Digital Signal Processing
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Lecture - 19
IIR Filters (Contd.)

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Good morning friends, we shall continue our discussions on IIR filter.

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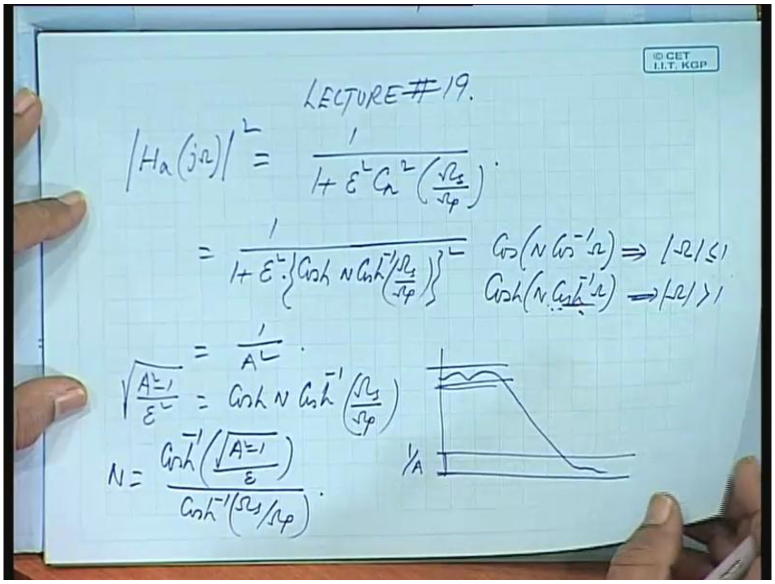

If you remember, yesterday we are discussing about chebyshev filter. Now, we can write the chebyshev function in the analog domain as; 1 by 1 plus epsilon square C n square okay, omega s by omega p is a function of omega s by omega p. This is in terms of stop band and pass band ratio. So, this can be written as 1 by 1 plus epsilon square, cosh as at I had mentioned yesterday; cos N, cos inverse omega for omega less than 1 and it is cosh N, cosh inverse omega for omega greater than 1 and both of them generate the same polynomial functions.

So, cosh N cosh inverse omega s by omega p whole square, C n square mean, this is C n and that is equal to 1 by A square where if you remember; we had mentioned in our earlier characteristics, the quantities A and epsilon, I will just refer back to the earlier notes. So, this was given as 1 by A and this was the tolerance in the pass band, okay. So, will refer to the same quantity A and that will be equal to 1 by A square when it is in the stop band, okay.

So, A square minus 1 by epsilon square is equal to cosh N cosh inverse omega s by omega p, which gives me N equal to cosh inverse root over of A square minus 1 by epsilon divided by cosh inverse omega s by omega p, okay. So, this is how you determine the value of N which is equal to cosh inverse 1 by k 1.

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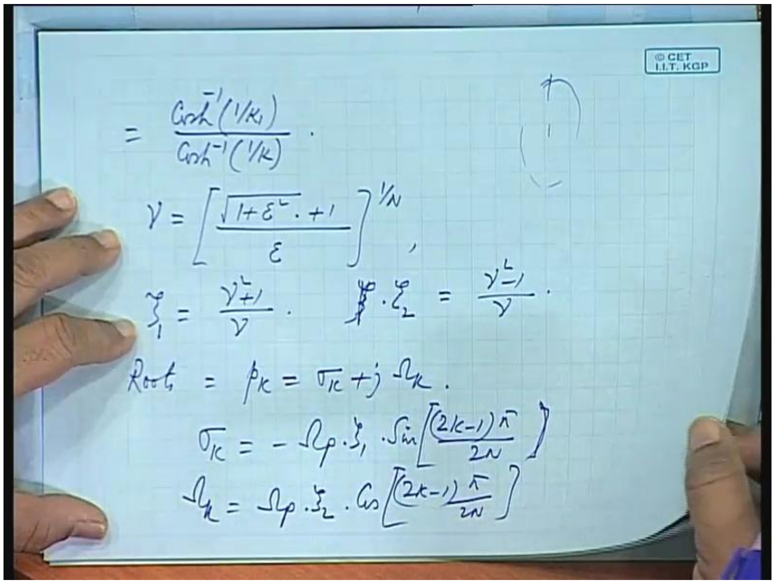

You defined the two quantities, k 1 and it comes like this. So, the polynomials can be framed, you have to get the roots. So, say gamma equal to; basically the chebyshev roots will be lying on an ellipse okay, so if you are given epsilon, if you are given epsilon then you can write this 1 plus epsilon square plus 1 divided by epsilon to the power 1 by N, zeta equal to gamma square plus 1 by gamma.

And they are two different entities; it becomes little difficult, I will call it 1 and 2 to distinguish because our hand writings are not matching with the prints. Our roots will be p k equal to sigma k plus j omega k; these are the roots k th root, so sigma k is given by minus omega p times zeta 1 sin 2 k minus 1 into pi by 2 N, okay. And omega k is equal to omega p zeta 2 cosine 2 k minus 1 into pi by 2 N, okay. Basically, they represent major axis and minor axis of the ellipse omega p times this.

There are other ways of looking at this chebyshev polynomials. There are many books where they refer to the direct design tables and we will discuss that at a later stage. Let us come back to what we are discussing yesterday. I had given you a small problem to try at home, did you try that?

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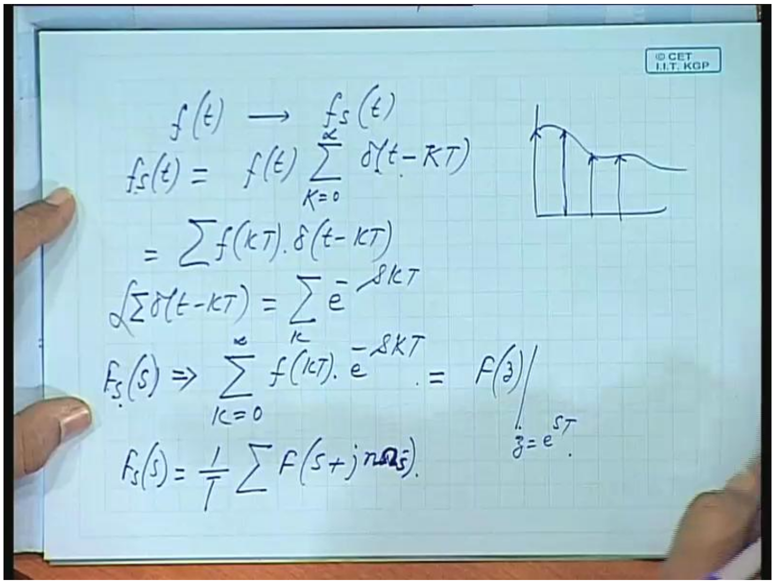

You can see, f t we are considering in the discretized form as f s t, okay. I can write, say sample values of a continuous function as f t multiplied by a series of delta functions at regular intervals of capital T, T is the sampling time, okay; k varying from 0 to infinity. So, if this is your f t, we are multiplying by delta functions.

So, these are the sample values. So, delta functions of unit magnitude at regular intervals that will give me; f k T as the weightage of that delta function, I can write like this. Now Laplace transform of delta t minus k T is e to the power minus s k T okay summation, is it not? These are shifted function, shifted impulse and series of shifted impulse; so it will be giving me a summation over k, this functions okay.

