Digital Signal Processing
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Lecture - 16
FIR Filters (Contd.)
Introduction to IIR Filters

Last time we discussed about FIR filters with only low pass specifications.

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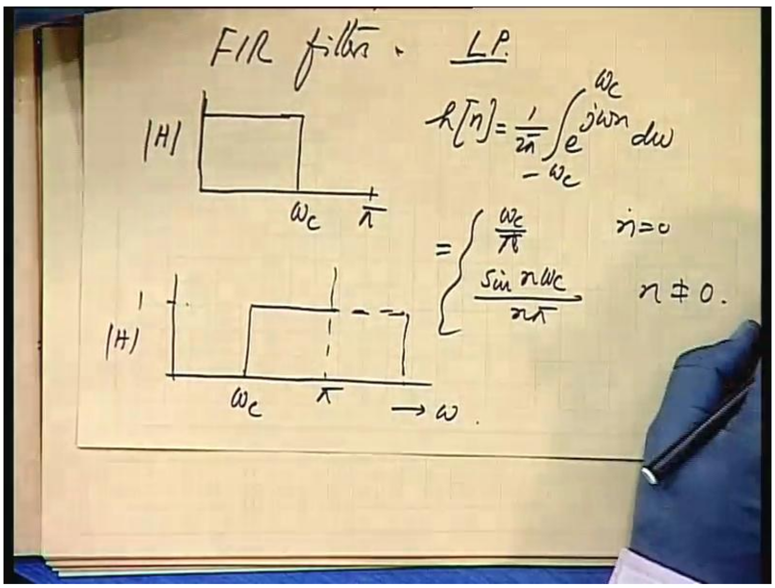

We shall be taking up today, a few more different types of filters; that is band pass and high pass filters. What will be the sequence h n, like other procedures are similar. Then we shall be taking up IIR filters, later on again we revisit FIR filter and we discuss about FFT. So, what we took was a low pass filter; this was the magnitude against frequency and we got h n as a sequence from this integration.

This is an ideal low pass filter d omega, which gave me omega c by n, omega c by p i for n is equal to 0 and sine of n omega c by n p i for n naught equal to 0, okay. Now for a high pass filter, it is like this okay, it continues like this. This is 1, so what would be H omega for this? Once again, you follow the same procedure; it will be see this one is omega c to p i. I have to take a range of 2 p i, so h n for high pass; if I write as high pass, it will be 1 by 2 p i, what should be the integration?

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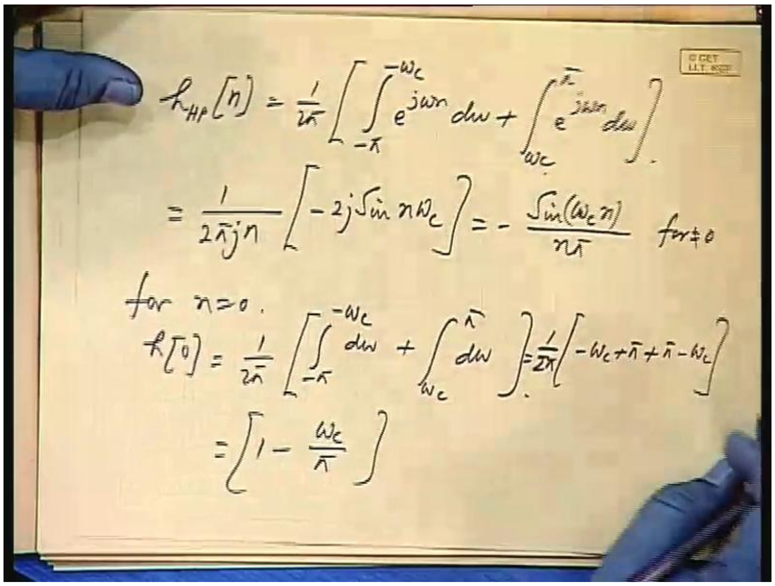

It will be minus p i to minus omega c and plus omega c to plus p i; in-between it is all zero. So, it will be minus p i to minus omega c, e to the power j omega n d omega plus omega c to p i, e to the power j omega and d omega, so do you all agree? If you compute this, finally if you permit me to write the final result; I rotate rather may be some slip, check the result.

What I have got is minus of sine of omega c into n by n p i for n naught equal to 0. For n is equal to 0? For n is equal to 0 when I take, sorry when I take n is equal to 0; this integration, this will be 1, this will be 1 minus p i to minus omega c and omega c to plus p i, what will it be? 0? Will it be 0? Okay, thank you.

So for n is equal to 0, h 0 let us see what it is like; minus p i to minus omega c d omega plus omega c to p i d omega, Agreed? 1 by 2 p i then this is minus omega c plus p i plus p i minus omega c, so it is how much is it? It is not zero, it is 1 minus omega c by p I; that should be h naught okay, let us find out for a band pass filter.

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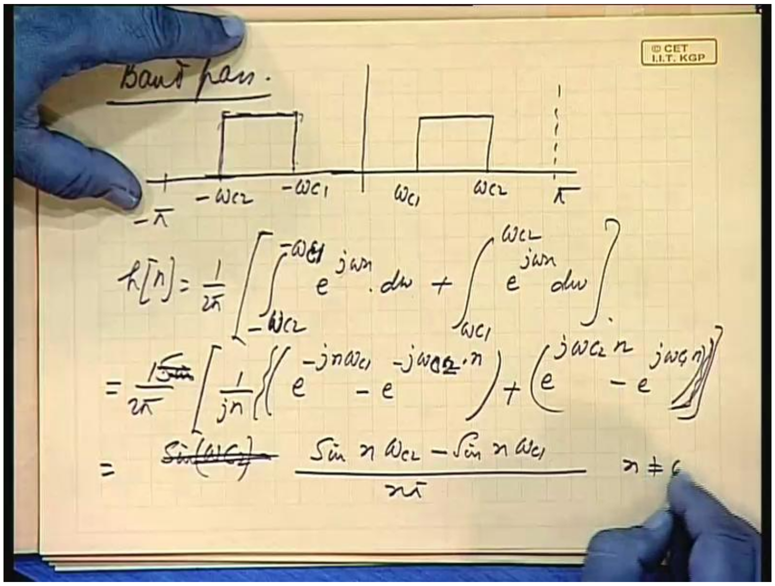

The coefficients h n for a band pass filter, define the band pass filter in terms of these two frequencies; omega c 1, omega c 2 and this is p i. So, it has its counterpart on this side and image to cover minus p i to plus p i. It will be like this, minus omega c 2 minus omega c 1, next one it repeats okay.

So, what will be h n? It will be 1 by 2 p i minus omega c 2 2 minus omega c 1, all right. e to the power j omega n into d omega because h omega magnitude is 1, plus omega c 1 to omega c 2 e to the power j omega n d omega, correct me if I am wrong, is this all right, first integration? Earlier one, this one? Upper limit, upper limit should be minus omega c 1, thank you.

So that gives me, sine of check whether you are getting this. See, e to the j omega n will give you j n in the denominator and this will be e to the power j omega n with these two values. So, again omega c 1 is appearing here; so this this one will be minus of e to the power j omega n j omega c, okay.

