0, this is 1, this is the first position then 0 and 1 second digit and then 0 and 1, okay. So, this becomes 0, 0, 0 then 0, 0, 1. Similarly, this 1, 0 and 1; so this becomes 0, 1, 1, 0, 1, 0 and 0, 1, 1. Similarly, here 0, 1 okay. Again you are having 0, 1. So, this will become 1, 0, 0 then 1, 0, 1. Then this is 0, this is 1; just the upper one and keeping is 0. Next one is 1 and if I arrange it like this, so it will be 1, 1, 0 and 1, 1, 1. So, this is basically the sequence of the digits, the decimal digits expressed in binary form.

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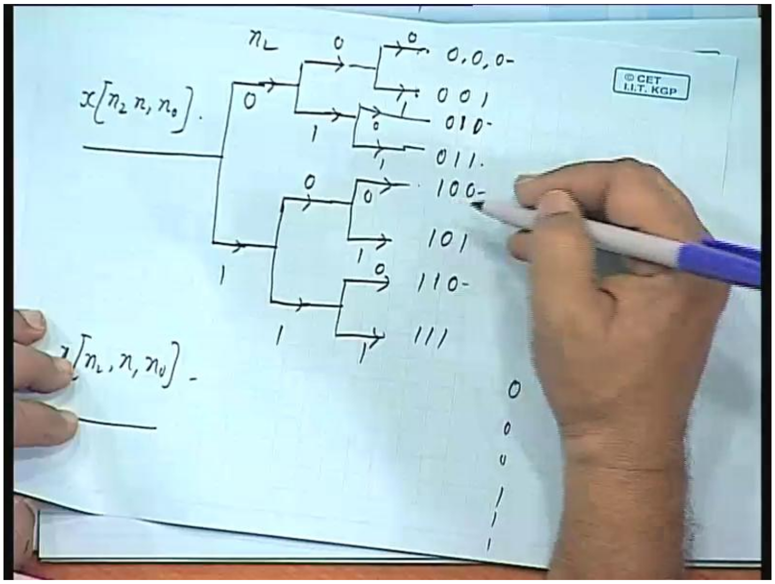

While realizing it, we are going back from here; we are starting from the last sequence. So, we are combining this 0 with 1. So, then after that it will be this position. Next it will be this position all right. So, basically the reversal takes place in this mode. So, x n 2 n 1 n 0; for example when you choose the even numbers and the odd numbers, what is the identity of an even number? The last digit will be, last binary digit will be 0.

So, you are taking all the zeros together and all the ones together, all right. That means out of this you are taking these zeros, selecting them first segregating them first; so those zeros are taking the first position all right. Then these ones are coming next; so in the next sequence, you have putting 1, 1, 1 etcetera all right. Then in the second sequence also, out of this the middle these are three digit number.

So, the middle one again you are selecting out of these zeros, the ones which are ending with zeros. You are selecting the middle ones; again with zeros and ones, so that is how thing is getting reversed. There is another technique instead of segregating them in even and odd parts; you can just half them, this is known as the earlier one is known as, decimation in time, okay.

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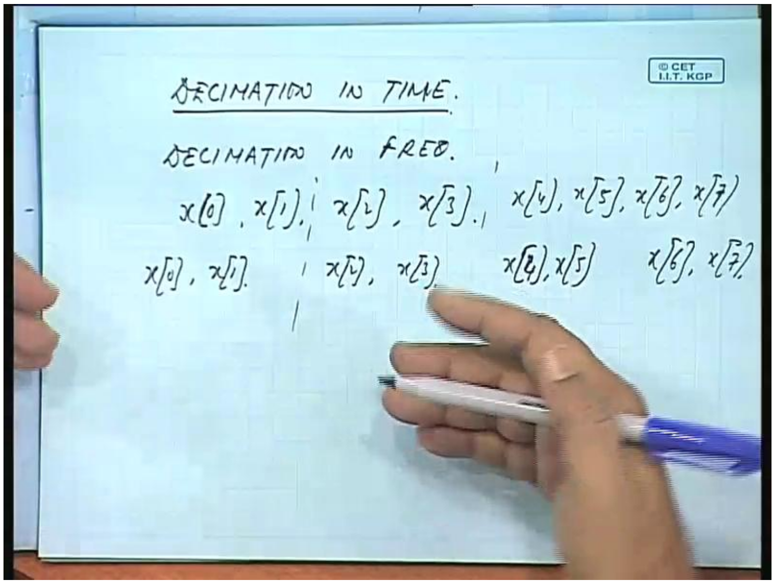

