Digital Signal Processing

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Lecture - 21
Computer Aided Design of Filters

Continue with computer aided design of digital filters.

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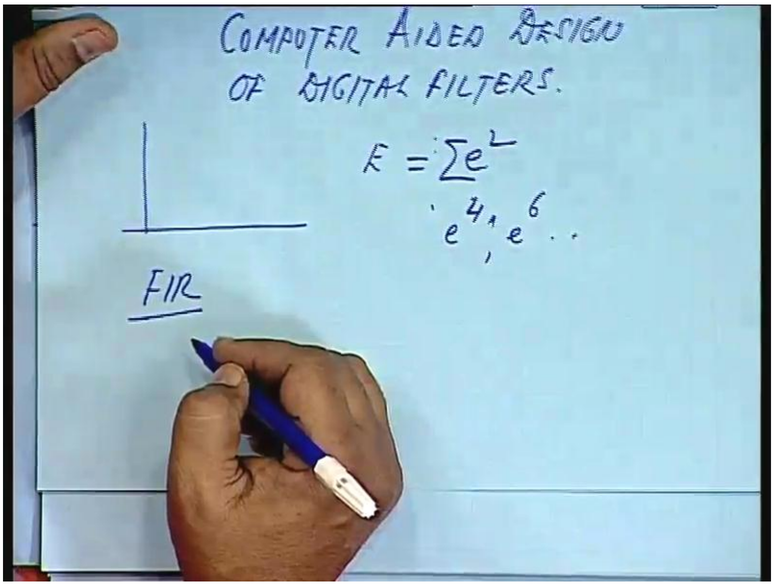

Last time we were discussing about a filter which is having both numerator and denominator polynomials all right; minimizing the error, of this error that we are defining the performance index was basically the difference squared, all right and summated, okay. Now, this need not be always a square, it can be even even, any even power of the difference, e to the power 4 or e to the power 6 and so on. You can take any of these as the criteria. Anyway today we shall concentrate on FIR filter, okay.

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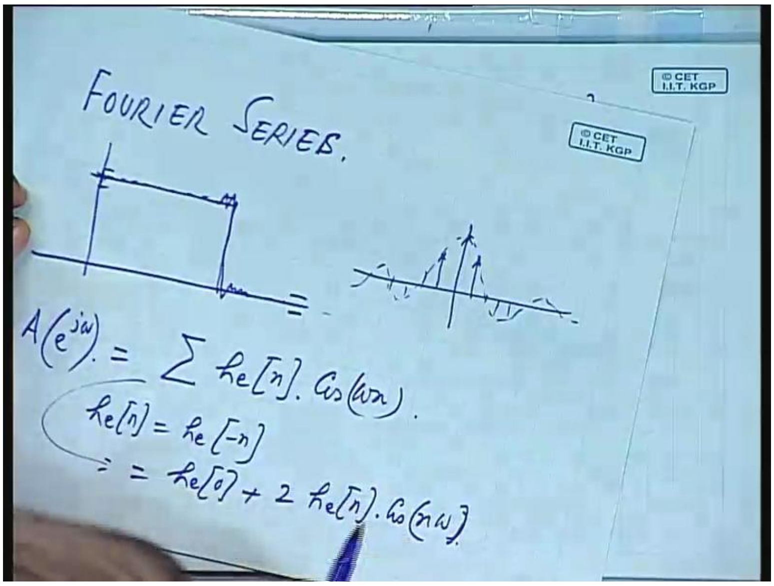

Now in the normal FIR filter design, what we did? We took the Fourier series okay and then truncated it at the desired length. Now, it has been, it can be proved, it is there in all the text books. If you take the Fourier series coefficients up to any desired length, then that will be the best possible, best combination of coefficients which will give you the minimum error. So, a rectangular window will give you minimum error, but we have also seen the effect of such truncated coefficients; trunk truncated series will give you at the transition a ripples, okay.

So, we wanted to reduce these ripples that are the transition should be smooth as far as possible. So, we went for those window functions other than the rectangular window, okay but mind you with those windows, you may have little bit of reduction in the ripples but the error will be more. Error is minimum with the actual Fourier coefficients, wherever you truncate. So, we did not have any control over the distribution of the error in this entire span okay, we did not have any control.

So, today we shall take up the other approach, that is if you can make some flexibility; if you can offer in the pause band and stop band tolerances all right or given as specified tolerance, whether we can approximate a polynomial, it will be a polynomial approximation which will give me best possible response. We will see, what it will be like.

We have say, a function A e to the power j omega; amplitude function as h e n is an even polynomial, cosine omega n, okay. h e n is h e minus n, all right. It is like this, it is an even function. We have seen in the tutorial class also one of the problems was like this. So, this will be equal to h e 0 plus twice h e n cosine n omega, all right. In the last tutorial class, we had that series minus 3 by 2, 1, 1, 1 and so on.

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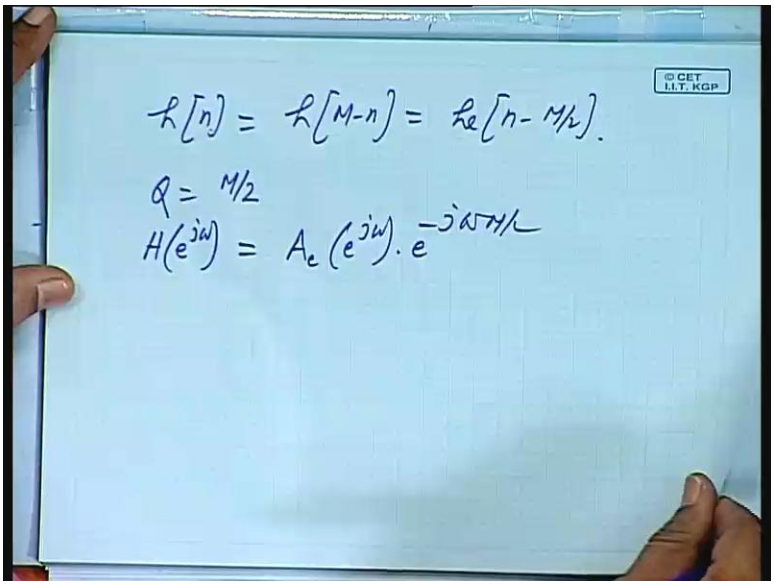

So, then we give a shift, so that actual h h n which should be equal to h M minus n is h e n minus M by 2, is it not? We gave a shift of a sorry in that nine point realization, we give we gave a shift of four steps all right. So, you obtain the final filter coefficients by giving a shift.

So, let us we define say Q as M by 2 where your total length is M plus 1, all right. So, we had nine point sequence; so 9 minus 1 divided by 2, 4, so the shift was 4. So basically, those 4 coefficients will decide the polynomial h 0, h 1, h 2, h 3, okay. So, h e to the power j omega will be A e e to the power j omega into e to the power minus j omega m by 2, this is due to the phase shift. Now, we will take up this particular structure of the function to represent the magnitude function in terms of a polynomial in omega.

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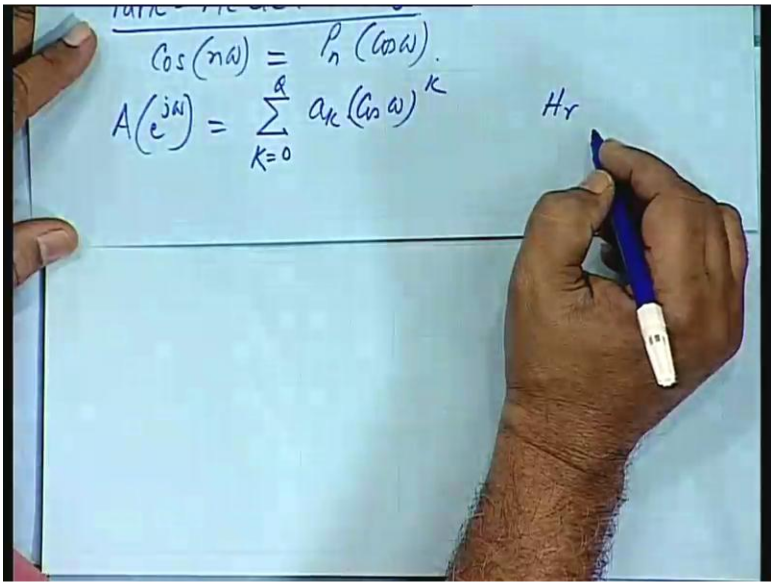

This is, this algorithm that we are going to discuss is known as Park McClellan algorithm, okay. So, what it does is cosine n omega, we can always write as a polynomial of n h degree in cosine omega. This we have seen in case of Chebyshev polynomial, if you remember; cosine n omega, we are writing cosine 2 omega, twice cos square omega minus 1, cos 3 omega and so on. So, it is basically that situation.

