Digital Signal Processing
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Filters Introduction
Lecture – 14

Last time we were discussing about functions, simple functions H z where only zeros are present, only zeros were present; that is only the numerator polynomial was there.

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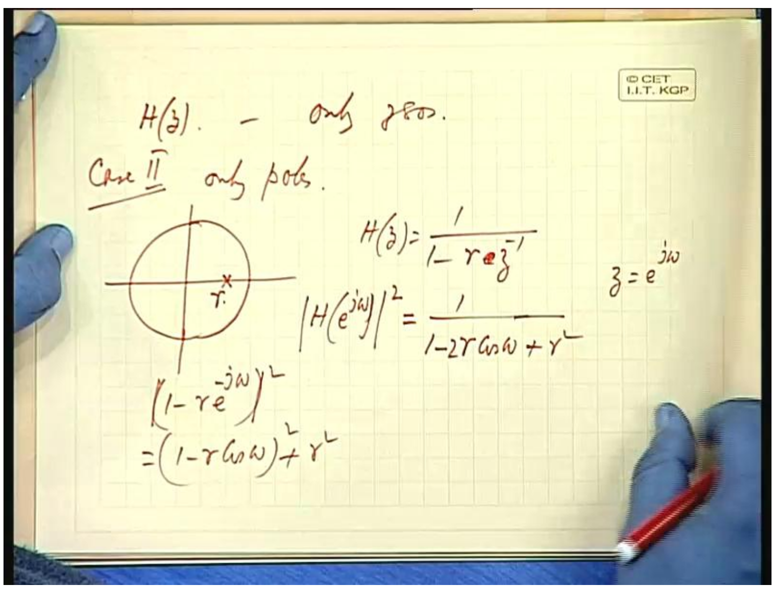

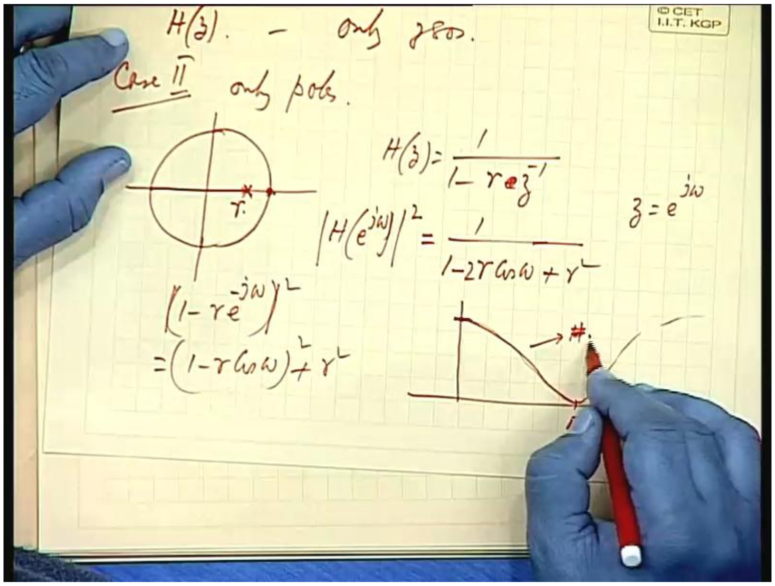

Now, let us take the second case when only poles are present that means, only the denominator polynomial as is present, numerator is 1. Suppose, there is a pole here okay, so H z is appearing as 1 minus some, okay I will call it r again; r e to the power okay, r z minus 1, r is less than 1. Then H e to the power j omega square will be; once again you can break it up, z is equal to e to the power j omega okay. So, how much is it? 2 r cos omega, all right, 1 minus r e to the power j omega magnitude square is 1 minus r cosine omega whole squared plus r square, all right.

So, 2 r cos omega plus r square and the root of this will give H magnitude; so as we are going along this when it is coming to 0, omega equal to 0 this distance is minimum all right. So that the distance is minimum that means, the function is maximum at that frequency. It is have been maximum value here. And when it is going to p i this distance is maximum, so this becomes overall function becomes minimum.

So at p i it is falling like this, it may fall like this but it is starting from maximum going to minimum going, will see the nature of variation later on, and again it will be going like this. So this is a high pass or lower pass? Low pass. So, a zero here on the real axis gives me a particular factor corresponding to that gives me, a high pass. A pole on the real axis, positive real axis gives me a low pass.