So, if I call the Laplace transform as F subscript s; this will be Laplace transform of this okay which will be f k T e to the power minus s k T, do you all agree? So, this is nothing but z transform, evaluated at z is equal to e to the power s T basically, is it not? This is by definition, z transform; if I substitute for e to the power s T z then this becomes z transform. So, F s s will be 1 by T summation, this is a proof that I asked you to try out okay. Let me write n s omega s T; instead of that I will write capital omega S, this is sampling frequency in the analog domain.

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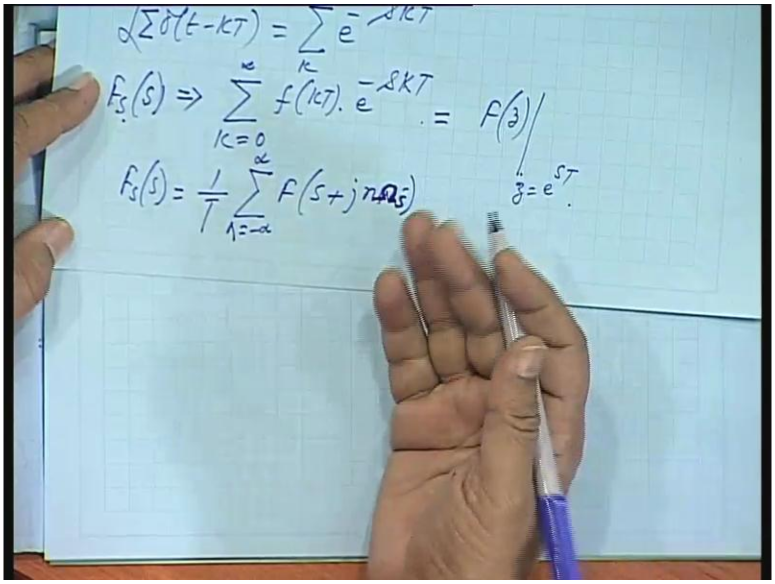

Will do that, that means whatever value of S you are having; you replace s by s plus j n omega s and n varying from minus infinity to plus is infinity, okay. This is what we have got, basically this is the the situation from where we are getting aliasing.

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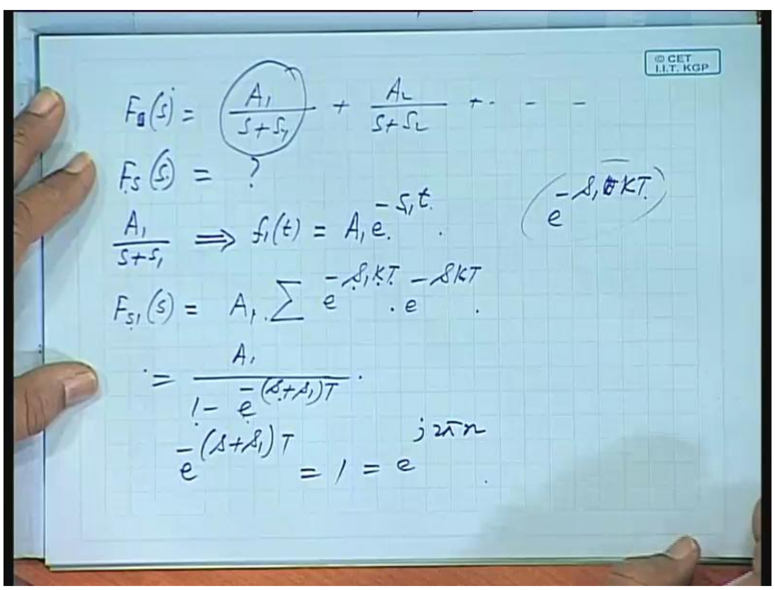

See let F S, F S be A 1 by S plus S 1 plus A 2 by S plus S 2 and so on. Let us consider only one of them. We are trying to find out, what would be F S by F S S. We have seen, F S S will be, F original F S replaced by S plus this with a with an average. So, you want to derive this relation. So, let us consider only the first term, okay.

So, A 1 by S plus S 1, what is its corresponding time domain response? If I call it f 1 t, that is equal to A 1 into e to the power minus S 1 t, agreed? So, F S 1 S that means corresponding to this, what will be F S? It will be A times A 1 times discretized version of this. If I take e to the power minus s 1 k T e to the power minus s k T, do you agree? Is that all right because, this is e to the power minus S 1 t; we are taking Laplace transform of the discrete version of this, mind you s 1 k T.

I have just put t equal to k T, so these are the magnitudes and this is the; just in the previous step we have done, f k T into e to the power minus s k T. So, what is f k T? The magnitude of the function that is e to the power minus s 1 k T, is it not? Because the response due to this is e to the power minus s 1 t that we have seen; so if I discretized it k T, this is the specific value at that point. So, this summation will give me A 1 by 1 minus e to the power minus s plus s 1 into T, is that all right?

So, a term like A by S plus S 1 will give me, correspondingly here A by 1 minus e to the power minus s plus s 1 into T, for the sample function. So, what are the roots of this? What are the poles of this? That means corresponding to a pole at S equal to minus S 1; the set of poles that will be generated here will be the poles for the discretized function. So, we are trying to find out the poles of this. So that means, equate this to 0 so e to the power minus s plus s 1 into T is equal to 1, all right is equal to e to the power j 2 pi n okay.

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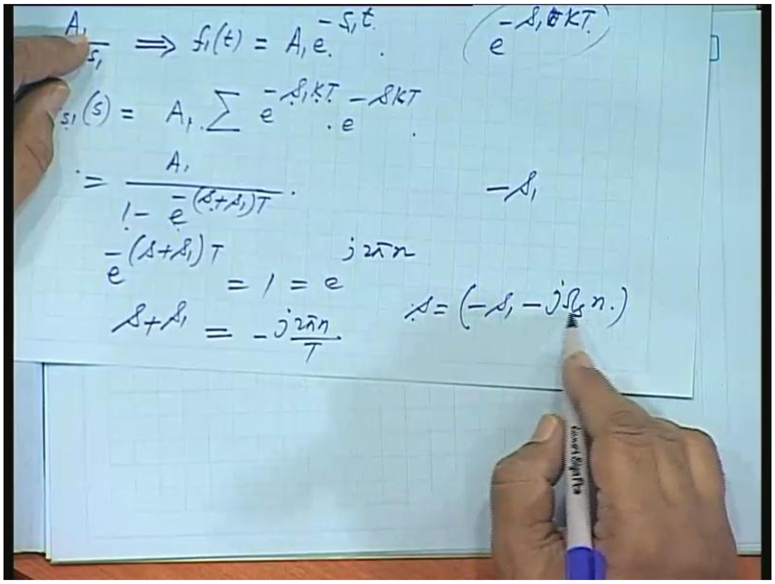

So, equate this two. So, s plus s 1 is basically j 2 pi n; actually plus or minus is immaterial, because I can always take negative integers also, 2 pi n by T okay and okay. Let me put it minus then s is equal to minus s 1, minus can be plus or minus j 2 pi by T is omega sample omega s, all right. So, s is equal to minus s 1, the root at minus s 1 will generate minus s 1 minus or plus n times j omega s, what does it mean?