This should be c 1 like this, omega c 2. So again this will be a negative sign. So, okay let me write it 1 by 2 p i 1 by j n e to the power minus j n omega c 1 minus e to the power minus j omega n c 2 plus or minus? omega c 2 n into n, okay. Will it be minus or plus? e to the power j omega and integrated from minus, but this is on the lower limit; so actually this minus this. So, minus of okay then plus e to the power minus j omega c 2 into n minus e to the power j omega c 1 into n, correct me if I am wrong, is that all right.

This should be j 1, should be a common term all right. So that gives me finally, j will get cancelled; this will be leaving you a sine term, this will also give you a sine term. So, you will get finally sine of n omega c 2 minus sorry, n sine of n omega c 2 minus sin of n omega c 1 divided by n p i; for n naught equal to 0, okay.

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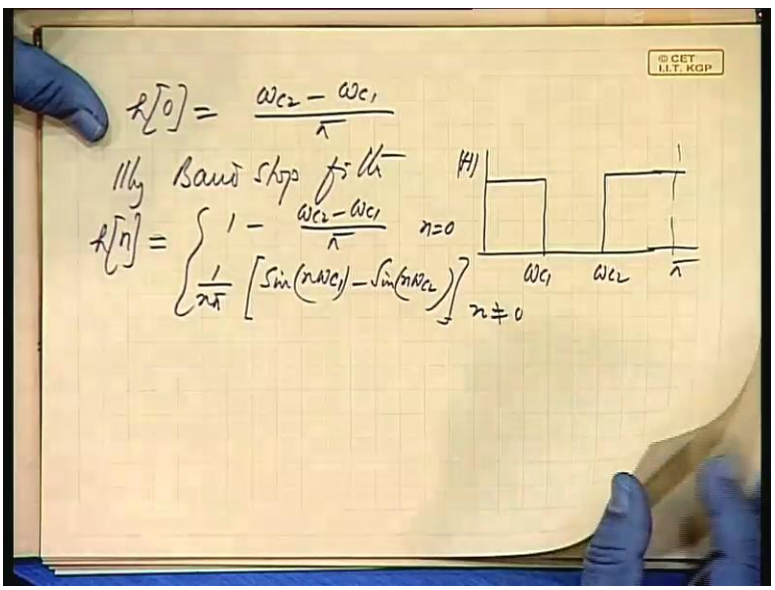

And for n is equal to 0, for n is equal to 0, just put n is equal to 0 this becomes 1 and this becomes 1. So, minus omega c 1 minus omega c 1, so twice minus omega c 1 and plus twice omega c 2, 2 will get cancelled in this; so you will be left with omega c 2 minus omega c 1 by p i, okay.

So, similarly for band stop filter; you can see it will be like this, this is omega c 1, this is omega c 2 up to p i, okay, it continues up to p i. And h n will be by the similar procedure, if you go it will be 1 minus omega c 2 minus omega c 1 by p i for n is equal to 0. And this one will be 1 by n p i sine of n omega c 1 minus sin of n omega c 2, okay for n not equal to 0, there are some other special types of filters.

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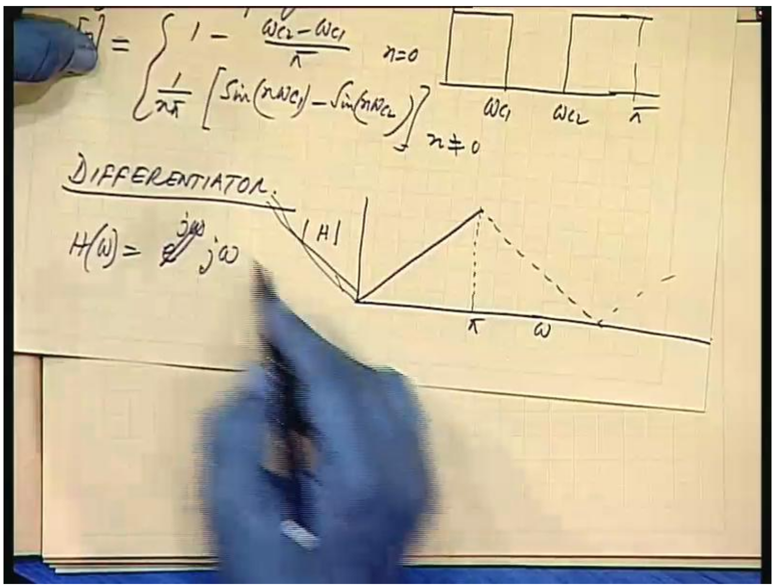

We shall be taking up only one of them and differentiator. Now, what are differentiators? In the analog domain, when you differentiate; you get a function s as a transfer function, you get s. Any signal in the s domain, getting multiplied by s will give you basically the transform of the derivative of the time function, so in the s domain.

If you take in the continuous frequency domain, therefore the transfer function will have value j omega, okay. In the discrete domain, therefore it should be j small omega; if you go along that particular argument. So j omega, if this is omega; H magnitude will be like this and its phase will be always ninety degrees j, so H omega is equal to e to the power j omega, is it e to the power j omega? No. e to the power j omega means what? Magnitude is always one.

Here it is just j omega, is that all right? So, magnitude is proportional to omega, so it is linearly varying with omega up to p I; obviously it cannot go infinity, go to infinity. And then again it will be repeating, similarly on this side okay, it will be following like this. So, H omega is equal to j omega between minus p i omega p i, is that all right?

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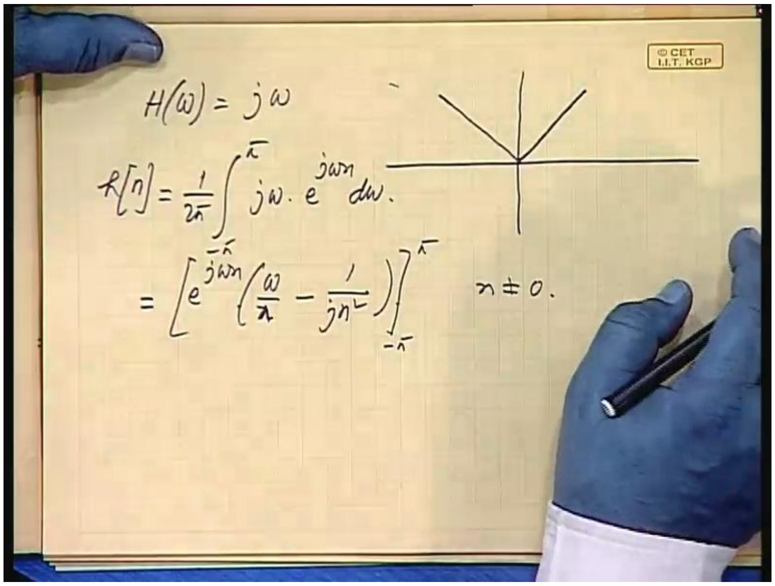

H omega is equal to j omega between omega between minus p i to plus p i, will it be a straight line like this or like this? Anyway, we are interested only in the magnitude, so it will be like this. Therefore, h n is equal to 1 by 2 p i is an interesting function; minus p i to plus p I, j omega a into e to the power j omega n d omega, okay.