Therefore that magnitude function, the amplitude function is a k cosine omega to the power k, k varying from 0 to say Q. This will give me that H realized; H desired and H realized there is a difference, so that will be the error all right. We try to minimize that error in some sense. Minimization does not necessarily mean square sum should be minimized, okay. We may be in interested in minimizing something else, let us see what could be other possible criteria.

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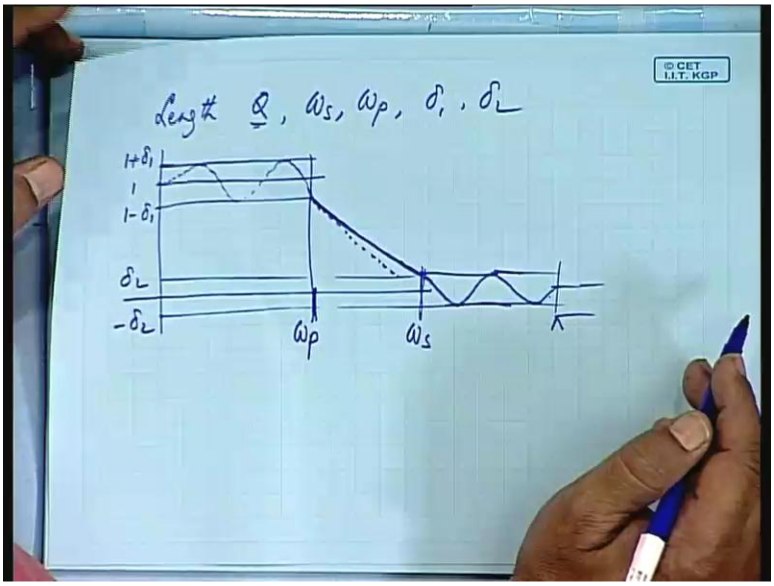

Now the length that is Q, length will be will be 2 Q plus 1, all right. So, length then omega s, omega p, delta 1 and delta 2, these are the variables. Some of them are fixed; others are determined by changing, okay. So, the amplitude varies between 1 plus delta 1 and 1 minus delta 1. This is omega p, this is omega s. This is delta 2 minus delta 2, okay. So, the response may be varying like this, sorry may be like this.

We are not really bothered about the transition, omega p is fixed, omega s is fixed. This tolerance is given, this tolerance is given and we may be given the length of the filter.

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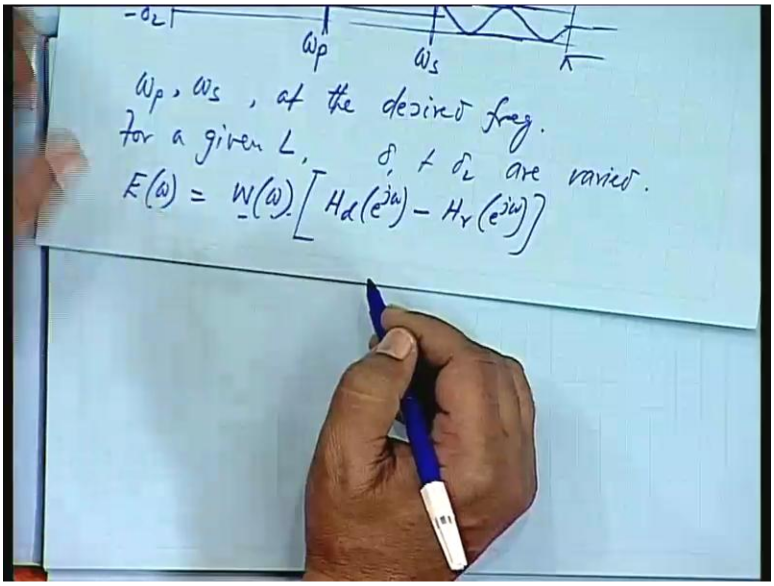

So the algorithm, algorithm suggested by park McClellan is omega p, omega s these are fixed okay; at the desired value, at the desired frequencies. So, for a given L we vary delta 1 and delta 2, okay. Now, let us define an error function. As now comes; a waiting factor which is frequency dependent, so we give different weightage to the derivation at different frequencies. It is not uniform.

Earlier we are taking the desired response minus the realized response square sum of this, is it not? Square of this error we are minimizing. Now, we are defining an error in terms of a waiting function; that means it is dependent on the frequency okay. Now for a low pass filter, for example for a low pass filter we are giving H d as 1 in this interval, yes please.

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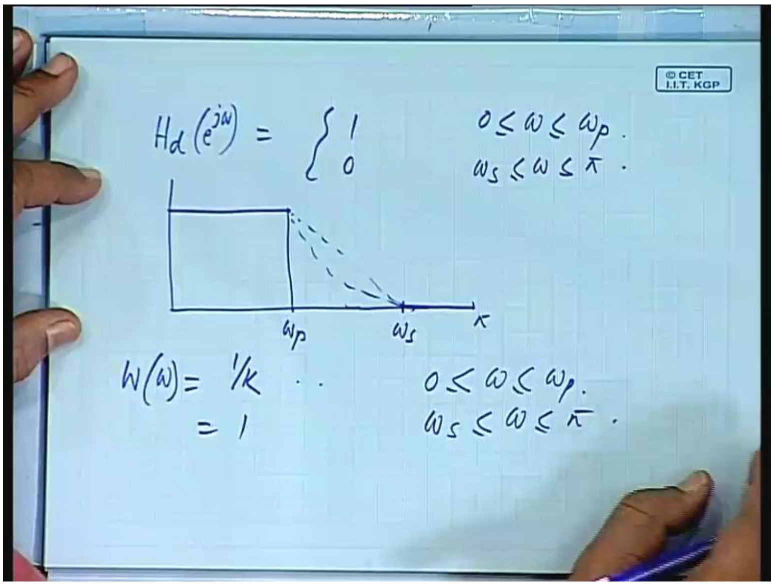

H d is equal to 1 and 0 between omega s and pi, this is the desired response. Like this, it should be it can be like this. We are not putting in this error function, we are not putting any condition here; that means this function will be quantified, this function will be measured only in these two subsets, all right.

So, we define the waiting function as 1 by k in this interval 0, omega, omega p and equal to 1 omega s, omega pi; where k we are defining as, delta 1 by delta 2.

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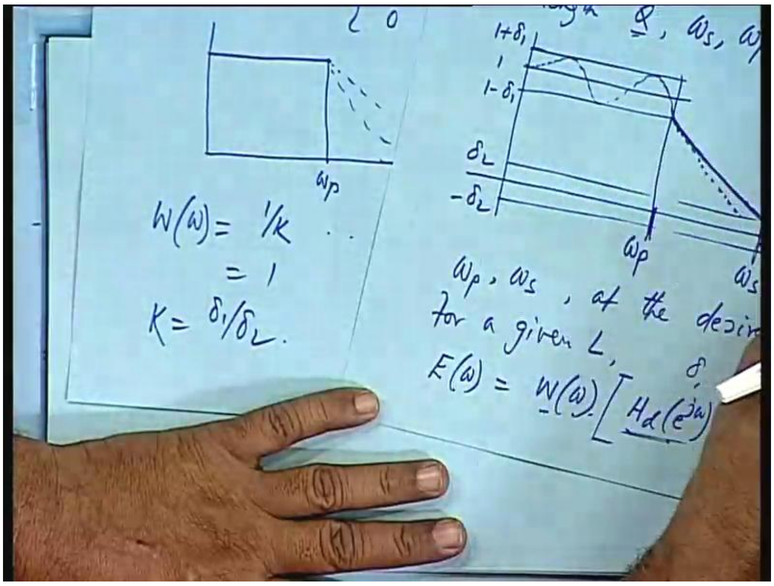

So, if delta 1 is more we are having a factor 1 by k inverse of that okay; that means we are giving less weightage to this derivation, okay. So, that basically we are normalizing, is it not?