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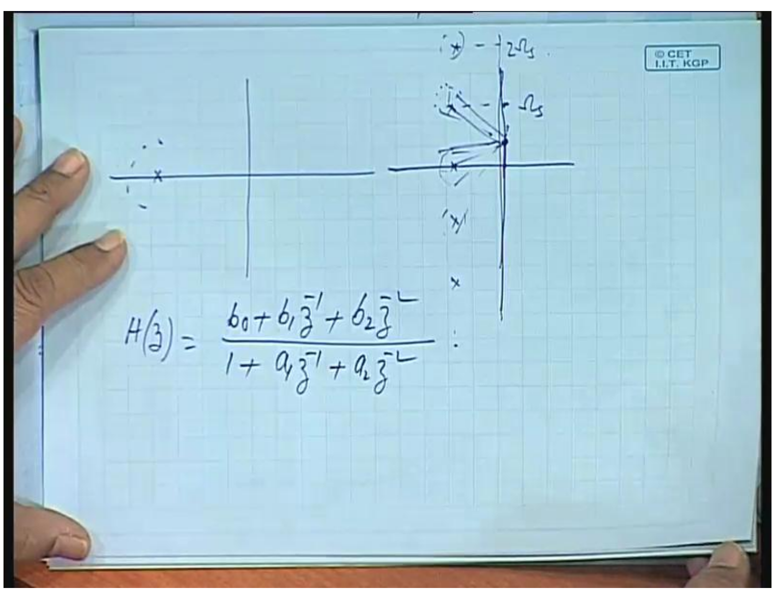

That means in the original time domain; if there is a pole here then for the discretized version this is for the continuous function, then for the discretized version in the s domain, we will have poles and infinite number of such poles at regular intervals of omega s, okay. Now, the effect of when I am travelling along this, you traverse the entire omega s of the imaginary axis. So, if I go along this at any point, what is the amplitude? I compute the amplitude by taking the zeros distances of zeros and poles.

Suppose there are zeros also, so there will be zeros correspondingly multiplied here; we are concentrating on one single pole, there can be a large number of poles in the family, large number of poles and zeros for the original one, so the entire set will be repeated. Now, at any point if I want to find out the magnitude in the discrete domain; if I want to find out the magnitude of the function then all such distances of poles and zeros will have to be taken out, then these distances will also influence, these distances will also influence, all right.

So, at any frequency if you consider the distances of poles and zeros and take that ratio of the products, that will decide the magnitude and if you do not eliminate the influence of these; that means they have to be widely separated okay, they are to be widely separated that means this omega s should be very very large, that means sampling frequency should be large, otherwise there will be aliasing. So, this is the logic for having omega s very large.

Now we will take up some examples, before we go there; there was one question raised after the class that is, can we avoid we are giving examples of impulse invariance and step invariance, for step invariance we found there are two delay elements all right, is it necessary to have two delay elements? So, it is not. So, suppose you have a function say H z, I will take a simple biquard; 0 plus b 1 z inverse plus b 2 z to the power minus 2, sorry okay. What would be the structure for this kind of function?

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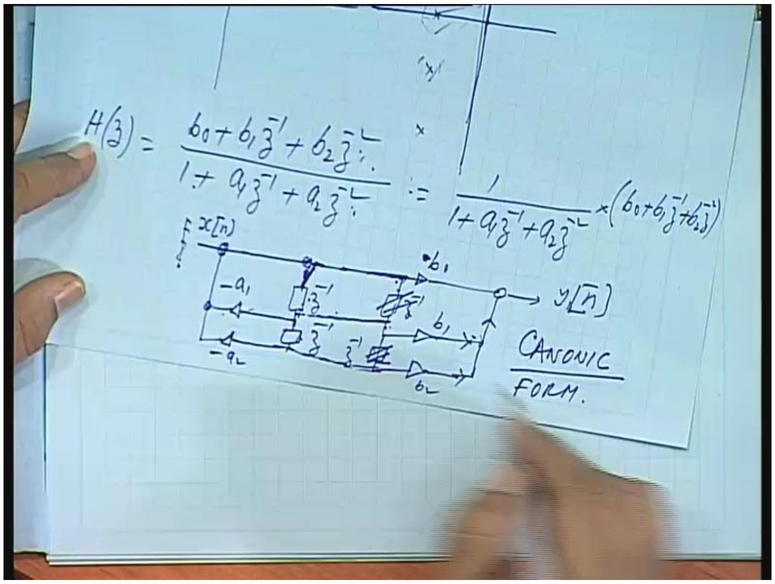

Now, I can segregate this into 1 plus a 1 z inverse plus a 2 z to the power minus 2 into another part b 0 plus b 1 z inverse plus b 2 z to the power minus 2 okay; if I consider this as part one okay, then I will have suppose x n is the input z inverse then I can have minus a 1, similarly minus a 2 another delay block z inverse and they are summated, then again they are summated here, okay.

Now, summation we normally note by a small circle and even the gain, we show by an arrow okay. Then on this side, you have b 0 z inverse b 1 then again z inverse b 2; added together that will give me y n okay, is the directional flow of information, sorry it not, minus. Now, since these two elements are appearing from the same signal, I might as well club them together. This is delay of this output, basically it is here; this is the delay of this output. This is also delay of the same signal.

So, this can be breached and I need not have these two delay elements separately. So, for a bi-quad I need two delay elements, all right. So, the number of delay elements in this arrangement will be depending on the highest order of the polynomials, either in the denominator or in the numerator, okay. So, this is known as the CANONIC form means; requiring minimum number of delay elements. So with this canonic structure; in the earlier case that is for step invariance will require only a one element. Let us take some examples for filter.

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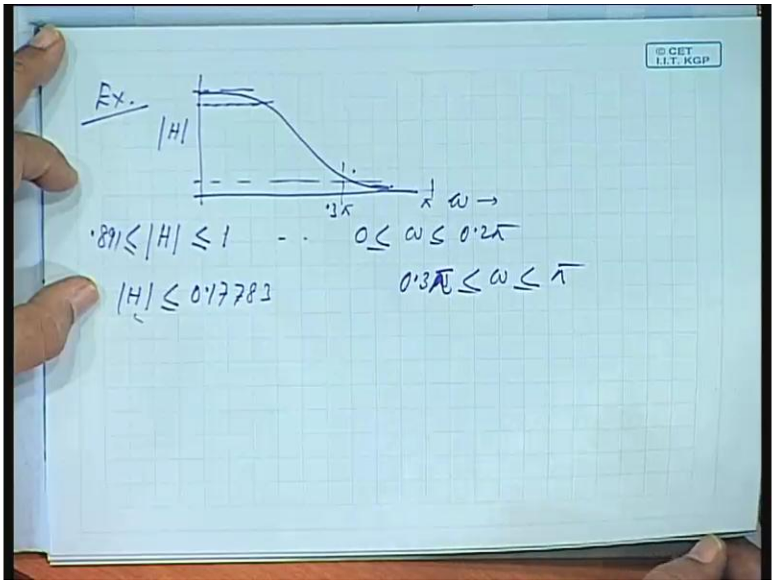

Let us have a filter with these specifications; a Butterworth filter, first is to be designed to the specification that magnitude of H is less than equal to 1, on the other side it is 0.891 that means between 1 and 0.891, okay. And this is for 0 omega 0.2 pi. And H magnitude is less than equal to 0.17783 for omega between sorry, 0.3 pi to pi, okay. That means from say 0.3 pi; suppose this is 0.3 pi up to pi, it should be within this value less than 0.177.