So that gives me, e to the power j omega n into omega by n minus 1 by j n square, okay. Evaluated at minus p i and plus p i, if n is not equal to 0, okay. If n is equal to 0, if n is equal to 0, if integrate this; it will be 0, is it not? It will be omega square by 2, omega square by 2 because it is an odd function, so it will be 0. H 0 is equal to 0. And this if you simplify, this will be cosine of n p i by n, one may write equal to minus 1 to the power n by n okay.

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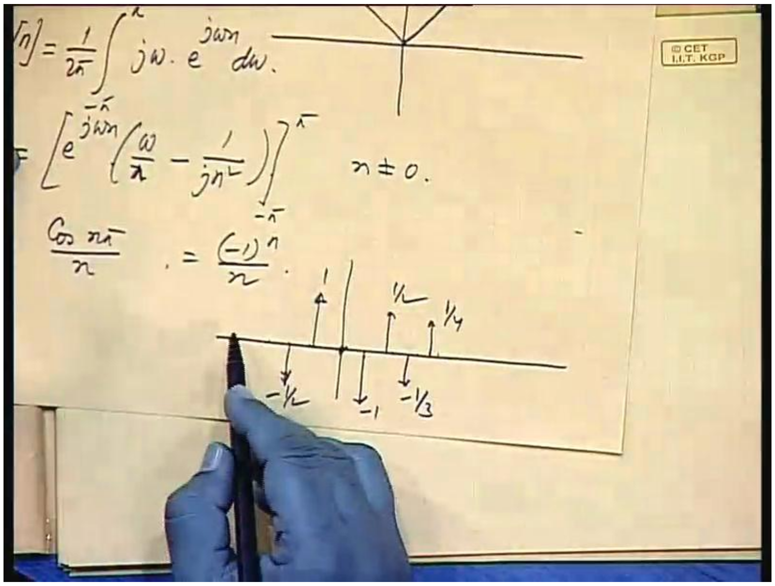

Cosine n p i by n, is it an odd function or even function? It is an odd function, because this is an even function, this is an odd function, so this will be an odd function. What will it look like? Okay, here it will be 0 then if I put n is equal to 1; cos of p i, so minus 1 by 1 then cos of 2 p i plus 1 by 2, so its magnitude is half. Then minus 1 third, then plus one fourth and so on, okay. And it will also have its counterpart on this side.

Can you identify this sequence? It is an anti-symmetric sequence where you are getting permanently, some ninety degrees continuation ninety degrees plus the slope. Now, this is the sequence that you get from here and then finally you shift it, is it not? This is what we did, if you are given a length, particular length; so if it is eleven point sequences, then you take five points on this side, five on this side and one at the central point. So, you give a shift of five steps okay and then decide a slope, okay.

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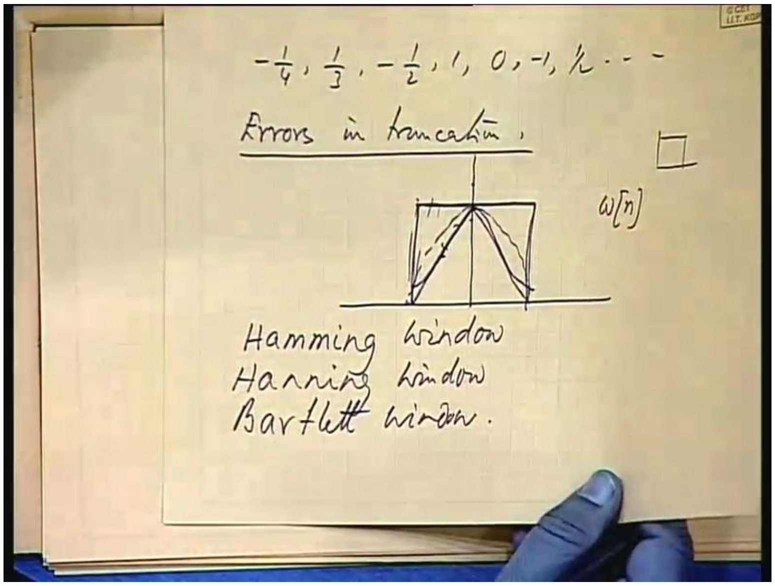

So, the sequence will be minus 1 by 4, 1 3rd, minus 0.5 then 1 then 0. Say, minus 1 by 4 plus 1 3rd, minus 0.5 plus 1, 0 then minus 1, 0.5, minus 1 3rd and 1 4th and so on. If it is given a shift of four steps, now we are also discussing about the errors in truncation. Now when we truncate the Fourier series, we are dropping the high frequency components; so there will be some errors, there will be ripples and we took first a rectangular window function, all right.

A rectangular window, if you take a rectangular window function sorry, the h n values are multiplied by the window function coefficients w n; and in a rectangular window this is one, you are multiplying all the coefficients that you obtain that infinite sequence Fourier series, you are dropping off at a particular point. That means, all the coefficients are retained as it is, multiplied by one and rest of them are all multiplied by zeros, this is what is a rectangular window mean.

Now, then we modified it by some other different types of windows where the window function was not exactly varying like this; but it was varying either like this a triangular function or may be a function like this. There are various types of functions that you discussed; depending on the individual characteristics we choose those functions, the standard ones are Hamming window and Han window or Hanning window in many books they write Hanning window, some books write Han window then Bartlett and then Kaiser and so on, there are many such windows.

Let us very briefly go through the performance of these window functions. Now in the time domain, in the time domain you are multiplying h n by the window function w n; so in the frequency domain, it will be convolved, okay. Now, what will be the rectangular windows frequency response? Rectangular window, rectangular window, will be giving you a sink function, it is like this.

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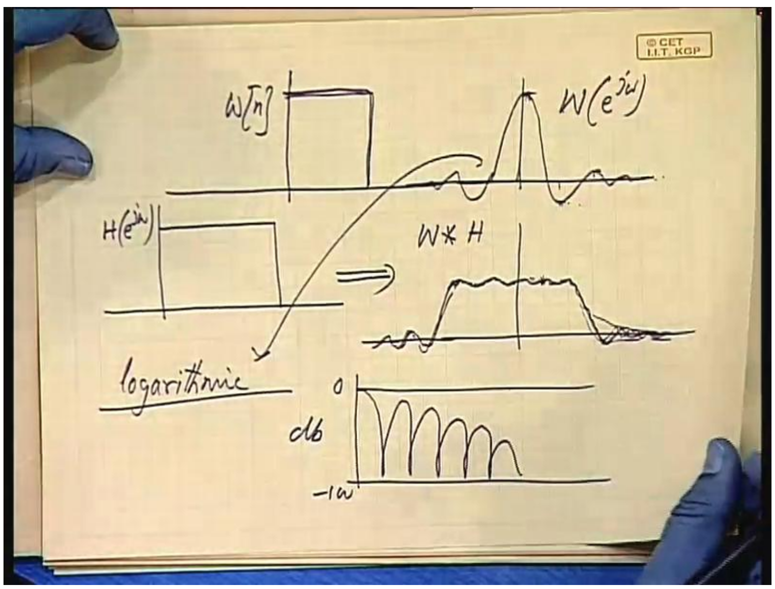

Any function of this kind, will give you the frequency response like this. And it is this distribution of the lobes okay, and their frequencies how fast they appear and what is their magnitude, relative magnitudes how fast they fall; that will decide the quality of the window okay, because this is to be convolved with h omega that is h n, sorry.