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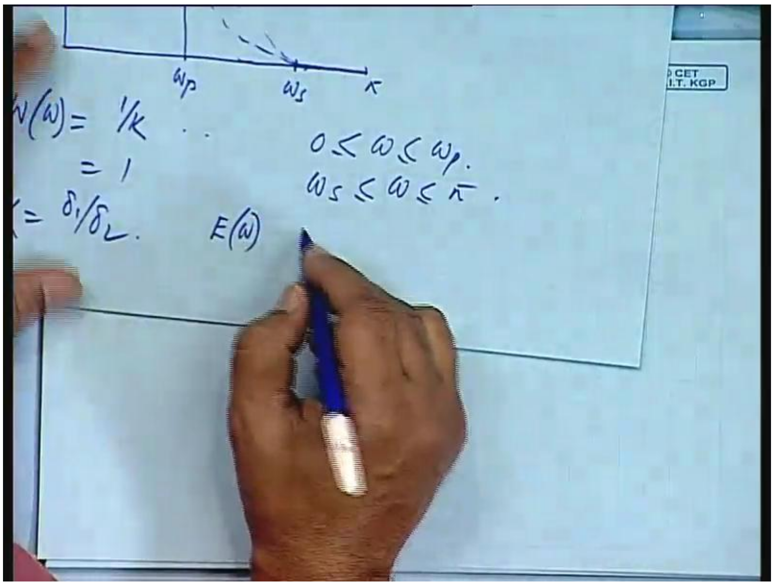

With this factor, if you see that error function will look this, I plot it here.

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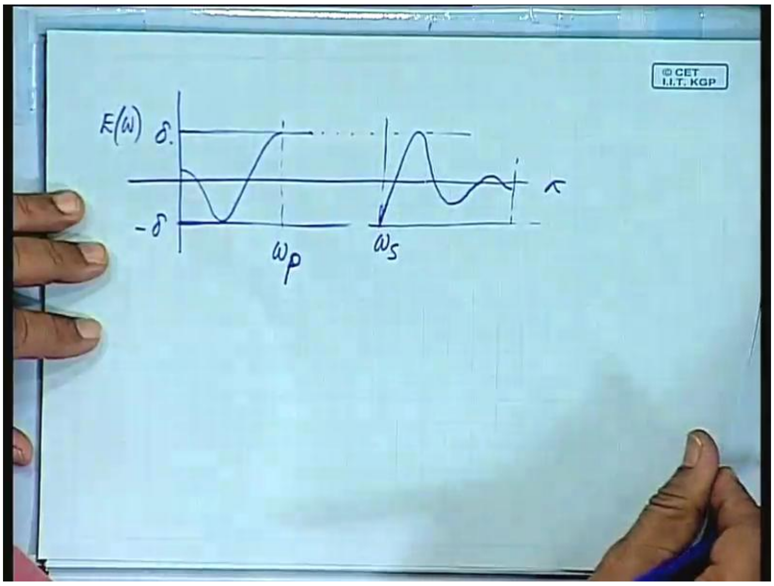

The error function may be like this, may be like this, okay. They will give the same bound some delta.

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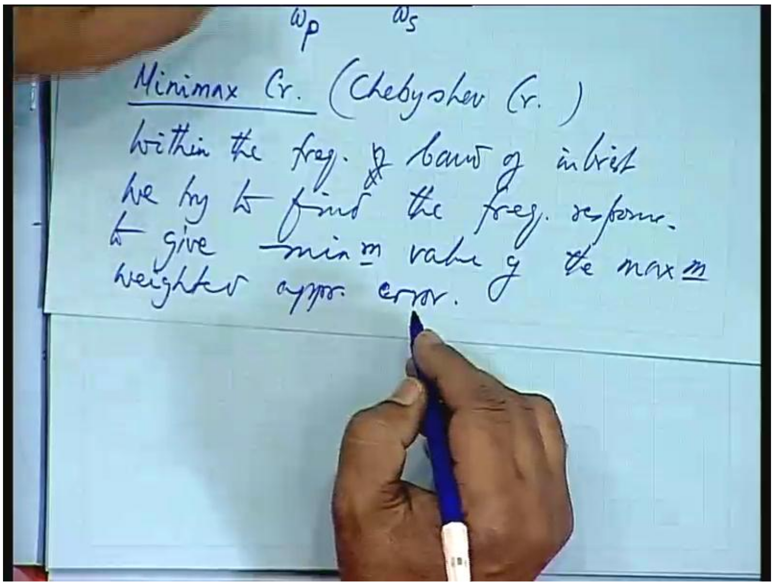

So, we are using minimax criteria, instead of squared error criteria; minimization of squared error, we are having minimax criteria which state is also known as, Chebyshev criteria. That is within the frequency band of interest, within the frequency band of interest, what is your frequency band of interest? It is the pause band and the stop band. The omega is to pi and 0 to omega omega p.

We try to find the frequency response, to try to find the frequency response to give minimum value of the maximum weighted approximation error, okay. That is we maximize this error; whatever is the maximize error, we try to minimize that. So, we will try to find the frequency response to give minimum value of the maximum weighted approximation error, okay. So, let us see what it will be like.

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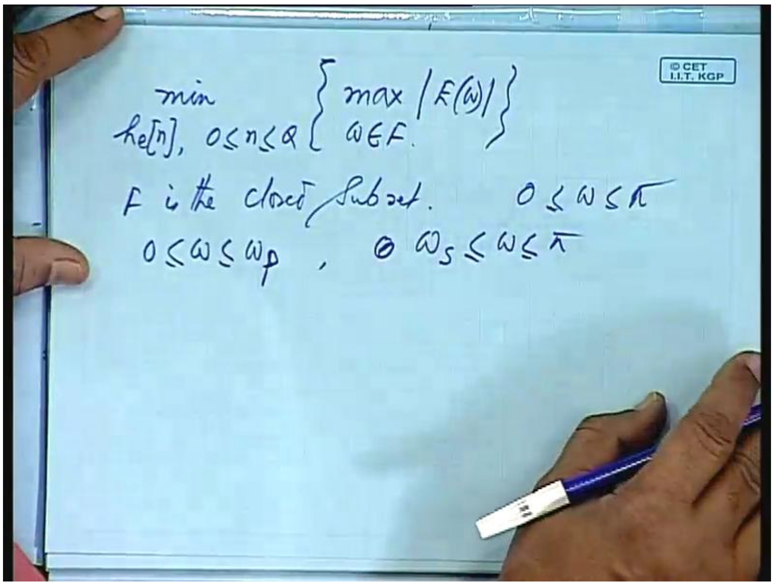

Or mathematically we can write minimize, the function 0 and Q maximum of E omega omega, in this closed subset; F is the closed subset, 0 omega pi okay which means, it is basically union of these two subsets 0 omega, omega p, omega s, omega pi. So, from h e n we will be determining h n.

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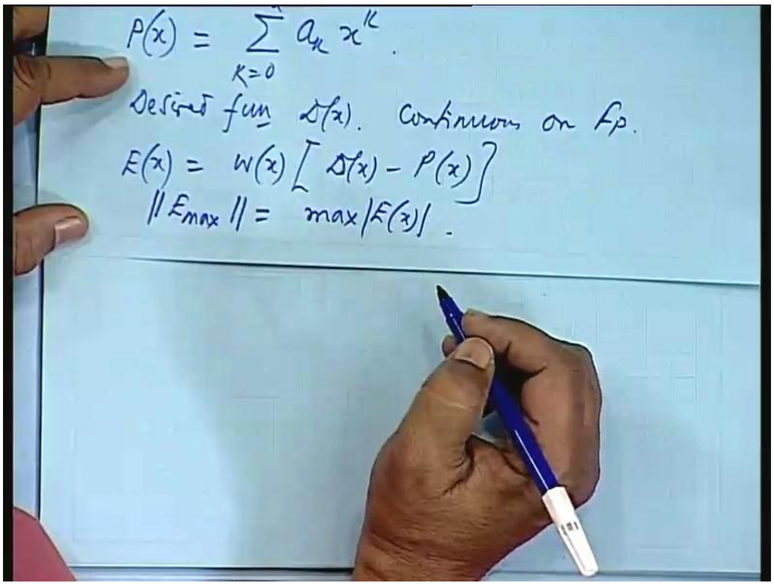

So, if you have P x as a polynomial a k, x k; k equal to 0 to Q. And there is a desired function say, D x which is continuous on F p this entire range of frequency. Then Ex is equal to w x into D x minus P x, that is the polynomial that we have approximated to, the derivation of this function from the desired function multiplied by the weighting function; we are defining as D x. And we define E max as maximum value of E x, okay. Absolute value of E x, whatever is the maximum value, we call it E max

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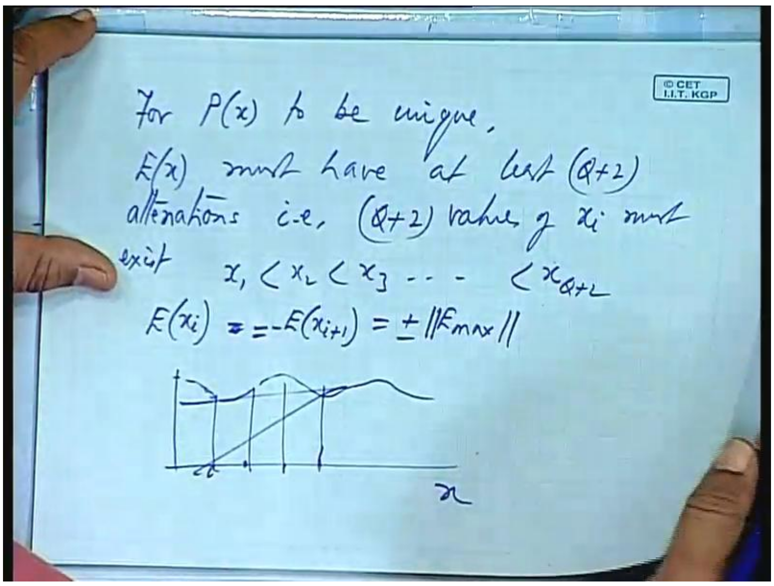

For P x to be unique, we are trying to find out that P x which will be giving me minimum value of this E max, is the problem clear? We are trying to find out that P x; is this coefficients basically a k's which will minimize this maximum value. So, for this condition that is for P x to be unique. E x must have at least Q plus 2 alternations, this is the alternation theorem.