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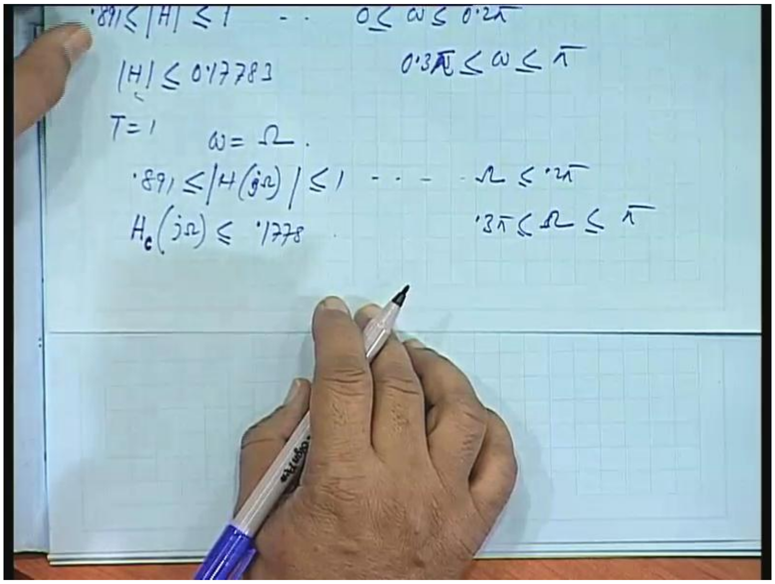

The specifications can be given in terms {abs} (00:22:58) absolute magnitude of the gain; some times in the logarithmic gain okay, so by impulse in variance method we are going to design this filter first. We can see in the impulse in variance, there is T coming there; so T will get cancel when we convert it, all right. So, we can take T equal to 1. So, we will put omega is equal to capital omega. So, these specifications we can write in terms of capital omega.

So, 0.891 is H j omega, all right and H j omega is less than equal to 0.1778 this is in the pass band okay; that is omega less than 0.2 pi and here 0.3 pi omega pi. So, H this is in the continuous domain. So, I will write H c okay, what is given was in the discrete domain; in the continuous domain by putting omega equal to small omega equal to capital omega, we are going into the continuous domain.

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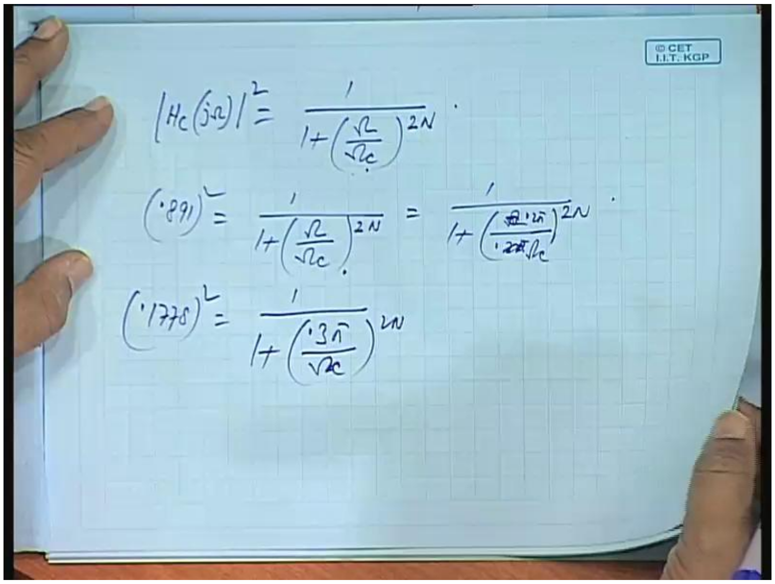

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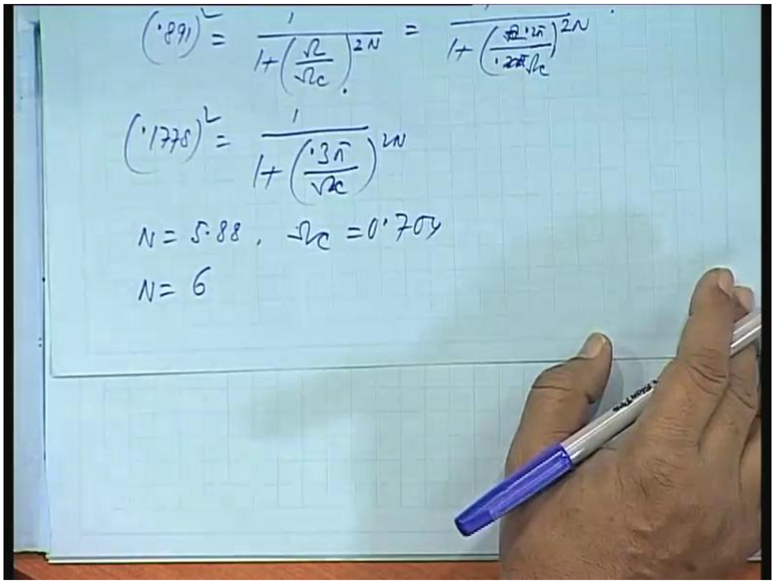

So, H continuous domain; the function is 1 plus omega by omega c to the power 2 N, this is the Butterworth filter of nth order. Now, we have to compute the two unknowns are omega c and N. So, you have compute these two quantities with these specifications. There are two inequality constants. So if you equate these, that is we are trying the border values; if you equate these two then 0.891 square, this is 0.89 square is equal to 1 by 1 plus omega by omega c to the power 2 N where omega c is corresponded point 2 pi, okay, the other way round, sorry 0.2 pi by omega c, is it not? Omega c is unknown.

In the other case it is 0.1778 square equal to 1 by 1 plus 0.3 pi by omega c to the power 2 N. If you solve these two; you get N is equal to 5.88 and omega c equal to 0.704 okay. So, you choose N is equal to 6, nearest integer.

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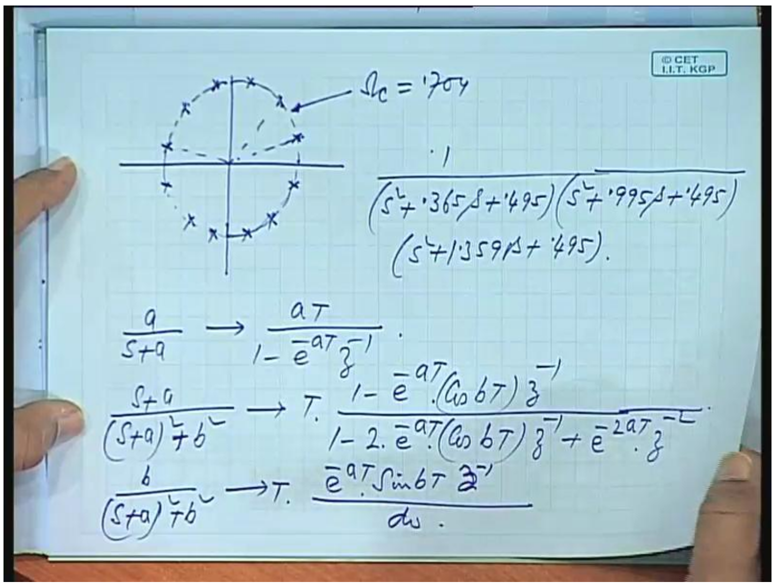

So, N is equal to 6 will give me the roots placed at an angle of three twenty, three sixty degrees divided by two into six, twelve; so thirty degrees, all right. So even number of roots, if you remember; discussed earlier even number of roots, six roots on this side, so there will not be anything on the real axis. So, at fifteen minus fifty degrees, fifteen plus thirty; forty-five degrees minus forty-five degrees, forty-five plus thirty, seventy-five degrees then again hundred and five, fifteen degrees; hundred and five degrees, hundred and thirty-five degrees and so on.

They just mirror images okay, one two three; so one, two, three, four, five, six, six on this side, six on the other side. So, you can form the polynomials and what is a radius of the circle? It will be point 0.704; you know it is the cut of frequency. You know it is normalized against this omega c; so this is of radius omega c equal to 0.7 naught 4, all right. So, you know the roots cosine of 15 degrees of this side, cosine of fifteen degrees with a negative sign and sin of 15 degrees and 45 degrees also and so on.