It is to be multiplied with h n and convoluted convolved with h omega. This is actual H omega, actual h n corresponds to this which is an infinite sequence; and that gets multiplied by a window function like this, okay whose frequency domain representation is like this. So, this use to be convolved with this, okay. The frequency domain they have to be convolved. So, you take this and you gradually move it; so the resultant will be after convolution W convolved with capital H, the result will be like this, okay. So this is the function, that we will be getting after realising the filter with the window function, this a nature of variation.

Now, depending on the choice of the window that you make; these ripples the width of these ripples and their fall, their transition will be changing, okay. So for a filter function like this, this window function for example; if you take in the logarithmic plot that is decimal plot, in d b this function with different types of windows will be varying like this. It will have a 0 d b value and then will be falling of like this, gradually falling, okay.

So this is 0 d b, this is minus 100 d b, all right. So, for a rectangular window, this fall is very slow for a rectangular window; this fall is very slow; that means the ripple continues for for a long period, all right. For a Hann window or Hanning window, what to be call it the function was cos of 2 n p i by 2 M plus 1, where M is the non-zero steps in one side, okay.

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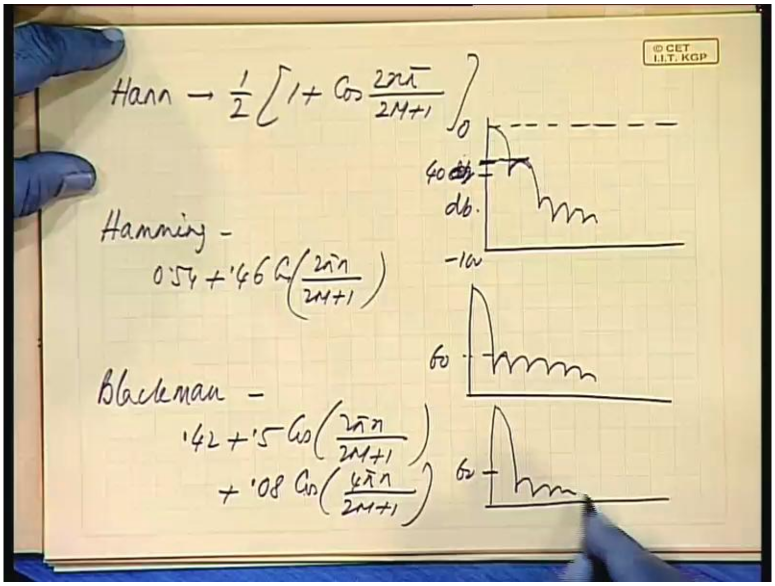

2 M plus 1 is the total number of points; it was this logarithmic plot will be, this is 0 d b and this is say minus 100, okay. The first peak here is somewhere around fifty or so, okay. This is about sixty, forty little a more than forty, actually it is little less than this, forty and so on. Hamming window, Hamming window the function is 0.54 plus 0.46 cosine 2 p i n by 2 M plus 1 okay.

This fall is much better actually; it is the first lobe which decides about the quality. This is going to almost 60 d b okay, routine of sixty degree. This was little close to fifty d b. Then black man window is 0.42 plus 0.5 cosine 2 p i n by 2 M plus 1; now in black man window, he takes a something like a second harmonic 4 p i n by 2 M plus 1.

See the window function extends to another cosine term okay and that gives me even better. It is only a relative comparison, the exact values I am not plotting; what I meant is, you improve further all right by choosing a little more complicated window functions. Obviously, it will be at some cost we have to improve.

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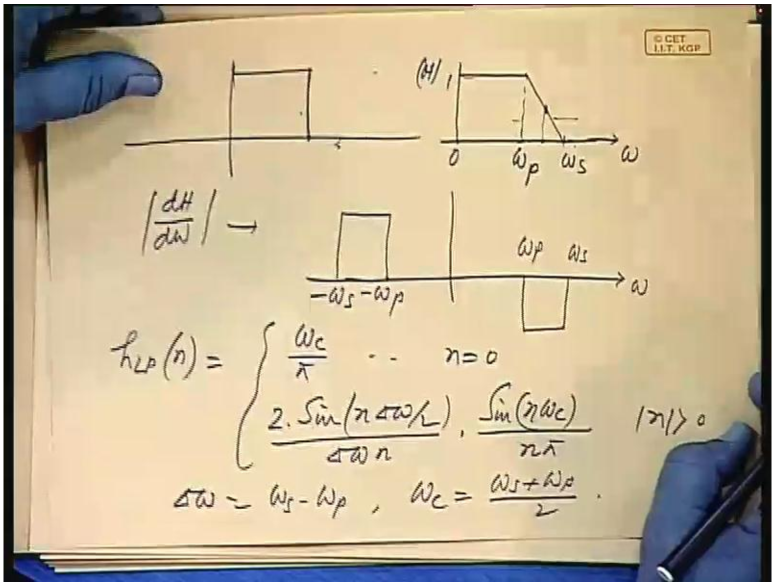

Now, impulse response we can improve the impulse response by having a transition, instead of having a very sharp transition from a gain of unity to 0 in the frequency domain; h omega we had a big wall structure for a low pass filter, the ideal filter characteristics we chose like this. So, this will be realised only by an infinite sequence long, long cos of system. It will be an infinite sequence.

So, we can improve upon this by having; no real filter will be having an infinite number of terms. So, if we can have a slow transition like this, the filter performance is improved quite considerably. If we take this as omega P, this as omega s; stop band frequency. This is H omega is 1 then what will be d H by d omega? It will be zero slopes okay.

Similarly on this side, I am not plotted the other side will be; minus omega is minus omega p degree, so from there and for such functions if you know d H by d omega, then by integration you get the corresponding terms for this, all right. So you know the properties of Fourier transform DTFT that you have studied; for the derivative function all right, in the Laplace domain also you have done, d f a by d s if you want sorry, sorry, so you know the corresponding n times f m, all right.

So, similarly in the discrete domain also you have studied the property, so we make use of that property and if you permit me, I write the answer right here. You can all derive this; it is not difficult or you can start even from the fundamentals, minus omega is 2 minus omega p, there is a straight line, so you have to write the equation of the line then again a constant, and then again equation of this line and then you integrate over the entire span, you will get the same result.

Twice sine of n delta omega by 2 by delta omega into n into sine of n into omega c divided by n into p i for n greater than 0, where you have delta omega is basically omega s minus omega p and omega c is basically the arithmetic average; omega s plus omega p by 2, okay. So, omega c is this average point and delta omega is this band, okay. So, in terms of those you will get this.