Now, we are decimating in frequency domain; decimation in frequency that is you keep on having them. So, you take x 0, x 1, x 2 and x 3 one set. Similarly, x 4, x 5, x 6 and x 7 in the second set; then again take these elements x 0 and x 1 as one pair, x 2 and x 3 as a next pair. Similarly here x 4 and x 5 and x 6 and x 7 okay. And then like the earlier butterflies you create outputs out of this and then combine in a certain fashion, what should that? It will not be definitely the earlier fashion of combining them. So, will see how this can be achieved we have X k

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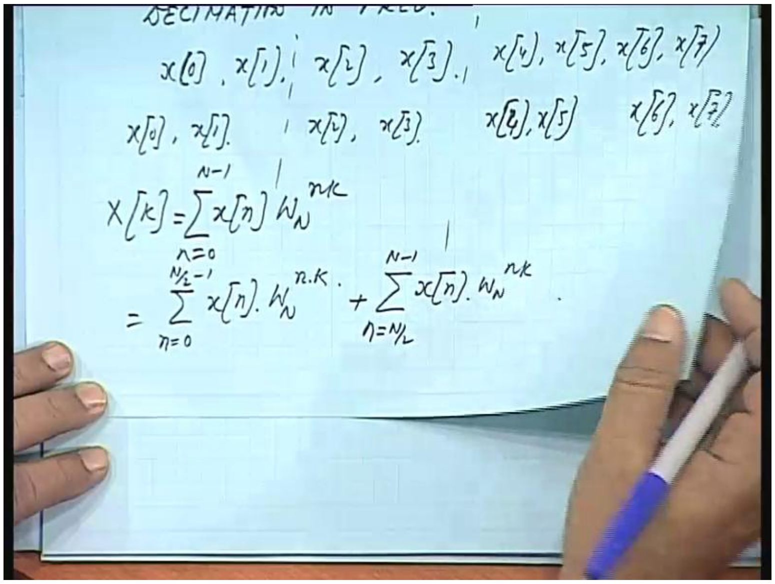

is equal to x n, W N, n k summation submitted over n 0 to N minus 1. So, if I take in to 2 halves it will be x n, W N sorry, n in to okay what we are doing? n is now, 0 to 3 all right. First four we are taking. So, n varies from 0 to N by 2 minus 1 plus x n, n varying from N by 2 to N minus 1 okay.

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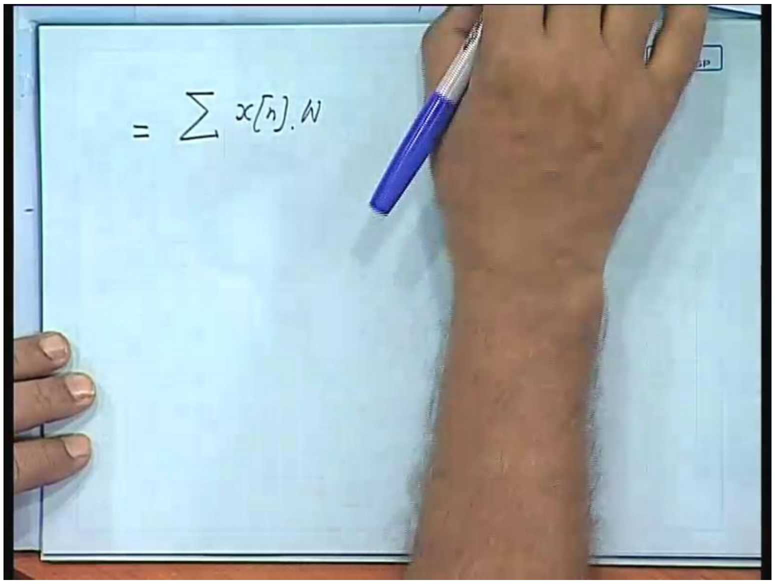