And if a pole goes to this side then the distance is maximum here; so then it becomes the high pass and reverse is the case, in case of zeros, we did not mention earlier zero coming to this side, will give you a low pass.

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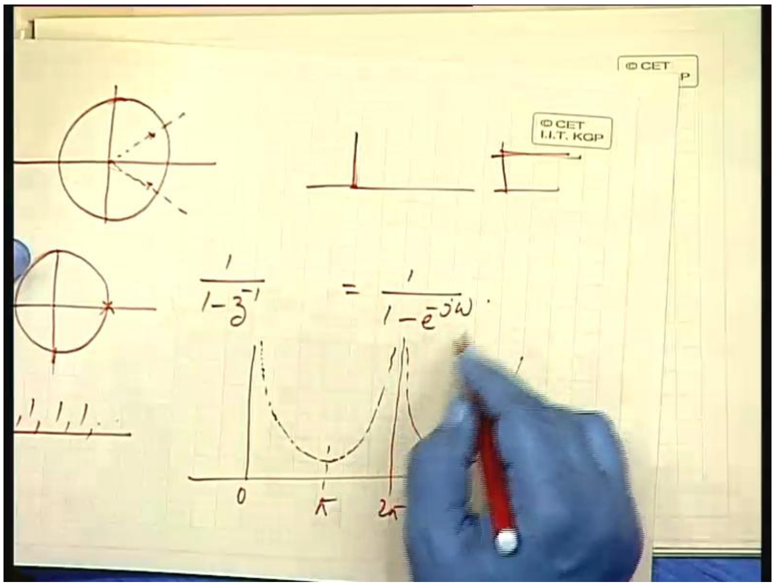

If I have a pair of poles, once again say somewhere here and here, if I have a pair of poles, okay. Before we take it up; suppose there is a pole on the real axis, if there is a pole here itself then the function is 1 minus z inverse, z is equal to 1, all right. So, this is 1 by 1 minus e to the power minus j omega when omega is equal to 0, how much is it?

Infinity, okay so it will be, and omega is equal to p i means; it becomes plus 1 to half, it will be like this, so this is 2 p i, this is p i all right. This will tend to infinity. Now, this type of filter function we do not like, why? You give an impulse, what is the response? What is the impulse response of this? One one one one, this is basically inverse of a unit step.

By giving an input, if I give a delta input, if I get an output as unit step what is it what is? It is an accumulator or integrator, all right. So, we do not want an indicator as such, there may be very very specific requirements in some situation where you require an accumulator. So, the output will be all ones. Anyway, let us get back to the original problem. So, 1 by e to the power j omega, it will have a characteristic like, what is it like?

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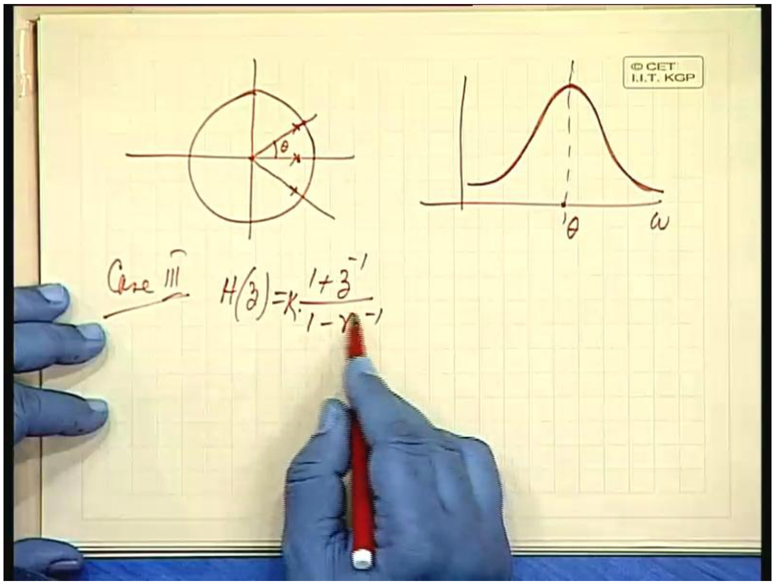