So, this is about a low pass filter with a transition, given transition rate. You can have, these are called a spline functions from here to here the roll off can be in different forms; this is a 0^{th} order spline. If you can have higher order spline or rather first order spline, this is a first order spline zeroth order will be a vertical line.

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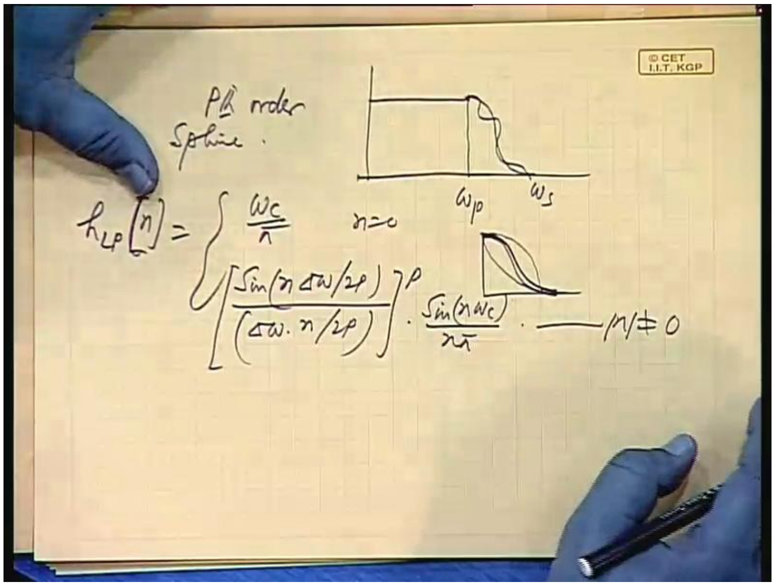

If you have higher order spline, say P th order spline functions are like this; I can go from omega p to omega s in this manner okay, or you can a higher order spline. Basically, higher order means you are trying to fit in more and more polynomial, I mean polynomials of higher and higher order okay and then make it go like this.

So, most commonly chosen higher order spline will be like this, cubic spline that is known as a cubic spline, okay. Instead of having like this or this, you have a smooth transition like this, okay. See, if you have a P th order spline then the low pass filter will be omega c by p i for n is equal to 0 and sine of; its is very simple delta omega by 2 p by delta omega into n by 2 p, whole thing to the power p. And the other term is say, sine n omega c by n p i, all right, this is for n not equal to 0. Now we shall start a general analogue filter design, will come back to the FIR filter later on.

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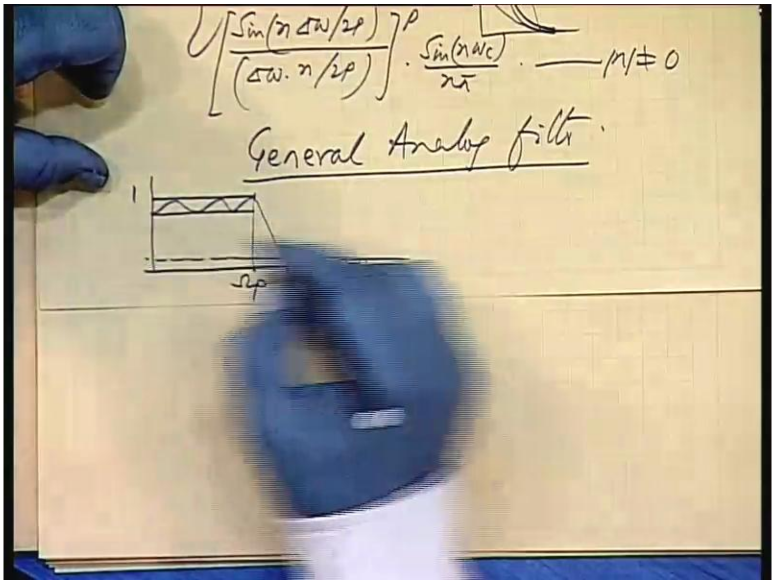

Before you go for the design let us see, the specifications for an analog filter. An analog filter will be specified, since no filter will be ideal; so we specify in terms of some tolerance, okay. Like this; some tolerance here and this is in the analog domain omega p and omega s, this is a stop band frequency. That means within a certain tolerance, this gain falls beyond this frequency omega s and up to omega p it remains close to a gain of one within a certain tolerance.

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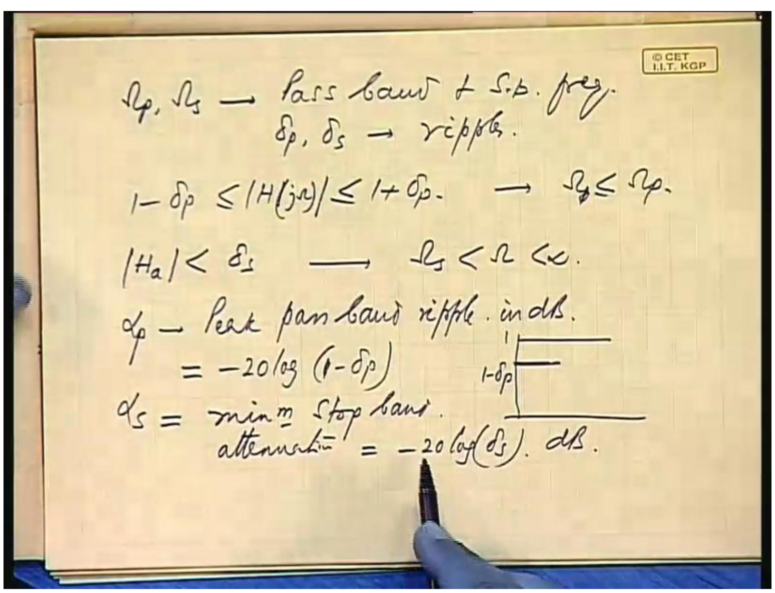

So, omega p and omega s, this will be pass band and stop band frequencies and stop band frequencies; and will also define delta p and delta s has the ripples, so 1 minus delta p H okay, j omega 1 plus delta p. So, about the mean value of 1, this is 1 plus delta p and 1 minus delta p.

Sometimes, that one plus delta p is taken as the reference one; that is normalized. So H a is less than delta s in the stop band. And here omega is less than equal to omega p; between omega s and omega p, it is not defined. And there you have some freedom to choose the path of transition okay. Now, we define alpha p as a peak pass band ripple in d B. So that will be, minus 20 log off 1 minus delta p.

If we define this, as I was telling you; if I take this as 1, normalize it and this has delta p then basically how much is the maximum tolerance? Okay. So, delta p 20 log of that okay, 1 minus delta p; if you take 20 log of that, so this will be alpha p its a negative sign, it is less than 1 so it will be coming as a negative term. So, negative into negative will make it positive. So alpha p, if I say 2 d B or 5 d B. Say, you know what is the value of delta p.

Similarly, alpha s is a minimum stop band, minimum stop band attenuation which is minus 20 log of delta s, so many d B. So, it will be a plus say it may be plus 40 d B, that means your minimum attenuation should be 40 d B, attenuation is just opposite of gain; how it is, sometimes we call it loss function. So, loss should be at least so much, that means loss should be more more than forty d B, all right that means gain should be very very small; it should tend to 0 in the stop band.