So, that is equal to x n, W. Now,

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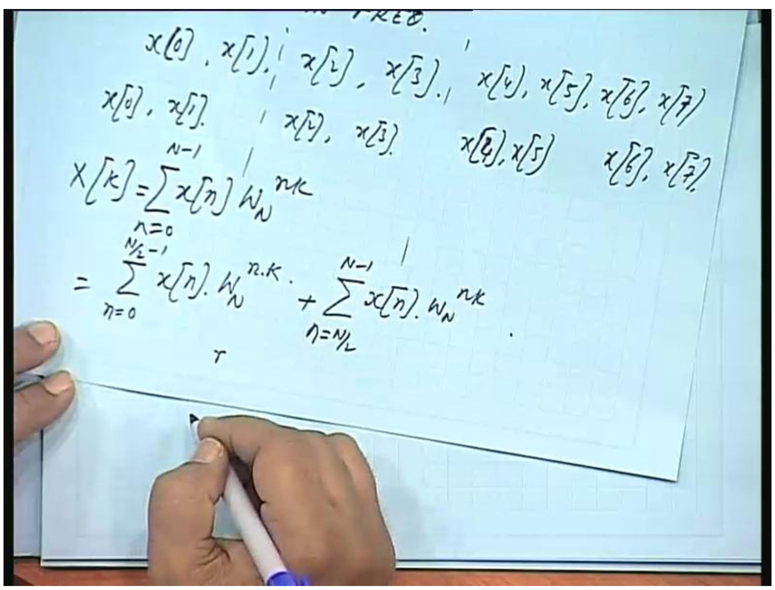

I replace this say; this is a four point sequence you see, I take this value of n as some r, this n I replace by r, r by

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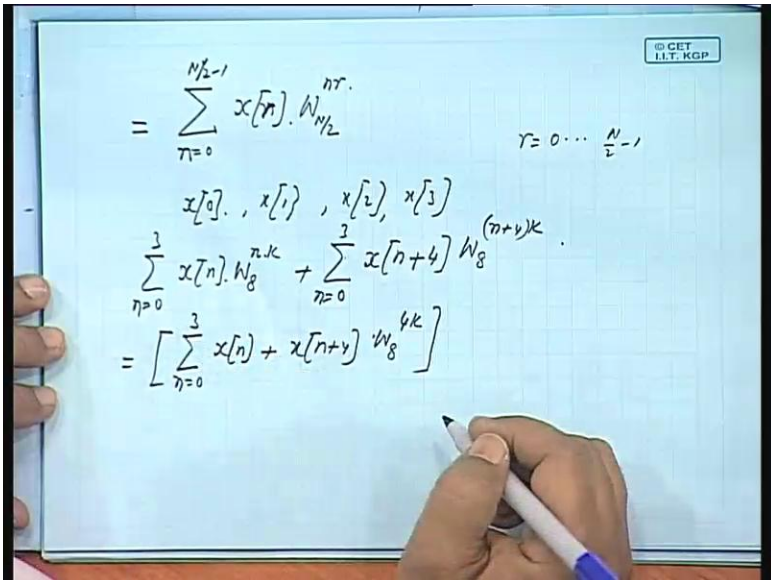

So, it will be x r, W N by 2, okay. r is equal to 0 to N by 2 minus 1. So, N by 2 n r is that all right; where n is equal to 0 to N by 2 minus 1. What I okay, let me take a sequence, it will be easier. What I mean is x 0, z 1, x 2 and x 3. We take together. So this will be a four point sequence, this will be four point sequence.

So, it will be x n in to W, four point sequence. How do you compute originally? Actually, it is clubbed with an eight point sequence. So, out of that we are taking only the first four terms. So, it will be W 8 n k but n varies from 0 to 3 okay and then how do I write this one? Will be x n plus 4 and again n varying from 0 to 3, is that all right?