We are going to discuss what is, Q plus 2 alternations that is Q plus two values of x i, that is Q plus two values of x i must exist in that range. It is x 1, x 2, x 3 I should be able to get x Q plus . And E x i must be okay, E x i minus E x i minus 1 x i plus 1 equal to; now equal to minus E x i plus 1 equal to plus 1 E max, okay. What does it mean? You will have successive number of points x 1, x 2, x 3, x 4 all right where the function will be alternating all right, okay.

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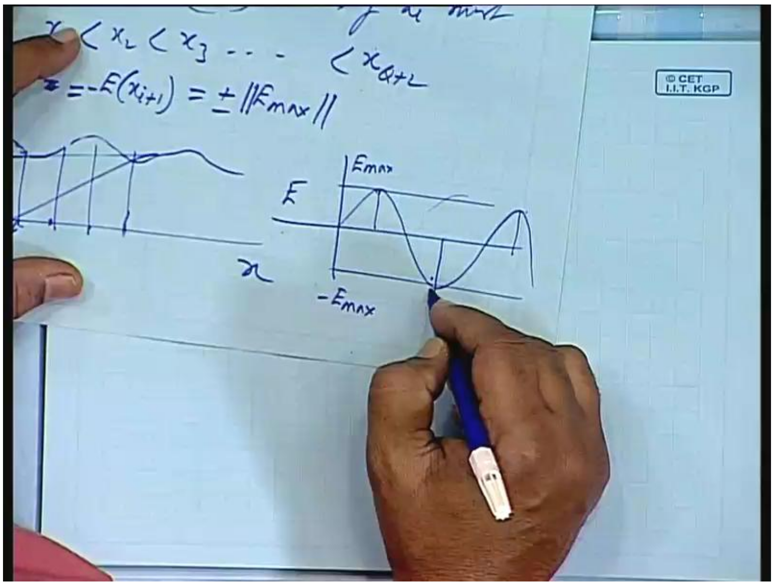

Let me just show it, where it will be hitting the maximum value plus maximum value or minus maximum value; that is the error function E. This is E max plus E max minus. So, you will have it may not start from 0, so this is x 1, this is x 2, this is x 3. So, alternatively they will be plus and minus, all right. You cannot have a situation like this; it will go to the extreme value okay and there will be at least Q plus 2 number of such points, mind you this is in terms of the error, in terms of the error. So, let us see what it will be like for a type one filter.

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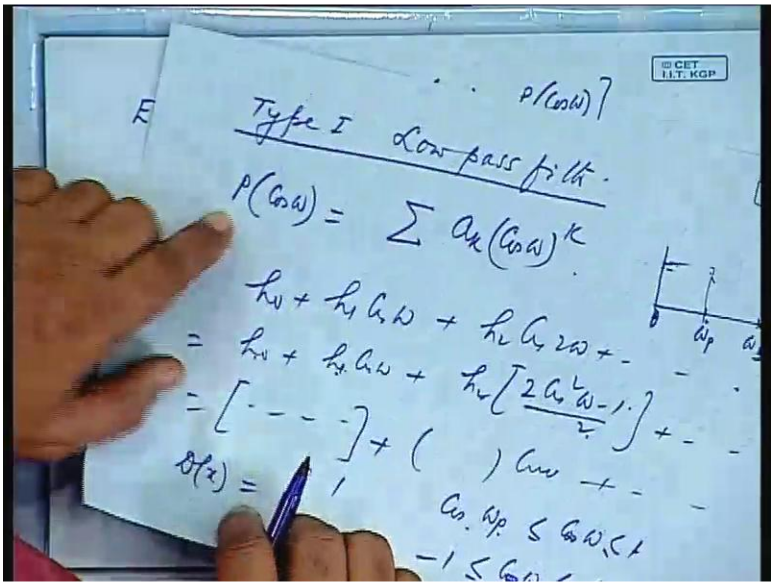

That is even symmetry and odd number of terms, all right, even symmetry and odd number of terms. So in that case, you remember that in an earlier class; the polynomial can be written as a k cos omega to the power k, we had derived these a similar expression. So, after summing up we can write in this form, is that all right?

For example, if you have h 0 plus h 1 cosine omega plus h 2 cosine 2 omega and so on. I can always write this as, h 0 plus h 1 cosine omega plus h 2, cosine 2 omega can be written as twice cosine square omega minus 1 by 2 and so on. So, finally h 0 plus minus h 2 by 2, so all these terms will be forming the first coefficients.

Then we will have cosine 3 omega term, will give you cosine omega also; so that will be clubbed with those coefficients will be clubbed with h 1, so something into cosine omega and so on. So, you get a polynomial of this type. We are not really concerned about the relationship between h 0, h 1 etcetera and a k that is not required.

So, D x is equal to 1 and 0, cosine omega p, cosine omega and 1. We are making a plot. Now we are trying to study P x okay. And this is minus 1 cosine omega cosine omega s. Why minus 1, cos pi all right. So, this entire range 0 to omega p, omega s to pi, what will be the value of cosine?

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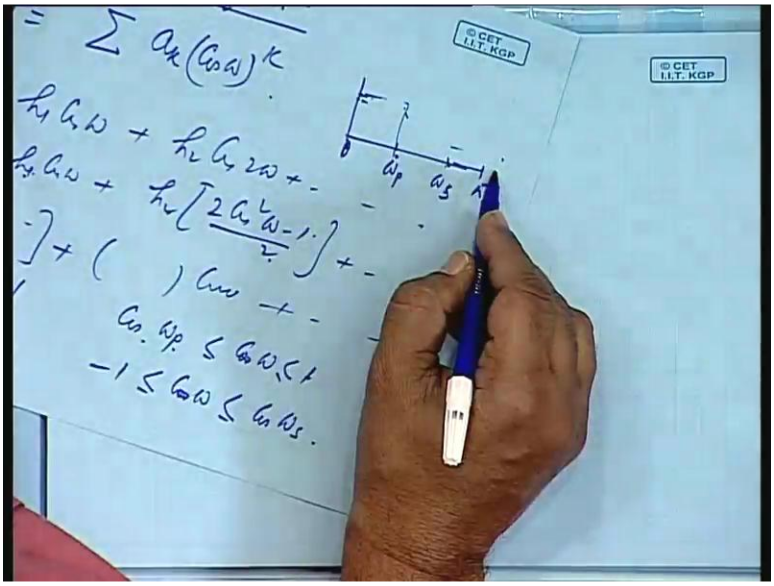

0, 1, then cosine omega p, so it will be lying between cosine omega p and 1; in this range cosine omega. Similarly, in this range it will be cosine omega s and minus 1, okay. So, that time the desired function should be 0

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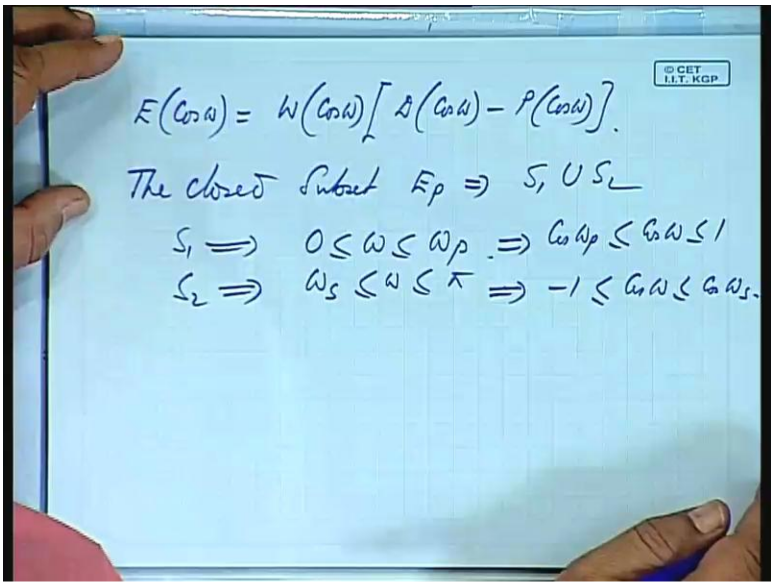

E, x which is now cosine omega should be W cosine omega, D cosine omega minus P cosine omega, okay. Polynomial is in terms of cosine omega x means, cosine omega. The closed subset E p; the closed subset E p is basically union of the 2 subsets, S 1 and S 2 where S 1 means S 1 corresponds to 0 to omega p and S 2 corresponds to omega S, omega pi; which means, this corresponds to cosine omega p, cosine omega 1 and this corresponds to minus 1 cosine omega and cosine omega s, okay.