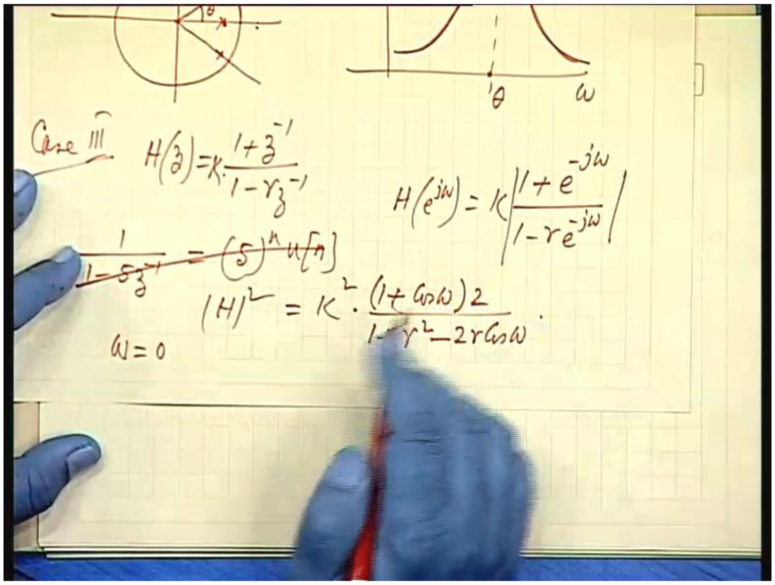

If I have a pole on the positive real axis, inside the unit circle somewhere here that we have seen. Now, we are having a pole here, so as we keep on changing the frequency somewhere around this point when it is equal to theta, the distance is minimum. That means a function becomes maximum all right.

So, it becomes a function like this, just reverse of the situation that you had in case of zeros; when omega is exactly equal to theta, we get a function like this, all right. Now, next this is a general case when you have both poles and zeros, we are considering for time being first order functions, H z equal to 1 plus z inverse by 1 minus r z inverse into some constant, K.

Now this r is necessarily less than 1, because we want this to be inside, poles must be inside; so the denominator must not have the roots outside the unit circle, numerator can have anywhere. So, you have chosen a very simple function, the zero is at minus 1. Then the system becomes, yes. Why cannot the pole lie outside the unit circle?

Okay, let us have that is a very good, fine question. Suppose, you have 1 minus 5 into z inverse what will be the z transform, inverse z transform? 5 to the power n u n, inverse means; it is a impulse response, so impulse response is unbounded, okay. So, it is an unstable system. So, you do not want poles to be located outside the unit circle.

Now, this is the function. What would be H e to the power j omega? It will be K into 1 plus e to the power minus j omega divided by 1 minus r e to the power minus j omega magnitude. So, simplify this. It is H square is K square into; if you square it up we will get one plus cosine omega, we saw earlier it is 1 plus cosine omega square, okay. 1 plus 2 plus 2 cos omega gives me at okay, is this all right, okay. And then 1 plus r square minus 2 r cosine omega, am I all right?

Now, suppose at omega is equal to 0, what is it? At omega is equal to zero, this is maximum, is it not? This is subtracting maximum portion from the denominator, so the denominator is losing and this is becoming maximum. So, this function is maximum here, all right. So, if I normalise it with the respect to the maximum value; that is omega is equal to 0, then H, I take H at omega is equal to 0, to be 1.

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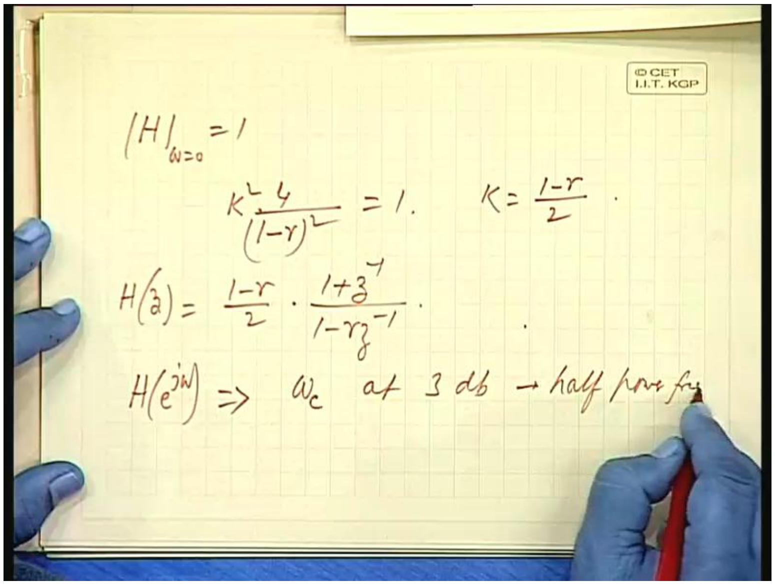

Then what should be value of K? That means K square into at omega equal to zero, this is four by 1 plus r square minus twice. So, 1 minus r whole square, that should equal to 1 or K equal to 1 minus r by 2, okay. So, I will take H z in a normalized form as 1 minus r by 2 into 1 plus z inverse by 1 minus r z inverse, okay.