W 8 n plus 4 in to k; so that will be taking care of 4, 5, 6, 7. So, it is summation x n plus x n plus 4 in to, it is x n plus 4 which gets multiplied by W 8; n k is a common term and W 8, 4 k. So, W 8, 4 k is the other term, n varying from 0 to 3 is that all right? Obviously,

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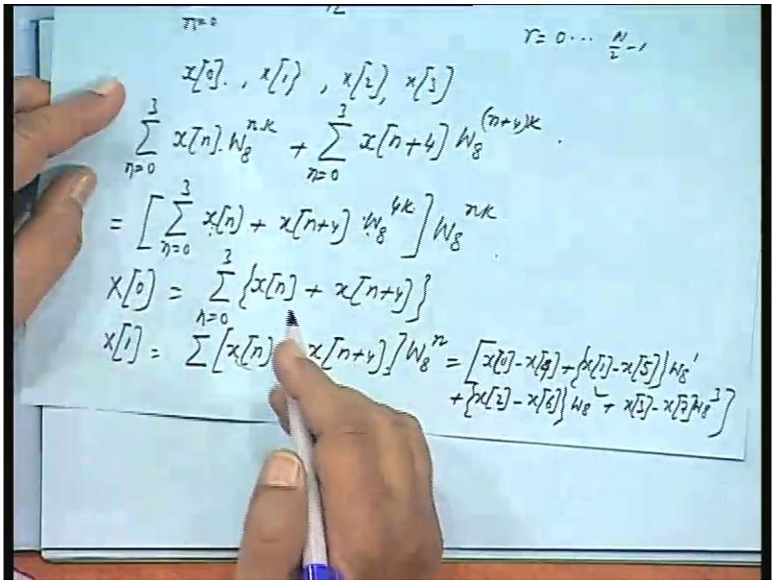

W 8, 4 k; W 8, 4 is minus 1. So in the first phase, it will be just minus 1 to the power k. So, x n plus x n plus 4, x n minus x n plus 4. So, alternately it will be changing, signs depending on k. So X 0. For example, X 0 will be sigma x n plus x n plus 4, W 8, 0 is 1, okay. n varying from 0 to 3. So, basically it is <a_side> (( )) (00:45:53) <a_side> Now in the first stage, thank you, thank you. W 8 n k yes, W 8 n k is a common term, okay.

So, x n plus x n plus 4 and W 8 n k and, when k is 0; this will be 1 and it will be just addition. And that is what we have seen, what would be X 1, X 1 it will be x n. Let me work out 1 or 2 then it will be clear; minus x n plus 4 and then see, W 8, 1. When I put 1, it is W 8, 4. So, that is minus that that gets multiplied by W 8. How much n? All right.

So, you have to take this n is equal to 0. So, that means; if I write the terms like this, it will be x 0 minus x 5 n is zeros in to 1 plus x 1 minus x 6. <a_side> (( )) (00:47:35) <a_side> I am sorry, this is four x 1 minus x 5 in to W 8, 1 plus x 2 minus x 6 in to W 8, 2 and x 3 minus x 7 in to W 8, 3 that will be giving me X 1, okay.

Similarly, x 2 etcetera will have a multiplier W 8, 2 k and then again W 8 n. So, you can keep on doing it, again you can break it down in to two parts. This one instead of computing from points I can go to 2 points and then combining, combine them in a particular fashion. Now, I leave it as an exercise, you find out what would be the signal flows, the butterfly structure for this in the frequency domain decimation, in the frequency domain.

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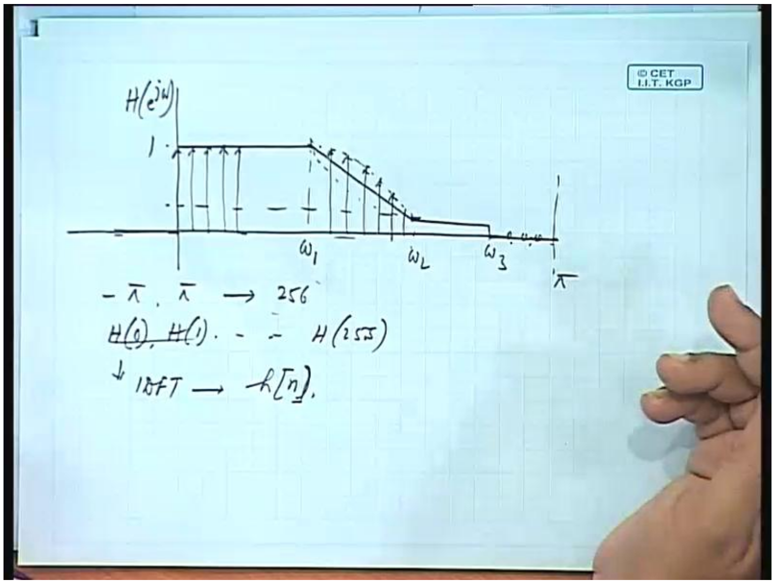