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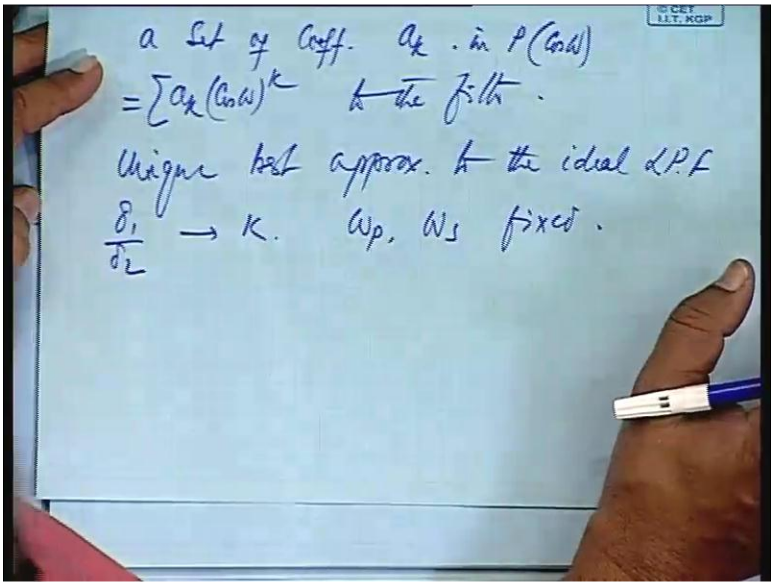

So, alternation theorem states, the alternation theorem states that for the unique function to exist, a set of coefficients a k; a set of coefficients a k in this polynomial cosine omega to the power k summation, will correspond to the filter, this will correspond to the filter representing the unique best approximation, unique best approximation to the ideal low pass filter, ideal the low pass filter with the ratio delta 1 by delta 2 fixed at k. And omega p, omega s fixed. We have omega p and omega s fixed, delta 1 by delta 2 ratio fixed at a value E k.

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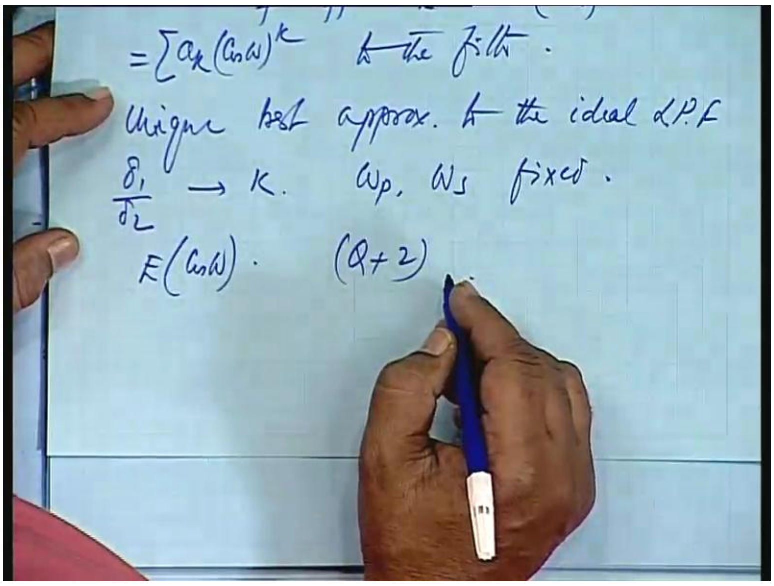

Then E cosine omega will exhibit at least Q plus 2 number of alternations on F p. It can be more all right, What can be the maximum value? Can it go to infinity? It should be at least Q plus 2, what could be the higher value? 2 Q plus 1? What should be the maximum number of maxima and minima that is; this error, error will be hitting maximum value and minimum value, how many times?

If your order is fixed, suppose it is nine point sequence that means, there are how many values? Four, four coefficients, is that all right or five coefficients? We can always normalize it. 1 plus something, okay. Suppose, we have five coefficients or four coefficients then what should be the value of, maximum value of alterations? The alterations can be let us see, how many points. Q plus 3? Q plus 4? Let us see.

Before we go to that let us also just sketch; what will, what will be the functions like say P x for example, let me take a little bigger page.

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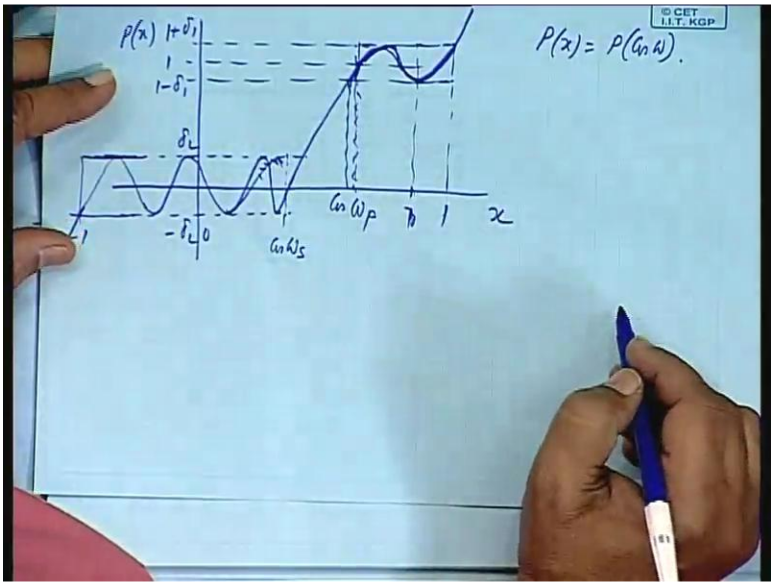

This is x, this is P x. What will be the P x value? P x is cosine omega, all right. P x is basically P cosine omega minus 1, 0. This is cosine omega s stop band. We are assuming that omega s is such that its value is positive; it could have been negative also, then pause band okay, then cosine 0 is 1. This is cosine omega p. This is 1 plus delta 1, 1 minus delta 1, this is delta 2 minus delta 2, all right.

So, it will be like this okay, sorry. It will be may be let me produce it a little more, make this B 1 here; because and this will be going down, remember the Chebyshev polynomial? We discussed beyond this range, it will be going up. So this is our P x values, it should have been here.

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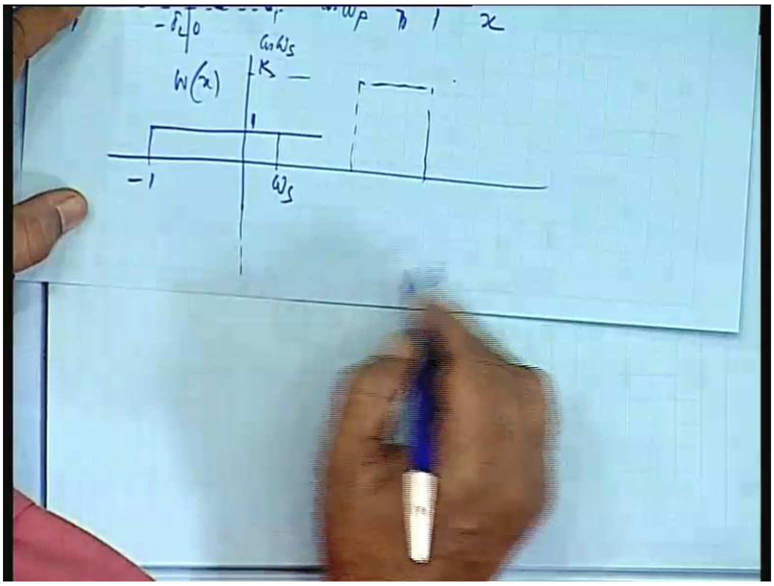

What will be the variation of k rather the waiting factor w x? It is from minus 1 to this omega s, it is at 1. And here it is k w x, any question? And E x, it will be too crowded.