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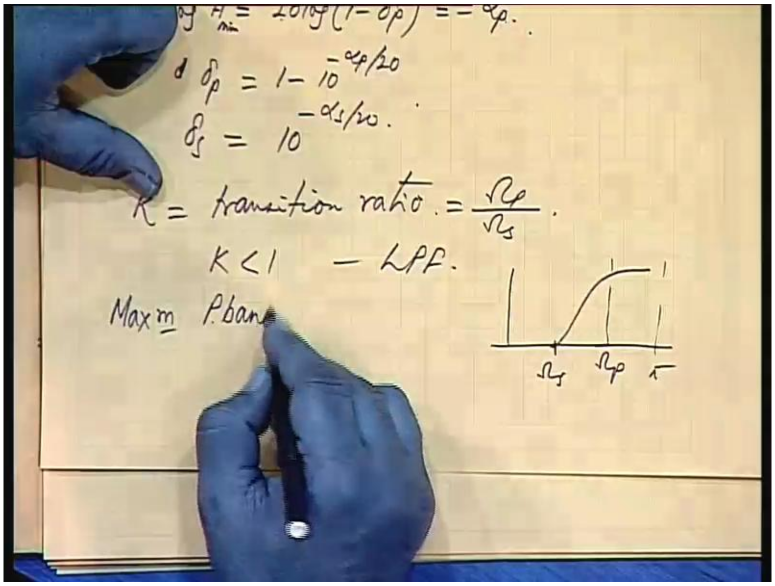

So, 20 log of H is equal to 20 log of 1 minus delta p in the pass band, okay; I will call it, H minimum and that is equal to minus alpha p, you are calling. So, H delta p what will it be equal to? Alpha p by 20 with a negative sign and if I subtract from here, so that will be 1 minus 10 to the power; check whether this is all right minus alpha p by 20, that gives me log of 1 minus d B.

So, delta p will be this much. Similarly, delta s I can write in terms of alpha s; 10 to the power minus alpha s by 20, is it all right? So from there we defined, K as the transition ratio, transition ratio that means; if it is falling off from delta p to delta sorry, omega p to omega s all right. So, omega s is how many times omega p, all right. So, equal to omega p by omega s, it's either way. It is obviously less than 1 for a low pass filter and greater than 1 for a high pass filter; is it greater than 1 for high pass filter?

Greater than 1, okay. Where is it going to? For a high pass filter, all right. This is p i, so this is omega s, this is omega p all right. Maximum pass band gain is 1.

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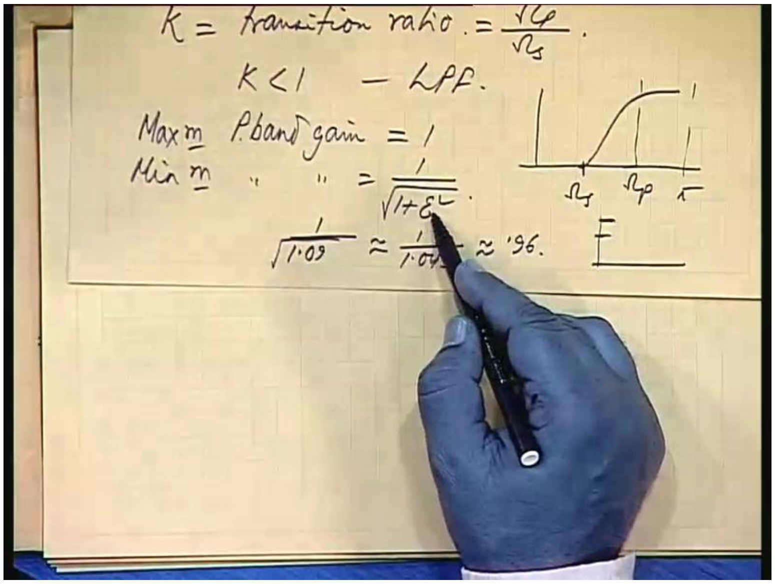

And minimum pass band gain is 1 by root 1 plus epsilon square, okay. What is epsilon? So, if remember in your earlier filter theory; we have used it, anyway; this is 1, this is 1 plus epsilon square under root, okay. So, epsilon is a parameter which decides this width, how much of fall will be there? Suppose epsilon is 0.3, epsilon is point three then epsilon square is 0.09.

So, 1 by 1 plus epsilon squared will be 1.09 under root, okay; approximately, 1 by 1.045. Approximately say, 0.96. So, the gain is between 1 and 0.96, in the pass band it varies between 1 and 0.96. So, epsilon is just a parameter which decides the tolerance in the pass band. We define and also maximum stop band ripple is defined as, 1 by A.

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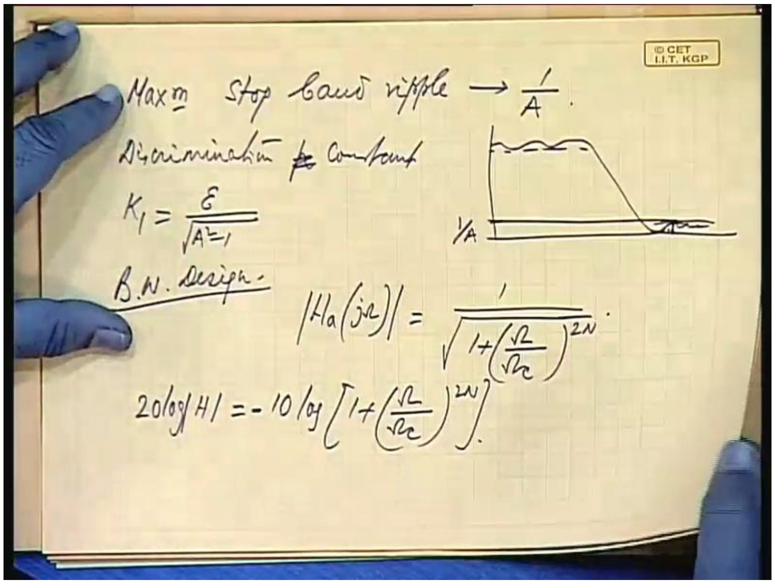

In the stop band; this we are defining as, 1 by A all right, that means A defines just inverse of this value, okay. So, discrimination parameter or constant, we define K 1 as epsilon by root over of A squared minus 1. So, what is this ratio? Epsilon means what? If epsilon is large, if epsilon is large; epsilon is large means this is large, so there is a large variation.

This in the pass band the variation is more, is it not if epsilon is large. And if A is large, this is small. So, if you want to have discrimination, have a low value of epsilon and high value of A, all right, it all depends on how you choose. Discrimination means, here and here in both the zones, how relatively you are trying to reduce these.

So, large A means in the stop band you are restricting this width, this tolerance. Now, let us come to a very standard simple design, a Butterworth design. A Butterworth function, a Butterworth function absolutely; sorry it looks like, 1 plus omega by omega c to the power 2 N, okay. So, 20 log off H, I am not writing j omega every time; if I take 20 log of this, it will be 10 log off 1 plus omega by omega c to the power 2 N that is all, minus okay.