So for computer aided design, specially in FIR filters we have just mentioned earlier; FIR filter specifications are given say it may be like this, I may be interested in a filter where this is omega 1, omega 2, omega 3 and this is phi, this is 1. This magnitude is given something, it may be a straight line fashion, it may be normally in this region in a analog filter; in an analog filter the transition is not mentioned, it is omega p and omega s which has of importance but in FIR filter if you are given a particular structure of the filter function, if you give me as plain function so like this or any other particular fashion, sorry in which the transition should take place. Then we can realize it because we shall be taking in the computer aided design, we are taking discrete values okay.

These magnitudes are taken okay. Now, you have you scan this entire region from minus phi to plus phi; this will be a an image okay, minus phi to plus phi this entire region in to large number of points, say 256 points or may be take more. And then take, what should I take? What is this? IDFT all right.

So 256 points you have got. Say I will call it H 0, H 1, H 2 and H 255; these are the points H 0 starts from minus 5, so so many points we have got. So take IDFT DFT and IDFT algorithm is same okay; except that w will be replaced by w star. So, we apply FFT algorithm; you get h n and that h n has to be done shifted by the number of steps that you have. So that will give you the linear phase characteristics. FFT is a very commonly used technique. What about these points? They are all zeros; so in that sequence some of these points will be initially, it will be starting with zeros also ending with zeros, okay.

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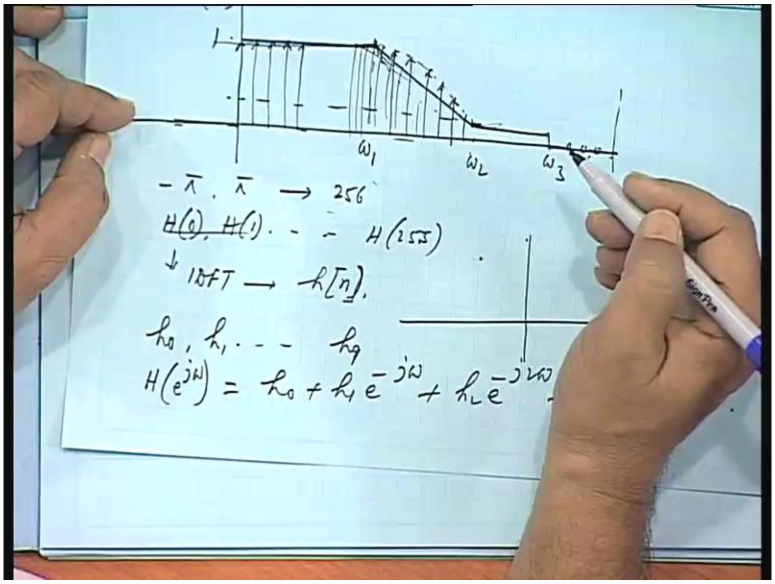

So you will get h n. Sometimes we may be interested in a particular order of the filter, a particular order, I want this filter to be realized by a 90 order filter or may be 10th order filter, then what we do you choose? Some sequence h 0, h 1, say h 9. So there are ten points, what are the best possible values of these? You realize this filter. So, H e to the power j omega is h 0 plus h 1 e to the power minus j omega plus h 2 e to the power minus j 2 omega and so on.

h 9 e to the power minus j 9 omega okay. And then you have got so many points. I need not have uniform, uniformly space points; I can choose large number of points near the transitions okay, wherever there are sharp discontinuities I take large number of points, and wherever it is flat or it is having a uniform characteristics, I take fewer points okay. You have as many points as you want and then you go by minimization technique, all right. What you minimize is the what thousand points, and you can write H e j omega 1 as this; similarly H at j omega 2 as H 0 and so on.