I just normalised with a maximum value, with respect to the maximum value. So, H e to the power j omega, you can write in the standard form; what is that frequency, what is the frequency at which power is half? That is half power frequency or in terms of d b, it is 3 d b. What is a frequency omega c at 3 d b? That is or I call it half power frequency, it is maximum at omega is equal to zero.

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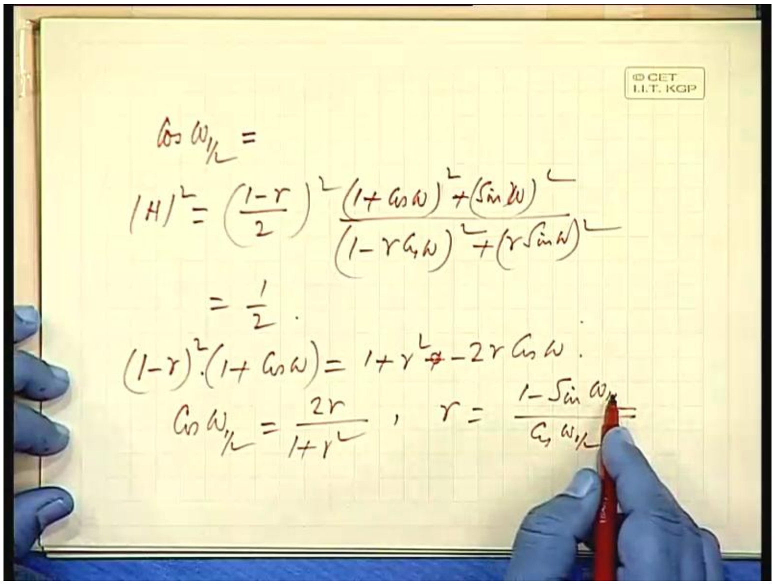

Cosine omega half, some people write omega half, some people write omega c okay; this is most specific pattern, half half power that is 3 degree point. Now, can you find out the numerator, should be half the denominator? Then only it is, H square, okay.

So, H square, H square we saw is 1 minus r by 2 whole square; this is K square into 1 plus cosine omega whole square 1 plus z inverse, plus sin omega square divided by 1 minus r cosine omega square plus r sin omega square and this must be equal to half, at that frequency. So, from there if you just square it up and then see, equate; we will get 1 minus r whole square into 1 plus cosine omega, there will be of term two square, so that will be 4.

So, that will be getting cancelled, with this is equal to 1 plus r square minus 2 r cosine omega, are you getting this? Therefore, cosine omega half solution of this is, cosine omega half is 2 r by 1 plus r square or r you can write as 1 minus sin omega half by cosine omega half. So, if you are given the half power frequency, that is something like a cut off frequency okay; you can you can design the filter, you select the particular value of r or given r you can determine what will be its half power frequency. This is a first order model for the filter that is both the numerator and denominator, r of power z to the power minus 1.

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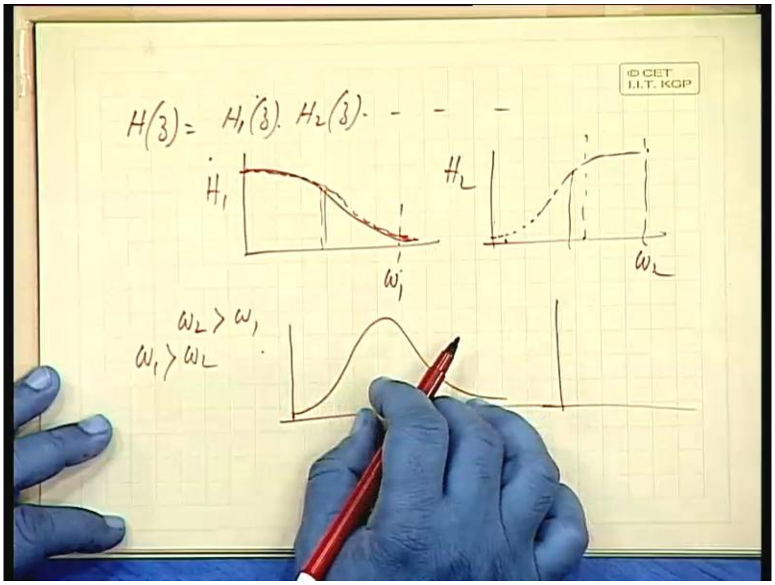