So, the polynomials if you take the pair of roots at a time, you will get S square plus 0.365 s plus 0.495, after de-normalizing s square plus 0.995 s plus 0.495 into S square plus 1.359 s plus 0.495, okay. Now for if you remember; we had earlier discussed, if I have a function a by S plus a, its corresponding discrete domain representation is 1 minus e to the power minus a T z inverse okay, all right.

Then S plus a by S plus a whole square plus b square, exponentially decaying cosine function corresponding z transform is; T into 1 minus e to the power minus a T cos of b T into z inverse divided by 1 minus 2 into e to the power minus a T cos of b T into z inverse plus e to the power minus twice a T z inverse square, is that all right?

Similarly, b by S plus a whole square plus b square will give you, T into e to the power minus a T sin b T z inverse, okay divided by same denominator all right. So, this we derived earlier. So, will be making use of this; making partial fractions of this h s and then making substitution correspondingly. We shall get this as

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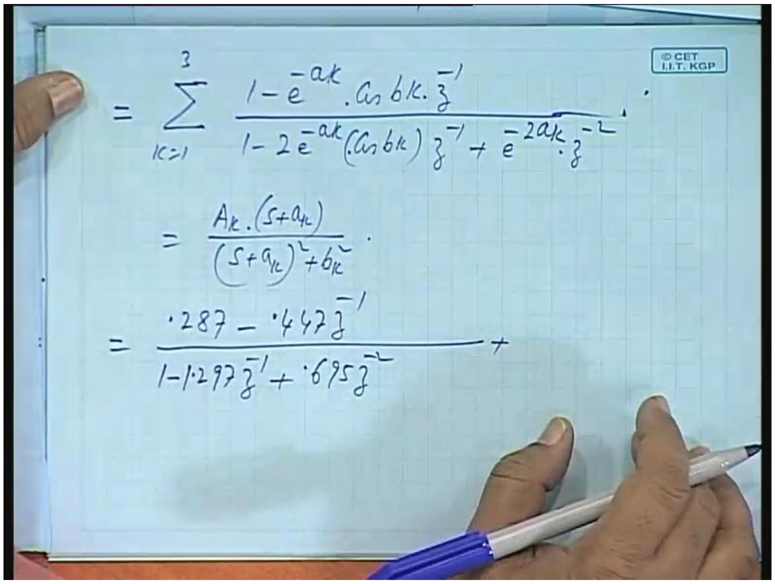

1 minus e to the power minus a k cosine b k z inverse divided by 1 minus 2 into e to the power minus a k cosine b k z inverse plus e to the power minus twice a k z to the power minus 2; k varying from 1 to 3 into, this is basically corresponding to S plus a k by S plus a k whole square plus b k square. So, corresponding to this term, this is the expansion.

So in this case in this particular example, we have already got these three factors. So, after doing this kind of partial fraction and simplification; it induces to 0.287 minus 0.447 z inverse divided by 1 minus 1.297 z inverse plus 0.695 z to the power minus 2 plus; three such similar terms I am not going to write all the details, similar terms will appear for the three factors. There are three quadratics, so there will be three such terms.

Now, the same problem, if you go for bilinear transformation; let us see what would be the effect. Now in case of bilinear transformation, we need pre-warping, all right.

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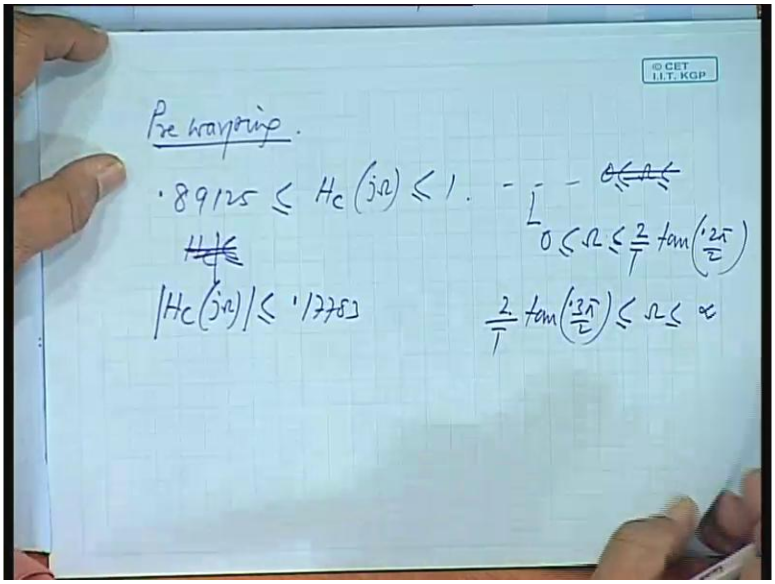

So, we change the specification for pre-warping. Let us see, how we do that. The same magnitude 0.89125 should be H in the continuous domain j omega equal to 1. And H c; this is for the pass band 0 omega and then what did you get in the bilinear transformation? What was the relation between discrete frequency omega and capital omega, small omega and capital omega?

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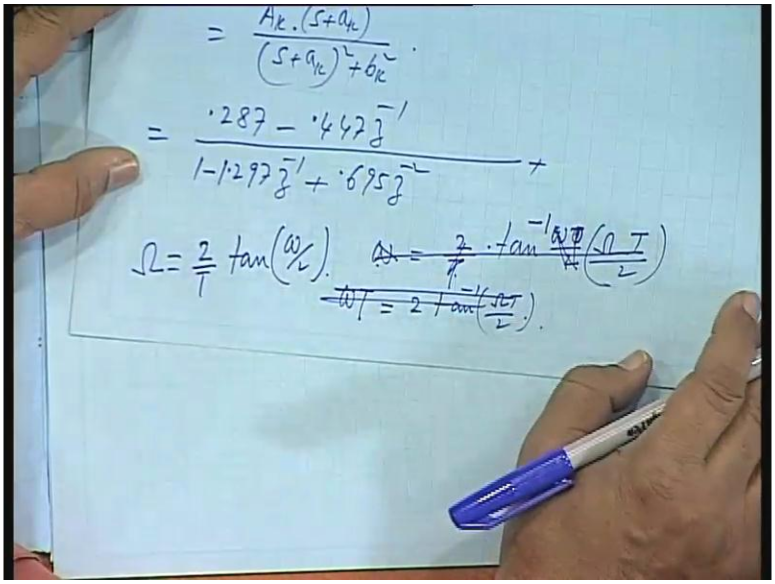

Tangent of omega T by 2, this capital omega into T by 2, is that all right? So, if we normalize if we take T to be 1 then 2 tan inverse, tan inverse or tangent? Are you sure, are you sure? Tan inverse, so omega into T is; omega into T is 2 into tan inverse omega T by 2, you mean this? Are you sure, okay. So, will apply this transformations or you can write omega, omega is equal to; it is the other way, you know.

T is not there. So, it is omega is equal to 2 by T, tangent omega by 2, all right. So, H c so this specification should be 0 omega 2 by T tangent 0.2 pi by 2, is that all right? Because omega is equal to this much, so we are replacing that omega limit of omega by this. Similarly, H c should be magnitude H c should be less than 0.17783 for 2 by T tangent 0.3 pi by 2 omega then this is infinity; earlier it was pi in the discrete domain, in the continuous domain it is infinity.