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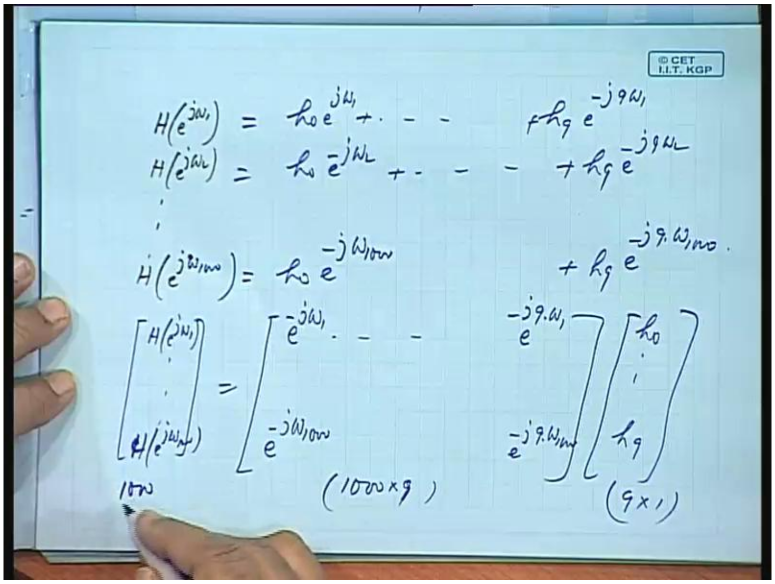

H 9 e to the power minus j 9 omega 2. Similarly, this 1 will be H 0 e to the power minus j omega 1000 plus h 9 e to the power j 9 in to omega 1000 and so on. So, this I can write in a metrics form. These are vector is the information vector, this are complex quantities e to the power H e to the power j {hun} (00:54:07) 1000 omega, 1000 and this side; I get h 0, h 9 and this one will be e to the power j omega 1, e to the power j, how much? 9 omega1. Similarly last one will be e to the power minus j omega, 1000 e to the power minus j 9 omega 1000 okay. This a metrics obviously. It is a nine element in this row and there are thousand elements here.

So, it will be 1000 by 9, 9 vector metrics. This is an 9 by 1 vector. This is a 1000 by 1 vector okay. So, the solution is h vector the optimum value of this vector.

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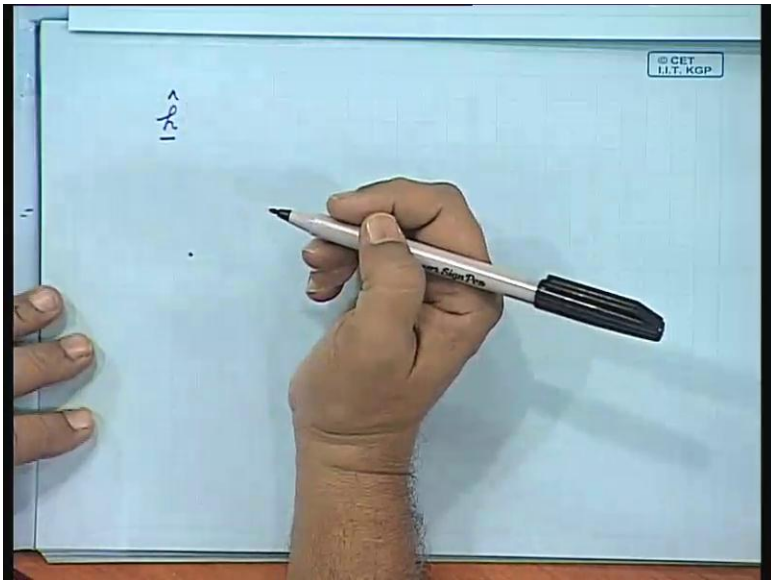

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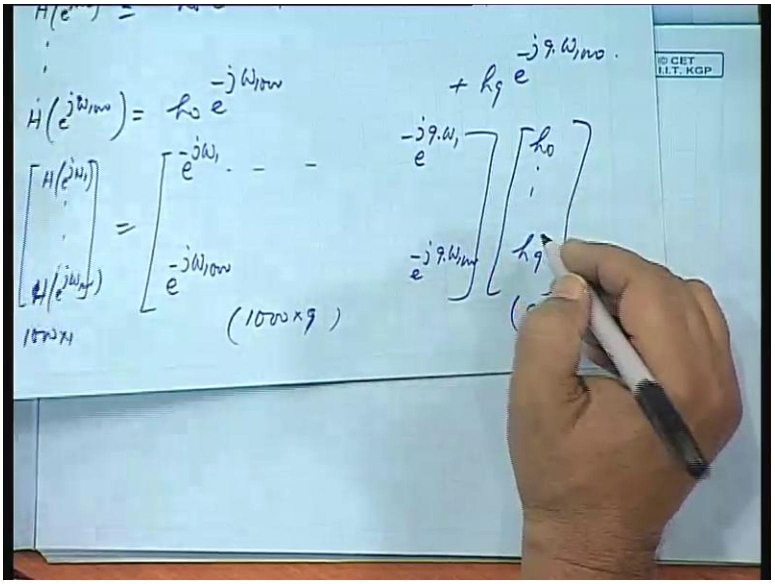