When omega is equal to omega c, this equal to 1; so 1 to the power 2 N is 1, so this will be 20 log off 2, so that will be.

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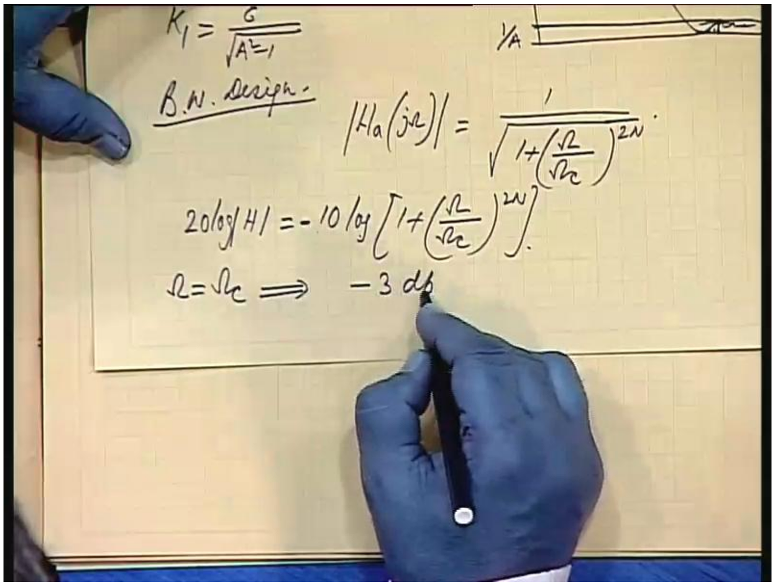

So, at omega is equal to omega c, this will be equal to minus approximately 3 d B. So, minus 3 d B point is omega equal to omega c.

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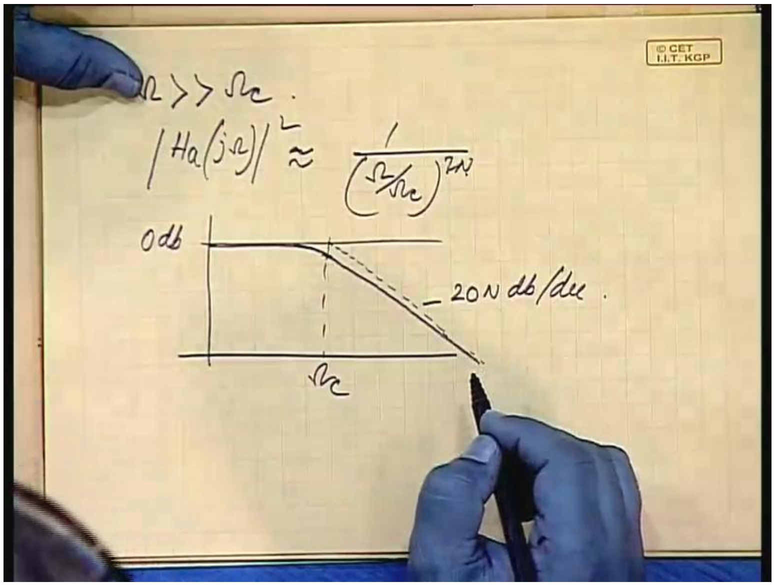

At omega much much greater than omega c, H a okay square; will tend to 1 by omega by omega c to the power 2 N, do you all agree? This quantity is more than 1, so to the power 2 N. If I take a frequency much larger than omega c, it's been quite all I mean, quantity much greater than 1; so this will be dominating. So, we will have this much of approximation. And in the logarithmic plot, it will have 20 N d b per decade fall, okay.

So the Butterworth filter, if this is 0 d b line; earlier in the pass band, this is the function you see as we go from omega is equal to zero onward, this magnitude is becoming more and more is positive and hence the fraction here sorry, this magnitude overall magnitude is gradually falling. It will be monotonically falling; there will not be any rise in-between. And when omega is very large, it will be falling with this slope. So at the most, it will be 1 when omega is 0.

So, 20 log of 1 is 0. So, it will start from 0 d b line, it will stay close to this and then after omega equal to omega c; it will have asymptotic fall of 20 degree per decade, that means it will be 3 d b here and then gradually catch up with this asymptote which is 20 N d b per decade. So, the rate of fall will be depending on the choice of N. The higher the value of N you choose, the sharper will be the fall, okay.

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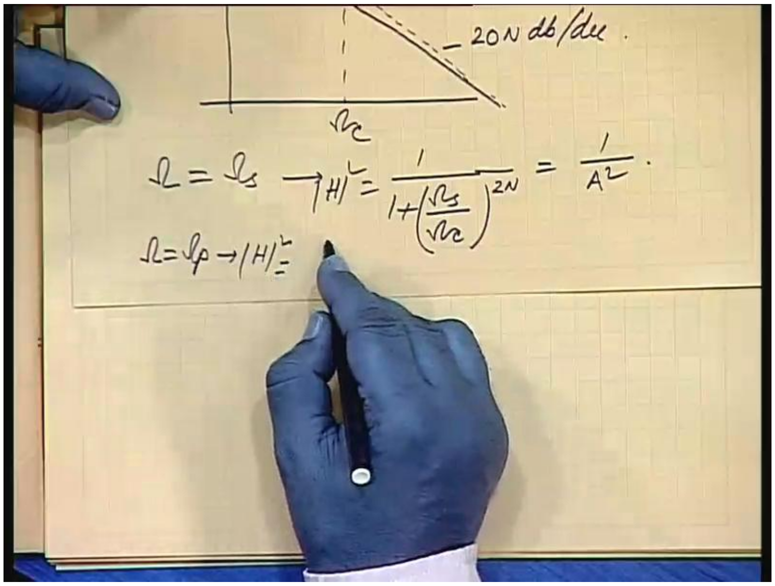

At omega equal to omega s, this will be 1 by 1 plus omega s by omega c to the power 2 N. And that should be equal to 1 by A square. A is the gain and this is H square, I am considering H square, so that will be 1 by A square, okay. And at omega equal to 0 or omega equal to omega p H square will be; okay let me take the next page.

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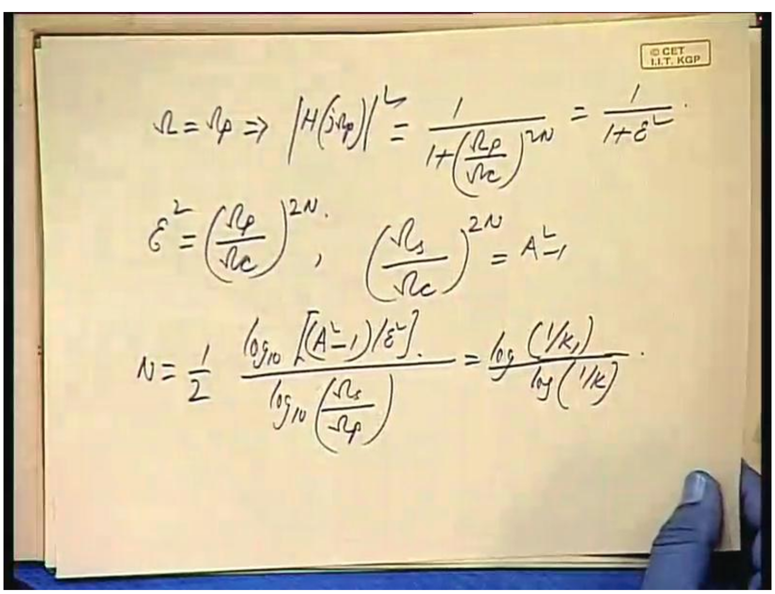

At omega equal to omega p, H j omega p square will be 1 by 1 plus omega p by omega c to the power 2 N, okay. And that is equal to 1 plus epsilon square, okay. Now, from these two specifications of epsilon and A, we can choose the value of N; depending on which, which quantity is as specified, epsilon square is equal to omega p by omega c to the power 2 N, okay.