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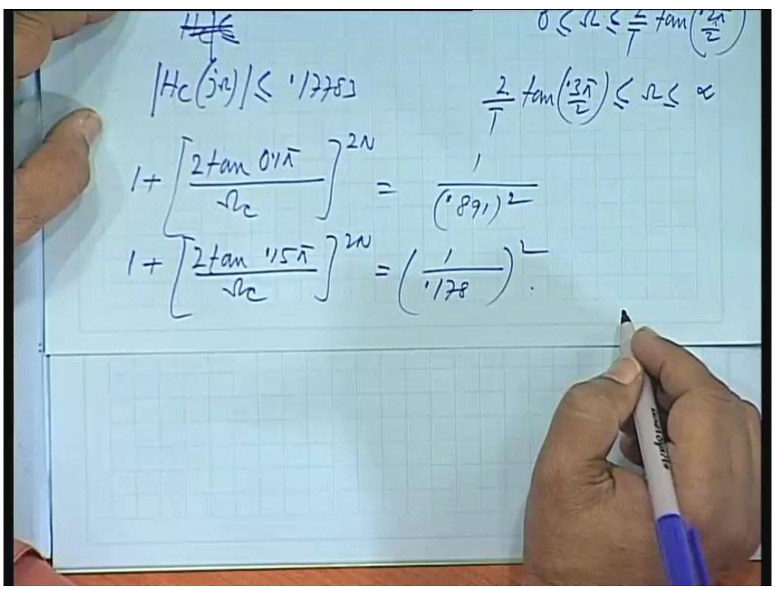

So, if you use this 1 plus let us convert this into equations, equality equations; 1 plus 2 tan 0.1 pi, this is 0.1 pi by omega c to the power 2 N that should be equal to 1 by 0.891 square, okay. This is s square and the other one is 1 plus 2 tan 0.1 5 pi by omega c to the power 2 N; should be 1 by 0.178 square, is that all right, T you have taken as 1, all right.

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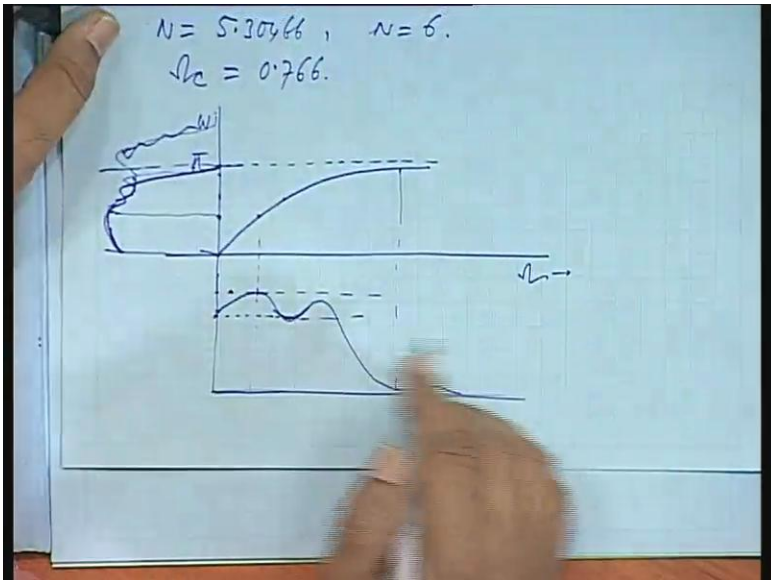

Therefore N, if you solve this two; N comes out to be 5.30466. So, we once again take the nearest integer, N equal to 6. If you substitute in the first equation then omega c becomes 0.766, okay. Now see, this is omega, this is small omega all right. This is the kind of graph we are getting; as capital omega tends to infinity, this omega tends to pi by 2, pi sorry, okay.

So, if I have a function specified like this, suppose this is the filter functions specified; these frequencies would be marked here, marked here, marked here, is marked here; so here it is much more compressed. Let me plot it to somewhat like this, sorry. It will be within pi because this entire infinite range is compressed within pi, so it will be, it will not go beyond this point. So, you can see there is a compression here, okay.

So, that is why while pre-warping, we have expanded this; so that after compression, it takes the normal shape. In any case it will get compressed, so while specifying you stretch it. If you stretch that then it is better to take the first equation. First equation N is more to meet the same same specification, if N is made more that means; omega c will also be more, this ratio has to be equated to the same constant, N is fixed. So, if you take higher values then this gets stretched, so that that is going to eat the design.

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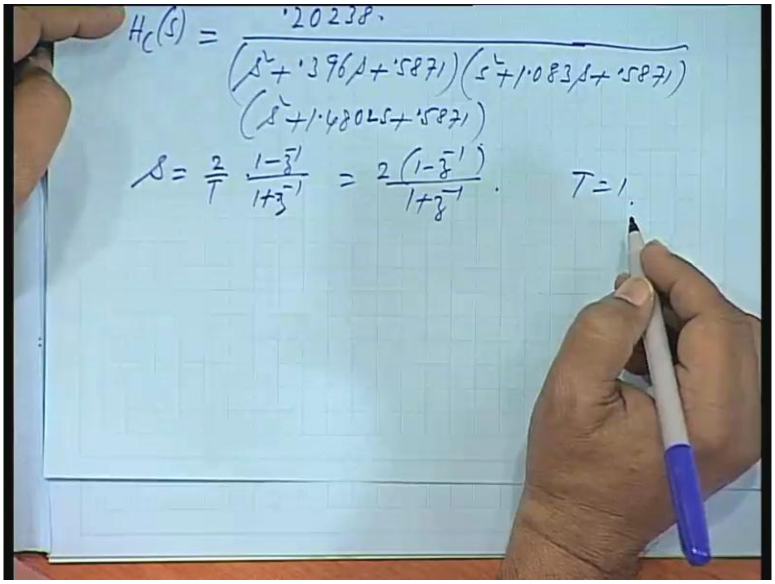

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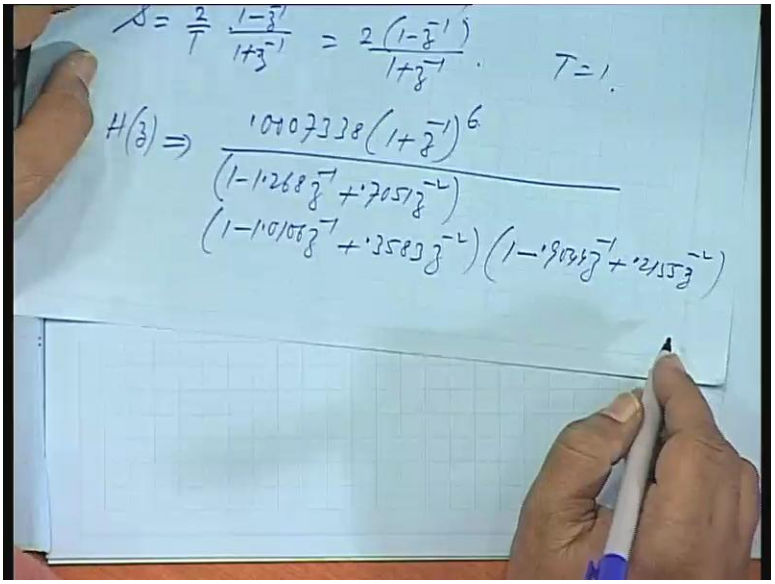

So if you follow, this H c s comes out to be 0.0238 divided by s squared plus 0.396 s plus 0.5871 then s square plus 1.083 s plus 0.5871 into S square plus 1.4802 S plus 0.5871, okay. You have got the three factors. Now, you are going to make a direct substitution, there is no partial fraction; unlike the earlier case, here you do not require any partial fraction, no substitution, no transformation in terms of impulse response.