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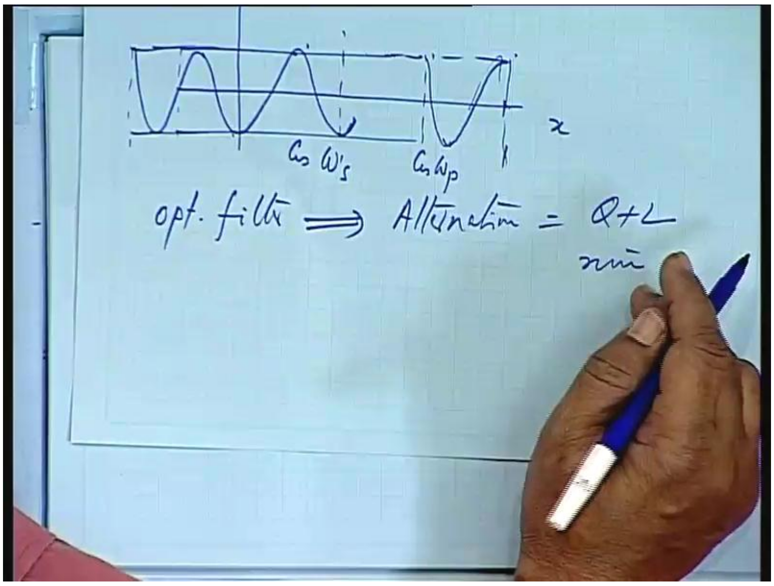

E x the error will be something like this. So like this; say this is omega s, cosine omega s. This is x and then again this is cosine omega p, this is 1, this is cosine 0. It may be like this or it could have been this way also. What I mean, that after multiplying by that waiting factor w x the error modified, waited error is this all right? That varies between plus 1 and minus 1. It is this function which is a foreign trust, all right is this all right? What we had expressed earlier if you remember, E x is sorry, this waiting function multiplied by this derivation;

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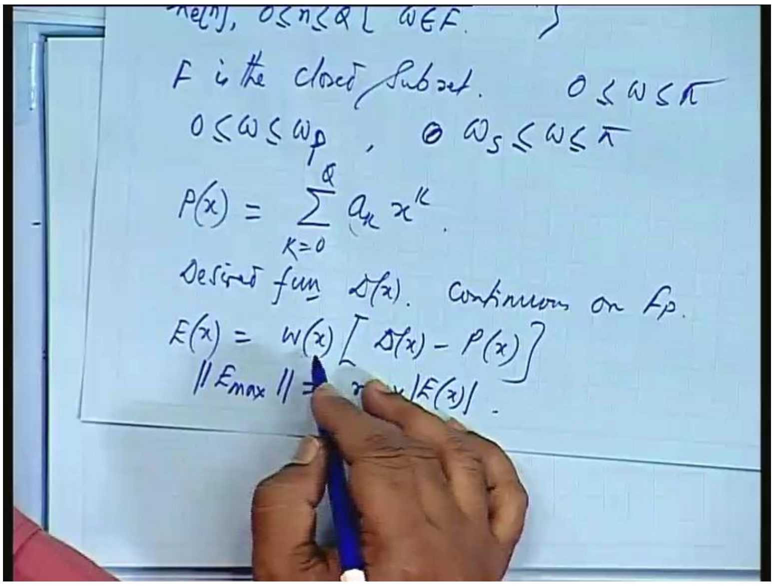

so this derivation after multiplying by this w x which is k, I mean 1 by k and 1, so we get a normalized error. So, alternation theorem states that optimum filter must have a minimum of Q plus 2 such alternations. Minimum number of alternations will be Q plus 2. Alternations equal to Q plus 2, this is minimum.

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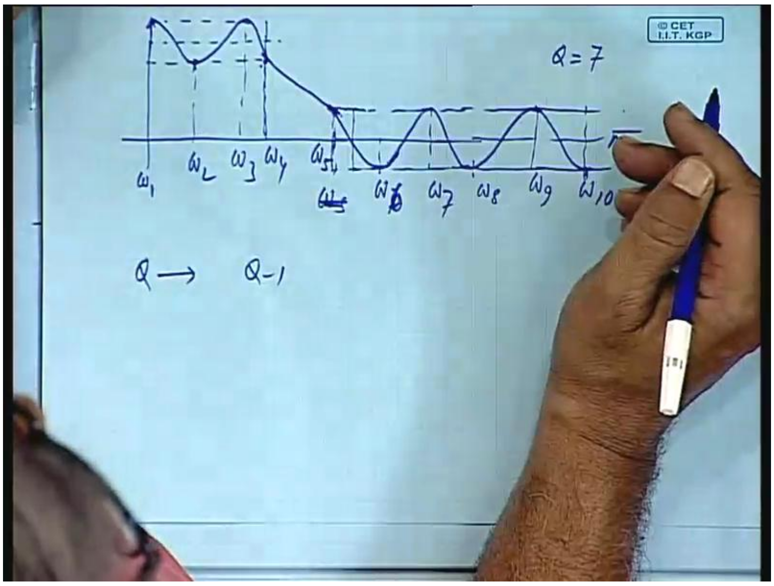

Let us sketch one or two typical examples. What are the possible values of alternations in a given set? Say if you have Q equal to 7, all right. Now, we are just going to sketch the original filter function, what it will be like? I can have starting with a maxima, okay okay. Let me write this is omega s, it enters here; it may be ending like this at pi, this is pi. So, what are the frequencies at which it is hitting the maxima or minimum values in that specification?

This is 1, 2 in this particular case, you see? Omega 1, omega 2, omega 3, omega 4, omega 5, omega 6, omega 7 sorry, omega 6, omega 7, omega 8, omega 9, omega 10. So, it is seven plus three all right, need not be seven plus two minimum; minimum mind you I just wanted you to raise this point, why should it be so many?

See, these are also the border cases. If you talk about these specifications in terms of the error functions; these are hitting the maximum or the minimum error, is it not? So, though the function does not have an absolute maxima or minima here but these are also the boundaries, okay. Now in a in an open set, how many; say a polynomial of degree Q, polynomial of degree Q, how many maxima and minima will it have? In the open set, Q minus 1 all right. And then you can have four extremes; 0, pi, omega s and omega p, all right.

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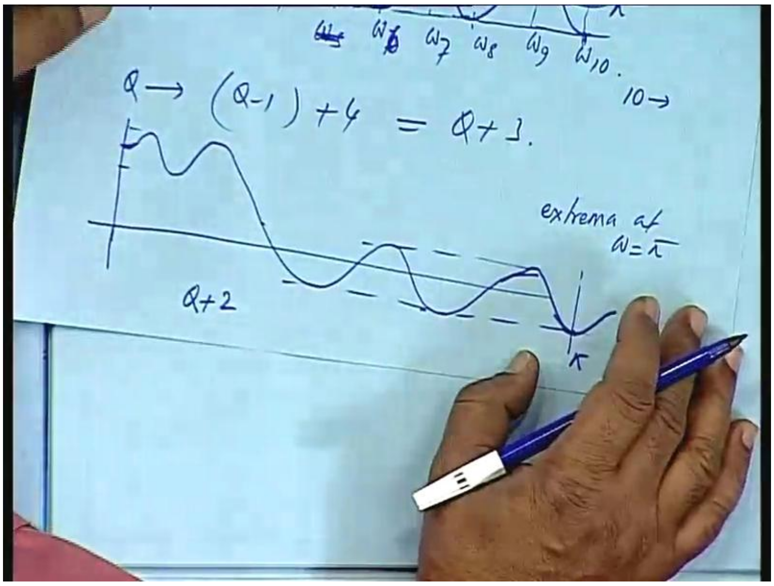

So, Q minus 1 plus 4 extreme that will be the maximum possible number of alternations you can have. So it will be Q plus 3. So, at least Q plus 2 means at the most Q plus 3, is that all right? So, let us sketch one or two more possibilities. This is extra ripple case where we have 10 such alternations.

You may have extrema at omega equal to phi only, that means it will not start with a minimum or maximum; it will start with somewhere in between and then it will be 1, 2 okay ending with a extrema at pi, okay. So, here you will find it would be one less that is Q plus 2. It may start with a maximum okay and may not end with an extreme point at omega equal to phi then also we will have 3. So maximum possible is Q plus 3, otherwise it should be Q plus 2.