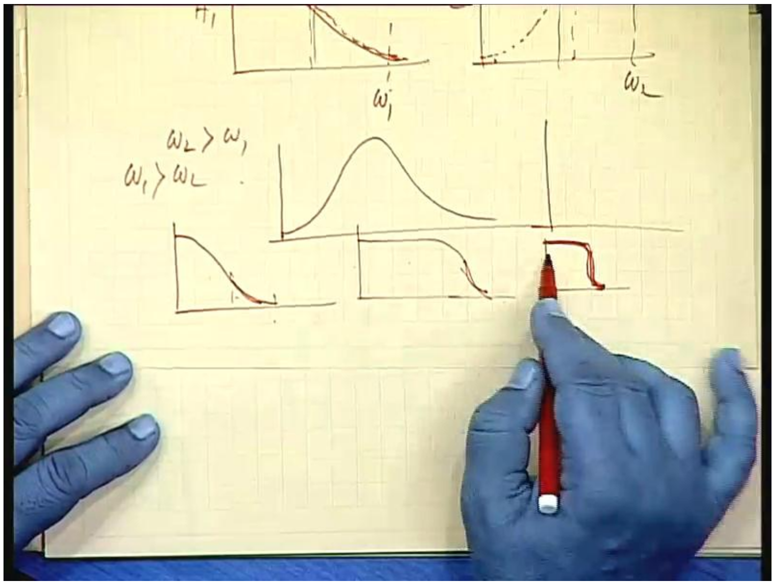

Now, any H z I can always write as, say you can factorize the numerator and factorize the denominator; so H 1 z into H 2 z into so on, so they will be all in cascade. So some may be low pass, some may be high pass and so on. So, it is a net product of all these characteristics that will be determining, the overall characteristics of the function.

So, I can have say a low pass, for example somebody may have a low pass filter like this; all the frequencies are allowed upto a frequency of say, omega 1, after that it tapers off all right. It is supposed, this is a half power frequency. And another 1 H 2 has like this omega 2, omega 2 is greater than omega 1 much greater than omega 1, then what will be the combination? Here it is high, but here it is low, the product is very low.

High means, 1 okay normalized then it will start from the low value. Somewhere in the middle, it will be two finite products, again here it is high, but this is low; so it will be somewhat like this, it will be a band pass. Suppose, omega 1 is less than omega 2 omega 1 is greater than omega 2, okay. Will it be same? Shifted, so you keep on changing this, all right, you can keep on changing this.

You have two low pass filters; one is like this the other one is like this. Then the product will be you see, at this edge it is a very small quantity, this is also falling very fast, so the overall product will be falling faster. So, you can get sharper filters, all right. So, if you want to have a very sharp cut off all right, almost tending to an ideal one then you can have 2 or 3 such low pass filters in cascade, all right.

I can have the same filter functions; square it, so H 1 square, H H 1 into H 1 into H 1, if I put the same H 1, I will get the same filter function but with much sharper fall, okay. So, the roll off can be adjusted by having the filter order increased, okay. Now, if in the same function we have say, H high pass is K into 1 minus z inverse by 1 minus alpha z inverse.