We make a direct substitution; s is equal to 2 by T and 1 minus z inverse by 1 plus z inverse and since we have taken normalizes T, T equal to 1. So, it will be just 2 into 1 minus z inverse by 1 plus z inverse, okay. T has been taken to be 1. So, if you substitute that here, you get H z which is the discrete filter as 0.0007338 into 1 plus z to the power minus 1 whole to the power 6 divided by; see 1 plus z to the power minus 1 square square and square, so that gives me it comes in the numerator as to the power 6.

Here in the denominator you get, 1 minus 1.268 z to the power minus 1 plus 0.7051 z to the power minus 2 into 1 minus 1.0106 z to the power minus 1 plus 0.3583 z to the power minus 2 into 1 minus 0.9044 z to the power minus 1 plus 0.2155 z to the power minus 2. So you get three quadratics in z square I mean z, z to the power minus 1 and 1 plus z to the power minus 1 to the power 6 in the numerator. So, the order is 6 in the numerator, 6 in the denominator. So, you require 6 delay elements.

In the other one, in the impulse in variance, how many did you get? How many delay elements are there? There also you got 6, is not? There you have to put all all of them in parallel mode. Sometimes the specifications are given in decibel, decibel gain.

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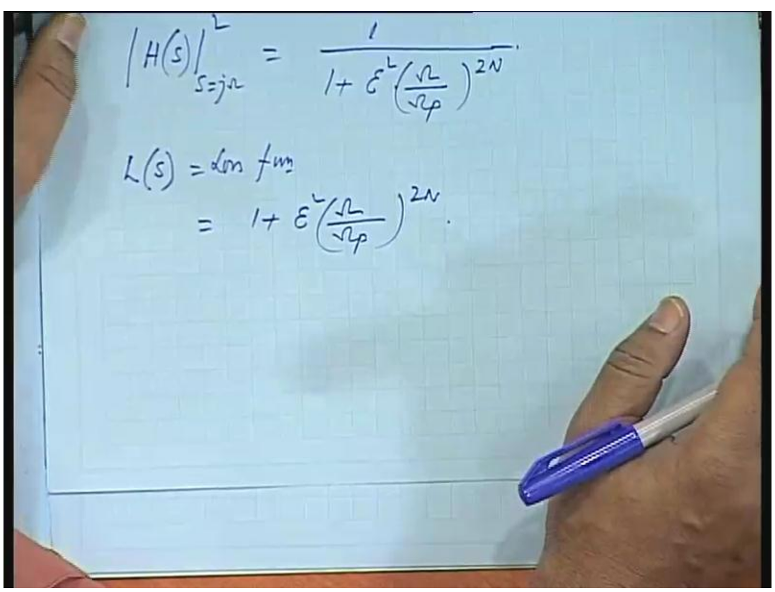

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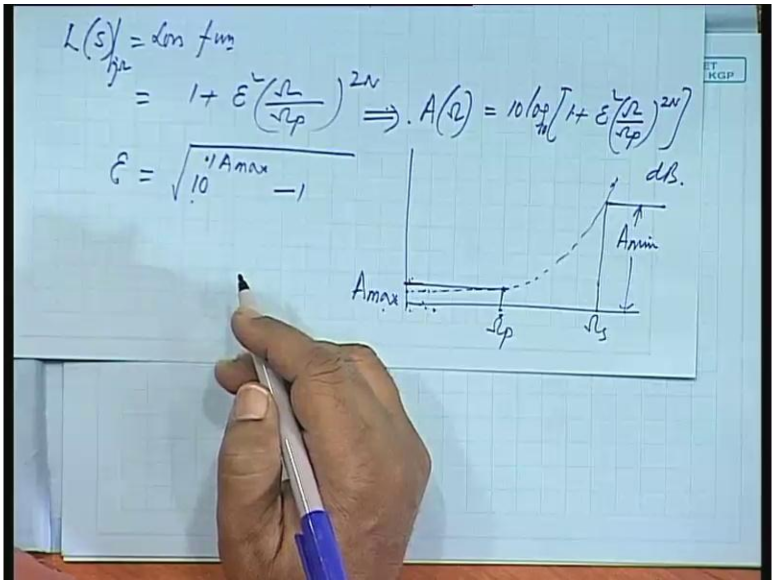

So, we may be given say; H S S equal to j omega magnitude as in the pass band, if the specifications are given in the pass band and stop band, we might as well write this in another form. Some people to some people like to take the frequency in terms of omega p, omega c is not known. So, we are not going to try to find out omega c, will straight away take omega p, okay.

So, we also define a loss function instead of gain, we define loss function. And that may be 1 plus epsilon square omega by omega p to the power 2 N. Basically, loss means in terms of power; so square of that 1 by H square, H s is basically output by input signal gain. So, power gain will be H j omega square. So, loss will be in terms of in frequency domain s equal to j omega, it will be this much.

So, epsilon is and if this loss is expressed in the logarithmic domain; so A omega will be 10 log 1 plus epsilon square into omega by omega p to the power 2 N, so many d B's all right. So, the loss function looks like this, I come to this afterwards. Loss function looks like this. Within a certain band this is the pass band; the loss maximum loss that the filter can suffer is specified A max all right, that means the gain should not fall below this point, that means the loss should not be above this.

Similarly in the stop band, we want the loss should be at least this much, it should be above this. So this is A minimum. So, A minimum is very high and A max is very very small; mind you A max is not necessarily greater than A minimum, never greater than A minimum, A minimum means in the pause stop band, this is the minimum loss, it should have. So, it is a very high value.

So, epsilon if you are given this specification; that in the pass band A max is the maximum loss then 10 to the power 0.1 A max minus 1 and that would will be, epsilon from here itself when omega is equal to omega p, that will become epsilon square. So, A corresponds to A maximum. So, A max divided by 10, so 0.1 A max. If you exponentiat with with respect to the base 10, so 10 to the power this minus 1; that gives me epsilon, okay.

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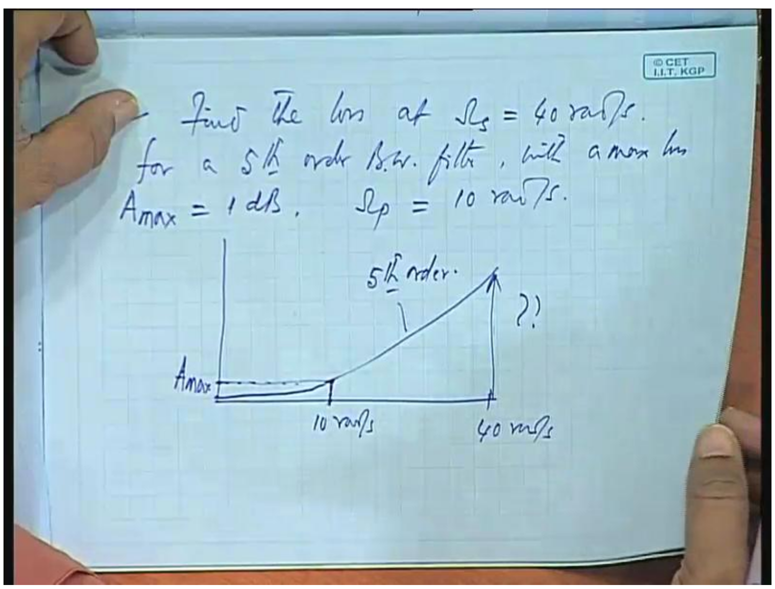