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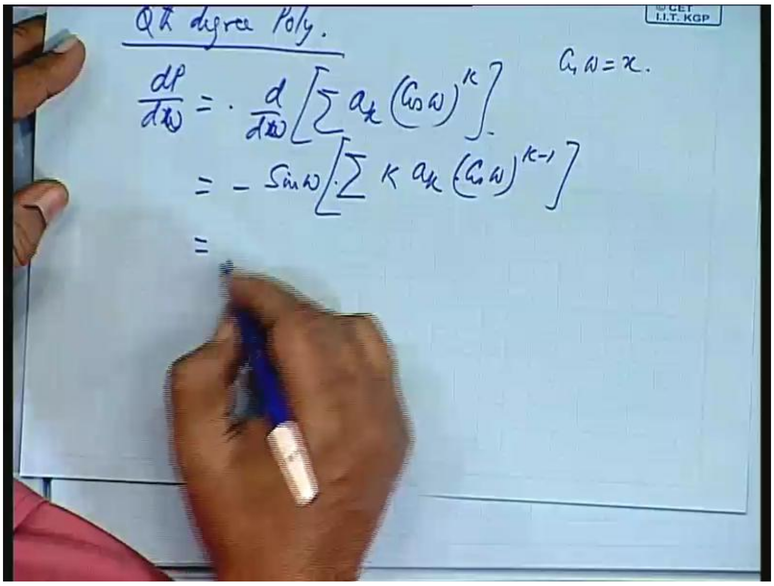

Now, Qth degree polynomial; you are having d p by d x equal to cos omega, okay. We have taken cos omega equal to x. So, d p by d x cos omega is x basically to the power k summation if we differentiate; it will be minus sin omega into k into a k cosine omega to the power k minus 1. Pardon, if I differentiate with respect to x, it will not be there if I differentiate with respect omega, it will be there, yes.

If I have d p by d omega okay, it will be minus sin omega, check whether you get this kind of a relation.

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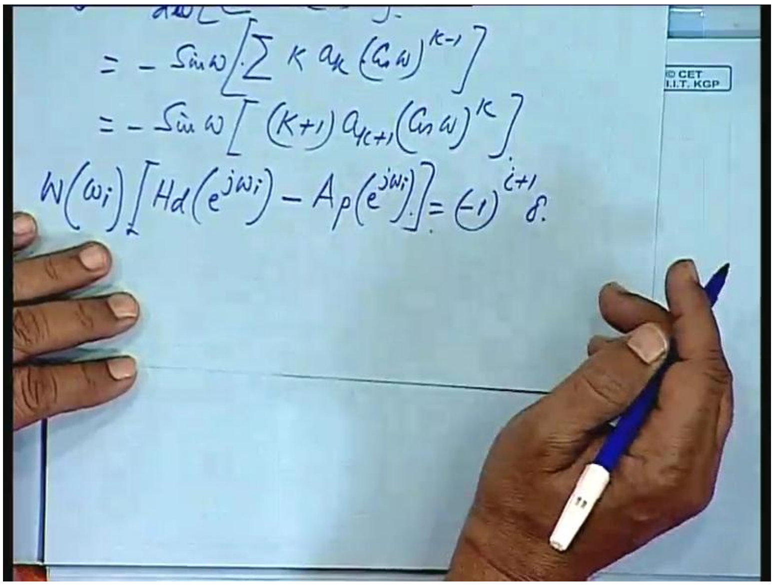

The algorithm that we are going to have now is a very simple, because we have already establish this relation. Say W say let us take any frequency W 1; H desired okay minus that function A p which we call the amplitude function j omega 1, equal to this is basically the error function, should be minus 1 to the power i plus 1, okay.

If you, if you take any ith value instead of taking 1 ith value, then i plus 1 delta. You have seen that normalized error. Normalized error is same okay. So, any value if I want to calculate it is negative of the previous value or alternately it is minus 1 to the power i plus 1, okay. So, this equation I can write in a matrix form for all the frequencies. So, it will look like this.

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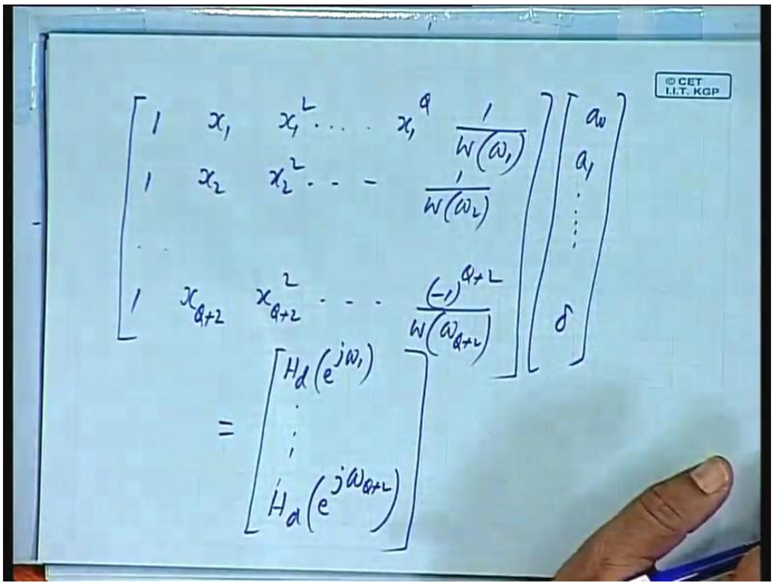

You translate this into the matrix form 1 x 1, x 1 square, x 1 to the power Q 1 by w omega 1. 1 x 2 x 1 x 2 x 3, these are the solutions all right; those frequencies that is cos omega 1, cos omega 2, cos omega 3 and so on. x 2 square and so on, 1 by w omega 2. 1 x Q plus 2, x Q plus 2 square minus 1 to the power Q plus 2 by W omega Q plus 2.

This side will have a 0, a 1 and last only delta equal to H desired e to the power j omega 1 and so on. This H desired e to the power j omega Q plus 2. So, x 1, x 2, x 3 you have already solved, you can calculate a 0, 1. Now, you will find there are similar algorithms for other type; that is type 2, type 3 filters and programs are also given. You can, I would like all of you to try this simple problem, with a bike ward say; b 1 z inverse 1 plus b 1 z inverse plus b 2 z to the power minus 2 divided by 1 plus a 1 z inverse plus a 2 z to the power minus 2 into, it has fallen into K.

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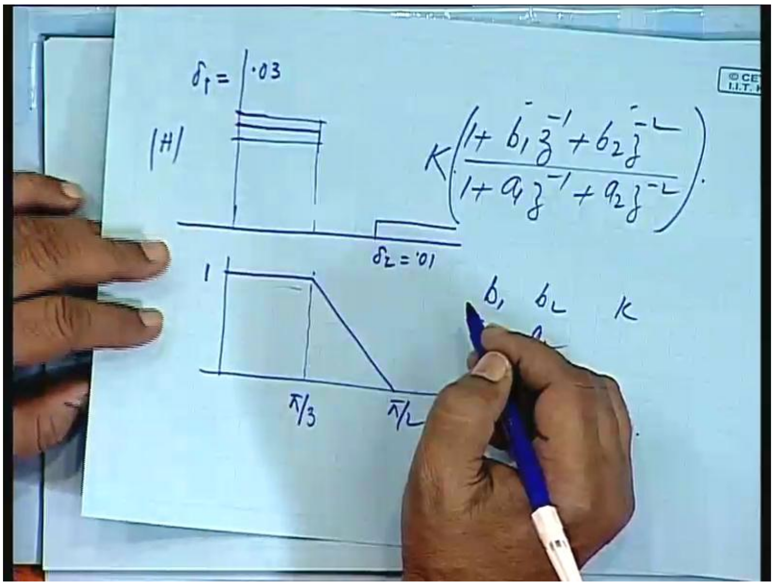

Try by the Greek stad technique; by the Greek stad technique, find out the filter H which will meet this specification, okay. We can take a few standard functions say may be this one; take any frequency say may be pi by 3 and pi by 2. This is 1, all right. We are not putting any restriction on the tolerance at this stage. What would be the values of b 1, b 2, a 1, a 2 and k; such that you get this type of characteristics, all right?

Minimize the error; minimize the error square sum okay. And then try also this problem that is for an FIR filter, this is for a general IIR filter which we discussed in the last class. And for the same one take delta 2, delta 1 to be 0.3 and delta 2 to be 0.1; for these specifications design a filter of length 6, Q equal to 6 by this parks McClellan’s algorithm, this one, okay.

So, I will leave it to you as an exercise and you submit it along with other problems that I had given you earlier or some more problems I will be giving you, may be next week also; sorry not next week, week after next, okay. I think I will stop here about this computer aided design. The other design that is left is based on FFT, I will just give an introduction to FFT.