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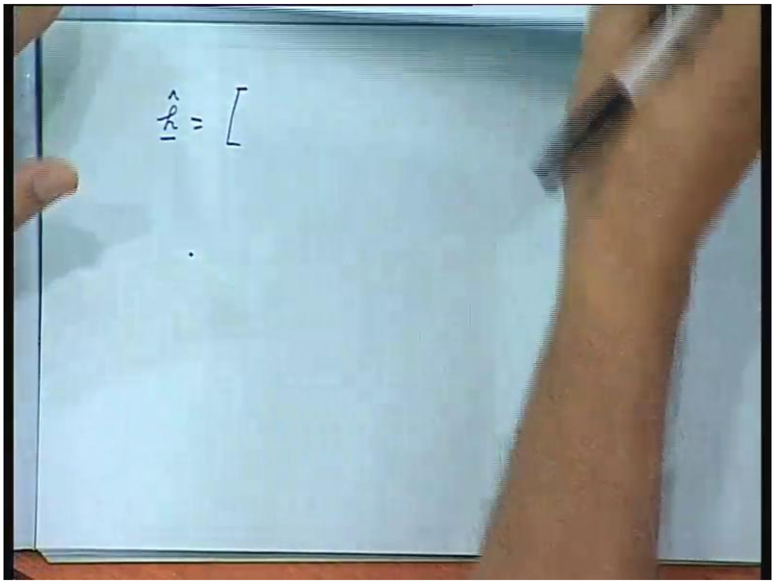

So, I will write the vector hat h, hat will be if I call this as some metrics say, a h okay let me call it what should I write?

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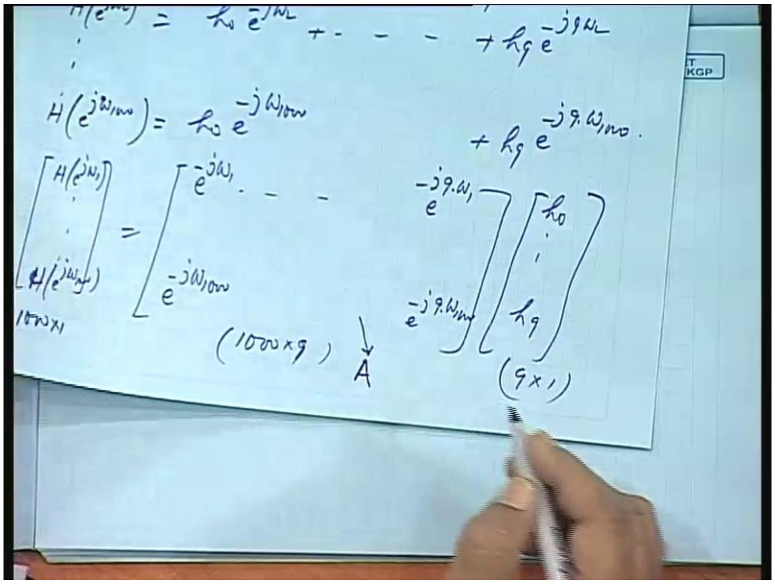

Some metrics A and let it be a transpose A whole inverse, a transpose b; where b is this.