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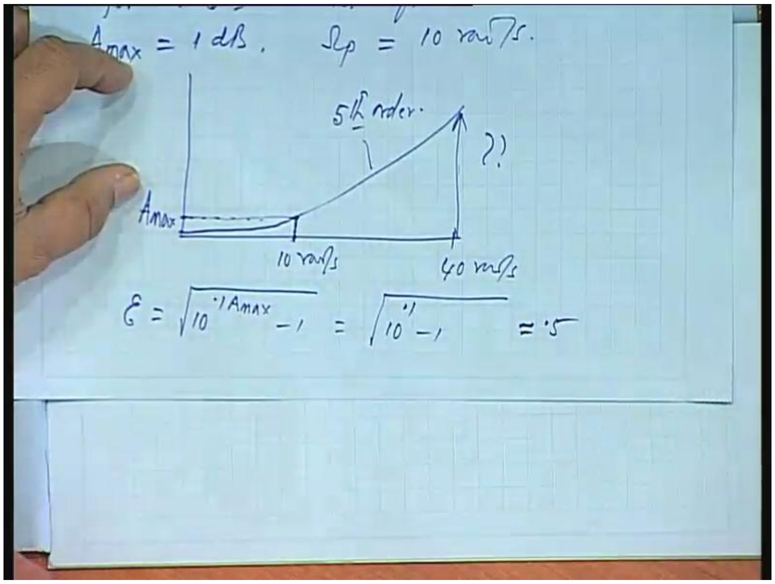

And let us work out one example. Find the loss, find the loss at omega s equal to 40 radians per second for a fifth order Butterworth filter; with a maximum loss which is A max, with a maximum loss A max equal to 1 d B in the pass band age, this is the pass band age frequency of 10 radians per second. That means, you are given this is 10 radians per second, this is 40 radians per second, this is the maximum loss A max. And you have to determine the loss here. If this is of fifth order Butterworth polynomial, okay.

So, epsilon is root over 10 to the power 0.1 A max minus 1; so that gives me 10 to the power 0.1 and A max is given 1 d b, so 0.1 in to 1 minus 1, is that all right? So, 10 to the power 0.1 is how much? If you take log, 0.1 into log 10 is 1. So, point 1, that is log of 5 minus log of 4 approximately; log of 4 is 0.6, log of 5 is 0.7, so this the difference is 5 by 4 is 1.25. So, 1.25 minus 1.25, so under root of that, approximately 0.5, okay under root of point 0.25.

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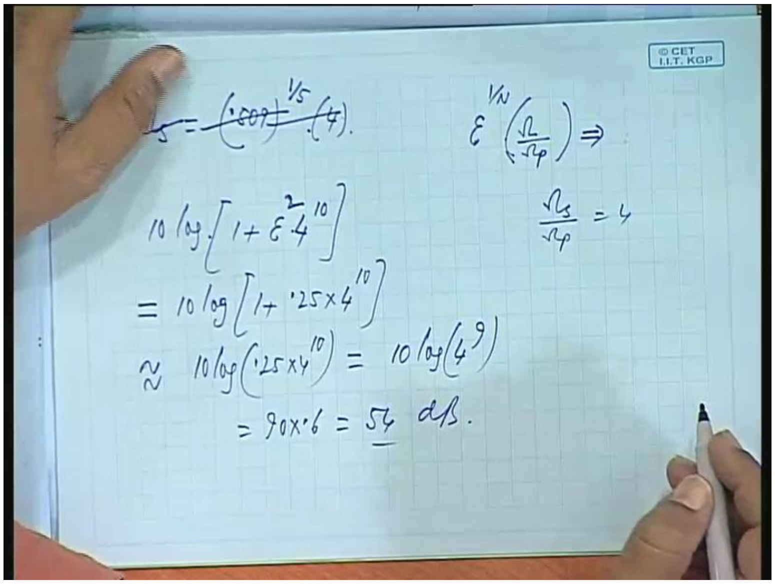

So, omega s will be what will be omega s? Stop band frequency? 0.509 to the power 1 by 5 into 4, basically it is epsilon; actually this is 0.5 was an approximation, actual value is 0.509, okay. So, epsilon to the power 1 by N into whatever was that omega s by omega p, omega by omega p, okay; will be no, this will be the normal omega. See, omega s was given as 40 radians okay. 40 radians, we have got epsilon.

So, if you can put in that equation, what was the equation; you can see, the loss function was 10 log 1 plus sorry, this is not I was calculating something else. So, 1 plus epsilon square omega s by omega p, and how much is that; is equal to 4. So, 4 to the power 10, okay. So, that is equal to 10, one can be neglected, 1 plus; and how much is epsilon, square 0.25 into 4 to the power 10. So, that is approximately equal to 10 log 0.25 into 4 to the power 10, okay. So, I can write 10 into log of 4 to the power 9. So, 90 log 4, log 4 is 0.6; so approximately 54 d B okay, okay thank you.

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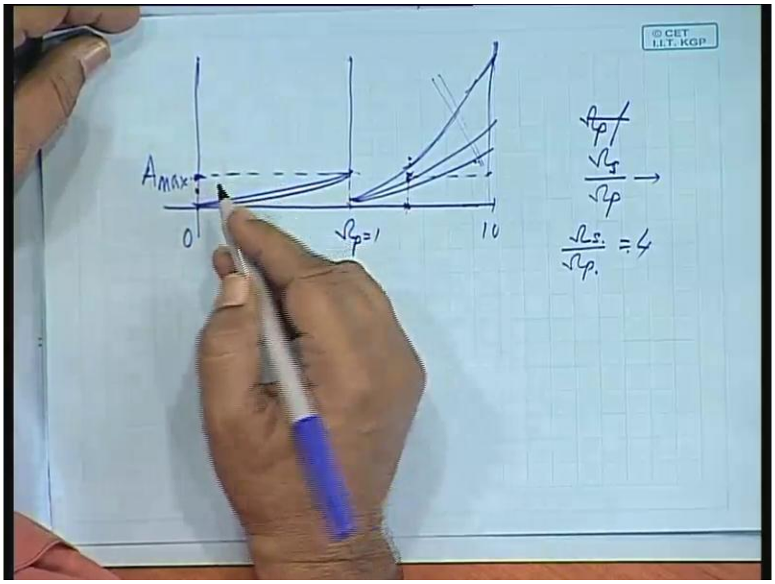

Now in the design manuals, you may refer to design manuals where for different values of A maximum in a normalized frequency, omega p by okay omega s by omega p, is known okay. You make a plot of omega by omega p, omega by omega p so omega p is taken as 1. You have for different orders, N is equal to 1, 2 etcetera; you have such plots and say this is between 0, 0 and 1.

And 1 to 10 makes decade, you have in the stop band you have the plots because omega s will be given; omega s by omega p, suppose the ratio is 4. So, I will go to 4. And for different values of N, as N increases it goes on increasing, all right. So, at N is equal to 4, I can read of that particular stop band d B; that is A minimum, if A minimum is specified I will just refer to the nearest integer value, okay, I will go to the higher value, next integer.

So, I select N. Once you select N, the polynomials are fixed all right. There are, all of the unit circle, you know the location of the roots; so you can calculate the Butterworth polynomial and even there also available in the table. So, the design is very very simple with the help of such charge for different values of A max.

A max will have to be given; so they give 0.25 d B 1, d B 0.5 d b and so on. So, these are specified and these curves are available; you can always select the particular value of N and design the filter okay. We will stop here for today; will continue with this in the next class. Thank you very much.