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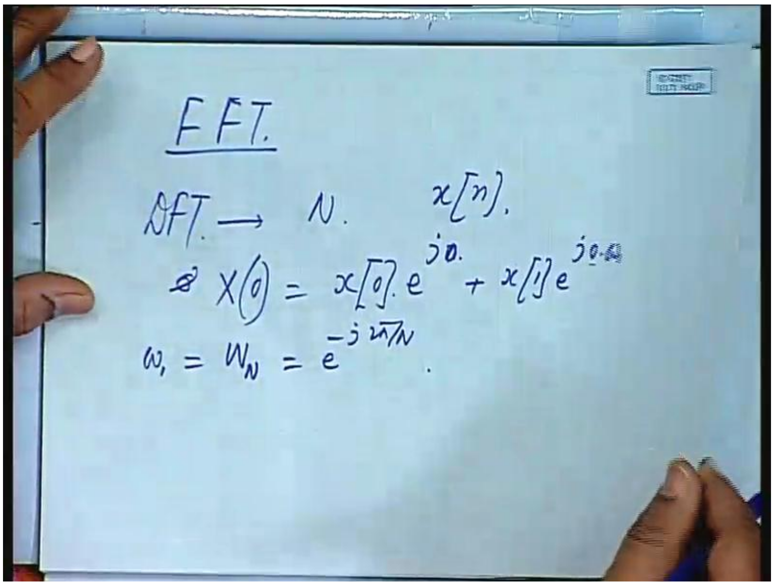

The DFT's that you have been computing, say we take a points N number of points, what will be the number of computations? How many additions, multiplications will be involved in this? Let us have an N point set then see for example; if I have a sequence x n then X 0 is, how do we compute? x 0 into e to the power j, 0 plus x 1 into e to the power 0 into omega and so on you keep on writing. And in DFT this steps of omega will be, omega N which is e to the power minus j 2 pi by N, okay; these are basic amplitude.

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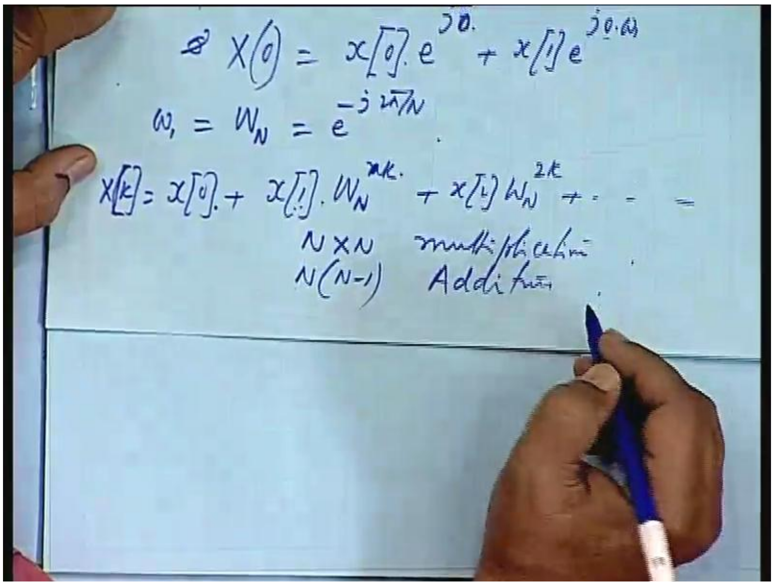

So, you get x 0 plus x 1 into W N n k sorry, 1 k plus x 2 into W N 2 k and so on. So, how many multiplications are there for computation of any X k, for computation of any X k, how many multiplications are there? This is multiplied by W N 0, there is a multiplication here; so how many such terms are there, N number of points, is it not?

N points having N multiplications and then for N number of such components. So, there will be N into N multiplications in total, is it not? We have to calculate capital X 0, capital X 1, capital X 2. So, N such components will require N into N multiplications. How many additions are there? N into n minus 1 because additions will be just 1 less N minus 1, additions; so N into N minus is 1, so many additions all right.

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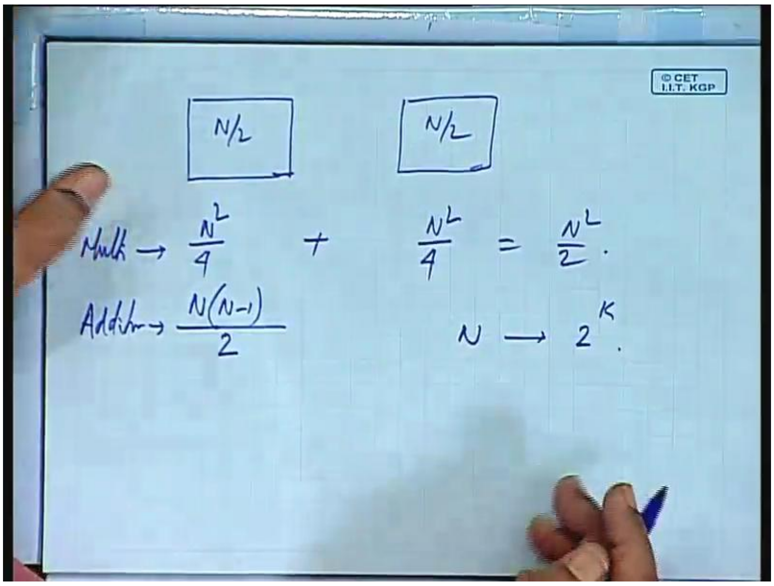

Now, let us see if I have N by 2 numbers of points then how many multiplications are there? N by 2 square again N by 2 square; so this plus this so many additions that means N square by 2, okay. So, this is the number of multiplications. Additions similarly N into N minus 1 by 2 by 4 and by 4, so again added together it will be additions of this order.

So, earlier it was N square, now it is N square by 2. Earlier it was N into N minus 1, now it is N into minus by 2. So, we have reduced the computation burden okay. We have reduce the computation burden by having the set and computing these separately, but then there will be also some addition operation when we mix them together, all right.

So that will not be the, burden will not be increasing again back to the same figure. It will be a little more. So, there is a substantial amount of reduction of computation labour, if we can half the sequence length. If you go down further, you can reduce it further. So, how long can you go? So, if you keep on halving it; that means it should be the original number, N should be an integral power of 2, okay.

So, if you have eight points, I can have it into two force then again each sequence of 4 can be halved into two point sequence. When it comes to a two point sequence then it becomes a very simple. What will be the FFT of a two, a DFT of a two point sequence? Say, H 0, x 0 and x 1, these are two points.

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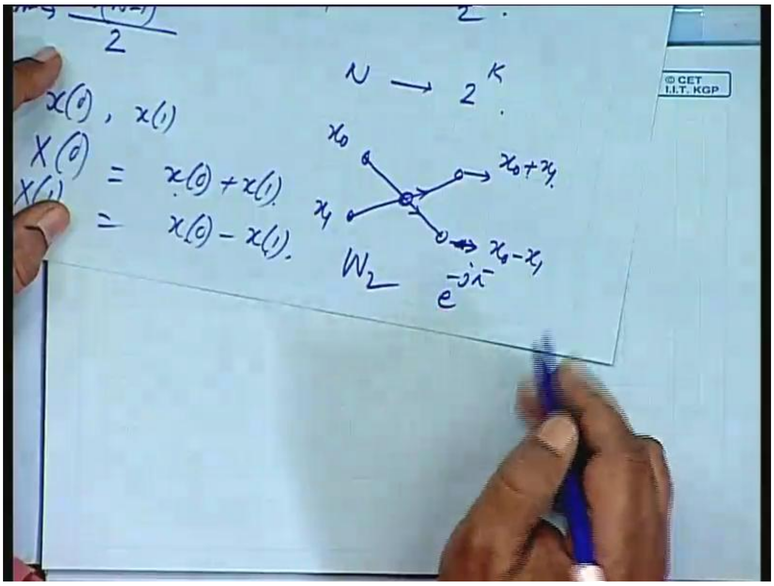

Then what will be capital X 0 and capital X 1? x 0 plus x1 and x 0 minus x 1, very good. So, it is simple addition and subtraction of two quantities, all right? So, we will start from this bottom end, we call it simple butterfly. If I have x 0 and x 1 as the input, the upper one we will take x 0 plus x 1, the lower one we will take x 0 minus x minus x 1, okay.

So, this is a butterfly structure where addition and subtraction will result at the output stage okay. So, taking this butterfly to a next higher level will be involving now; complex multiplications and additions, okay. How do you get minus and plus? It is basically W 2, is it not? e to the power minus j pi, all right. Now, if I have going for a four point sequence then it will be e to the power j pi by 2, e to the power j pi by 2 will involve j, plus j, minus j, plus 1 minus 1, okay.

So, in the next we will have multiplication by imaginary and real numbers both next to next step, it will be forty-five degrees. So, these are the different multipliers that will grip in and we will have to make keep on making additions and subtractions. So, in the next class will take up this algorithm. Thank you very much.