Could you tell me, what would be the expression for the other quantity? Omega s by omega c to the power 2 N is equal to A squared minus 1, Are you getting this? See, A square is equal to this much; both are 1 by this quantities, so if I subtract one from here, so omega is by omega c to the power 2 N, try to eliminate and get hold of N.

Our first task will be to determine the order of the filter, that is N from the given specifications. You are given either A and epsilon omega s and omega p. So, log 10 A square minus 1 by epsilon square divided by log of 10 omega s by omega p. Actually, omega s and omega p need not be given explicitly; it can be given in terms of the ratio all right, that is what I said either you may be given all of them or it may in terms of those two parameters that we defined. That is log of sorry, this is log of 1 by K 1 by log of 1 by K; you may be given even this, okay. So, this will be the design of Butterworth filter.

Now before we take up other types of filters, now will be stopping here today; I just give you in brief the idea for designing an IIR filter. In IIR filter what you do; we start off from analog filter with the same specifications, we design an analog filter first. Then we make a substitution for s all right, what is it? In terms of s, that means we are going from the continuous domain to discrete domain.

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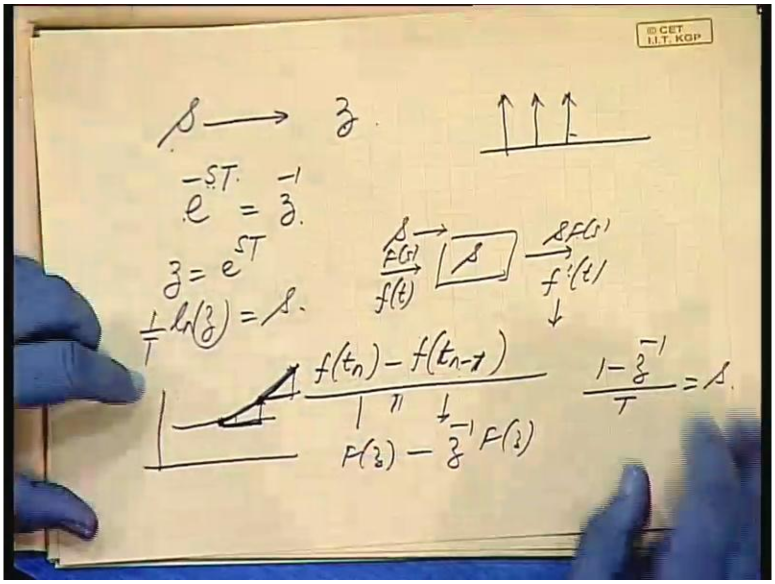

In the continuous domain the transform is s, what is its equivalent in the discrete domain? Z. So, you establish a relation between s and z, okay. Now, there are different types of relationships; one is the actual relationship is e to the power minus s T is equal to z to the power minus 1, this you can prove, any discrete sequence you find out what will be its Laplace transform. And what you have replaced is e to the power minus s T by z inverse.

So, this is the ideal relation. e to the power minus s T is an infinite sequence, all right. I can write z is equal to e to the power s T all right. So, what is s? l n z 1 upon T is s. So, in the analog domain wherever s appears; if I replace it by z of this kind then I will get the z domain function, it is very complicated. l and z what is it? It will not get in terms of z inverse, any polynomial for l and z all right, so this is the problem.

Representing s in the z domain by various approximations, there are number of approximations; for example I will give you just one. S corresponds to derivative operation all right; so if I have an f t here output is f dash t, so basically I have put an s in the s domain, is it not? f s multiplied by s will give me, s times f s whose time domain representation is f dash t. So, this block if I now try to have f dash t in a very approximate form, it is f t n minus f t n minus T that is n minus 1, okay; n minus 1 divided by t and if I normalize that, that is divided by 1 okay. So, if I take the corresponding z transform, it is F z minus z inverse F z.

So, it is it can be or if you put divided by T; so 1 minus z inverse by T is some approximation of s, all right. While doing this approximation, suppose this is the slope; you are taking this value minus this value divided by this as a slope, somebody may take this slope at this point as the future value minus this divided by this, so that is another approximation. Basically, for a curve the slope at a point you are approximating by just a straight line.

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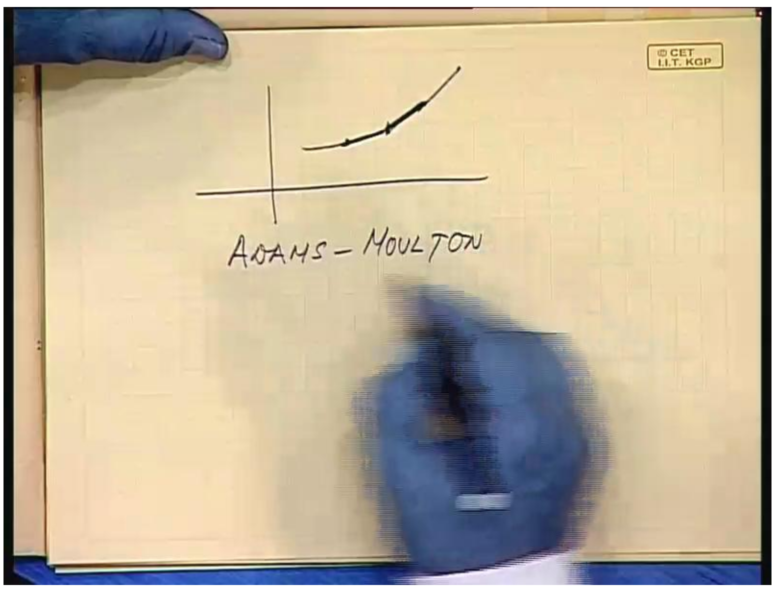

So, a slope here either you measure from this side or from this side; neither of them will be true, okay. So, we will see by having further modifications on this instead of a single straight line; I can go for second order, third order, fourth order and so higher order approximations of a derivative function and then what will be the corresponding z domain function? So, s will be converted to z by different transformations.

The most commonly used transformation is bilinear transformation. 1 minus z inverse by T is one type of approximation. Similarly the other one, if you take the forward point; if you take higher order term the general, the general transformation is Adams Moulton transformation. Bilinear transformation is the simplest that is the lowest order of Adams Moulton transformation. So, we will take it up in the next class, thank you very much.