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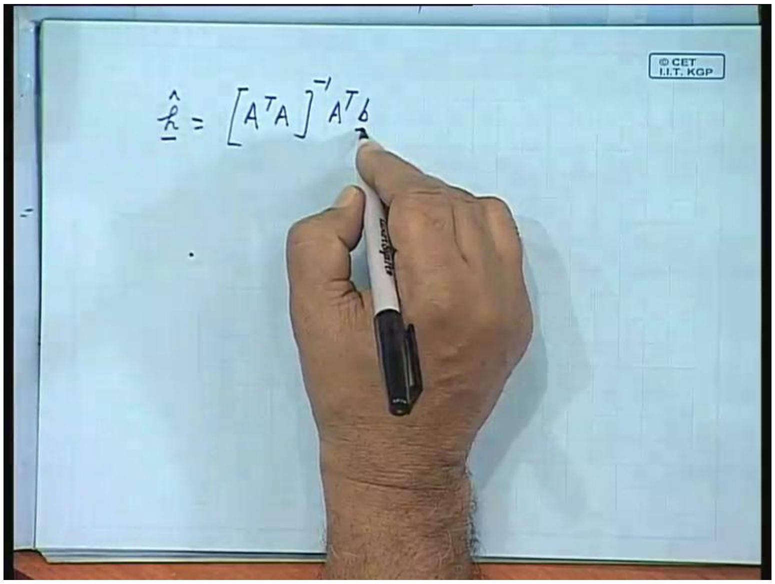

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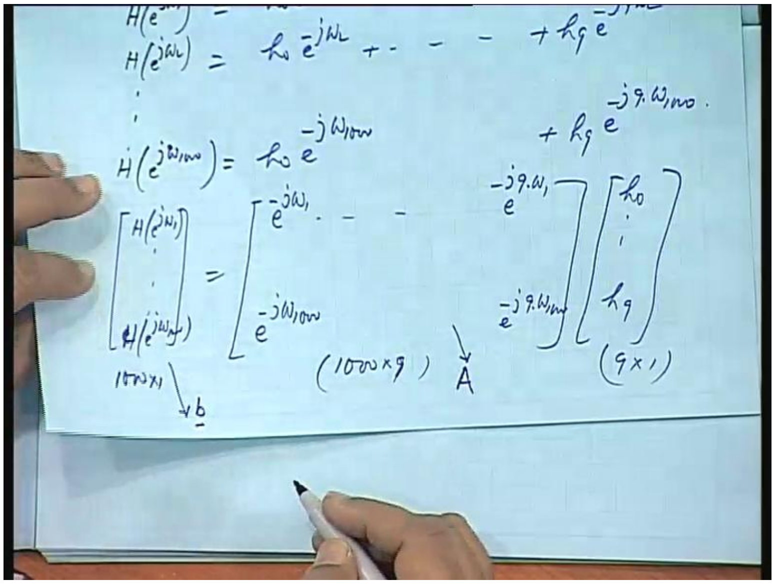

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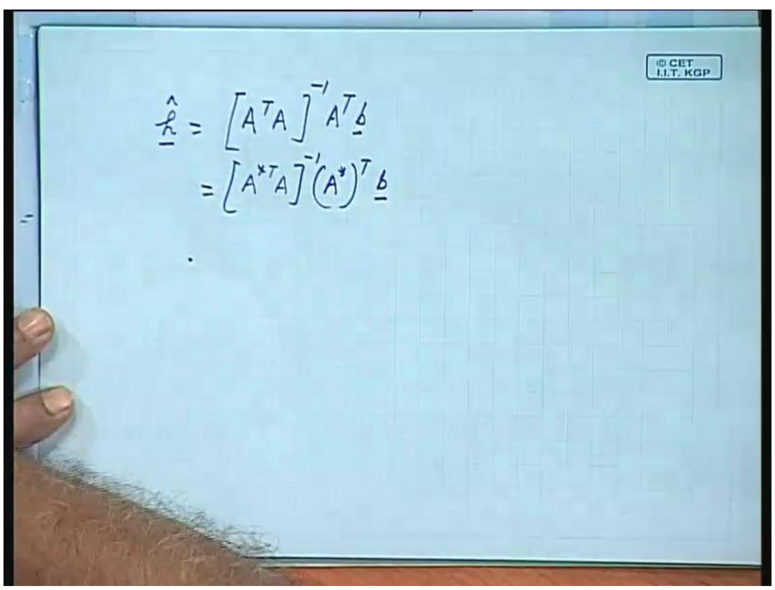

When A is complex, when A is complex it will be A star transpose A, had it been a normal metrics? Then the best possible solution for h is A transpose A whole inverse, A transpose b okay. Otherwise you take conjugate of A object this. So, this will give you the elements of h. Thank you very much will next take up other forms of filters.