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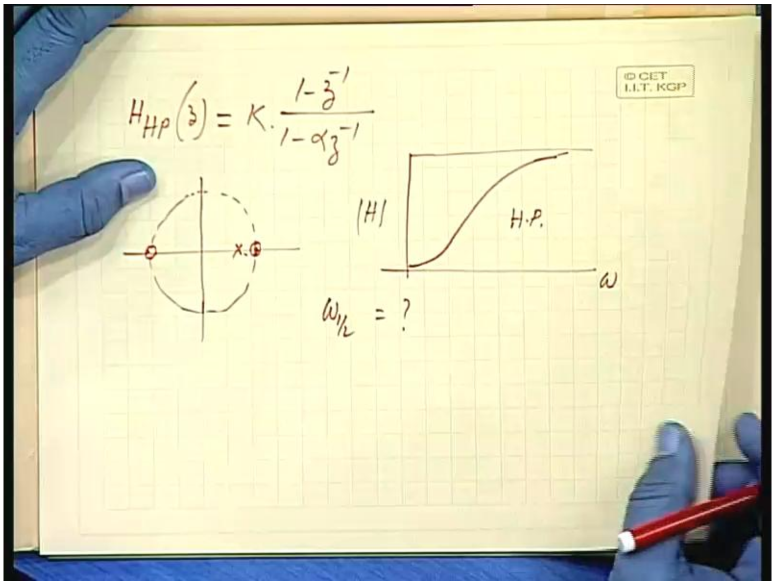

The last one that we had; if you remember 1 plus z inverse, now I am shifting the zero to the other side, these I have written as it is then what do you get? Just like the previous case of only zeros. Now, I put a zero here; earlier it was sorry, earlier the zero was here, now I put a zero here, okay. So, now z is equal to 1 is a 0 and there are there is a pole here.

So in the first case, it was like this. In the second case, it is like this. At, omega equal to 0? Now, this is minimum 0; so it will start from zero will go like this, so this is a high pass characteristics, the earlier one was falling. Here also you can determine, I leave it as an exercise, find out the half power frequency omega half, all right. Now, let us take some interesting distribution of the filter coefficients.

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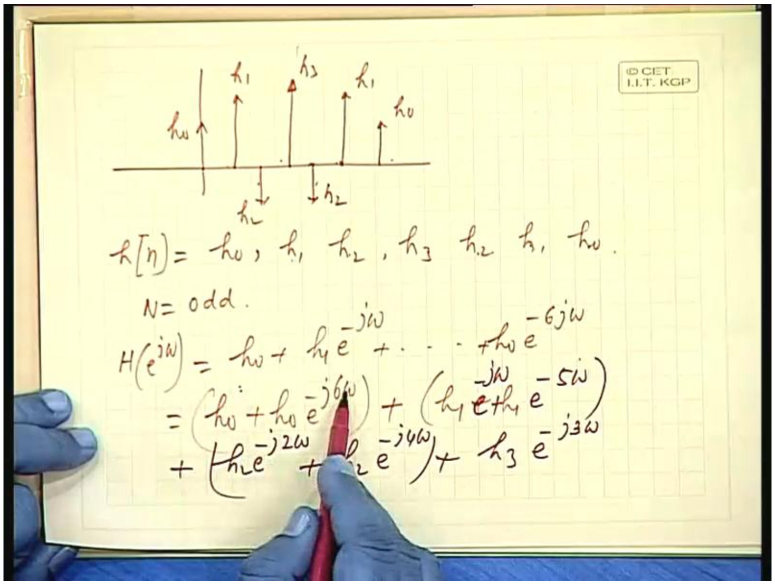

Suppose, we have a symmetric distribution okay h n is a finite length; that is only the numerator, again only the zeros, which is h 0, h 1, h 2, h 3, h 2, h 1, h 0 that means. Suppose, h 0 is this much, h 0, h 1, h 2, h 3 then again h 2, that means h 4 is h 2, h 5 is h 1, then h 0. How many points are there? One, two, three, four, five, six, seven, so number of points N is odd, okay.

We are starting with an odd sequence or number of points, and it is symmetric about the middle point. What will be H e to the power j omega like? It will be h 0 plus h 1 e to the power minus j omega and so on. Last term is one two three four five six, h 0 e to the power minus j 6 omega sorry, is that all right? Now, I can pair these two terms, these two terms, these two terms; write them together, put them under the same bracket.

Plus h 1 plus, sorry h 1 into e to the power minus j omega h 1 into e to the power minus j omega plus h 1 into e to the power minus j 5 omega, is that all right? Plus h 2 e to the power minus j, how much? 2 omega plus h 2 e to the power minus j 4 omega, okay? Plus h 3 e to the power minus j 3 omega, all right.

Now, you add these two together. You can take the common term out, h 0 and if I take the mid value, this this is e to the power j 0. So, 0 and 6 omega, what is the average, 3 omega. So, I will take e to the power minus j 3 omega outside, so it will be h 0 into e to the power minus j 3 omega, outside okay.

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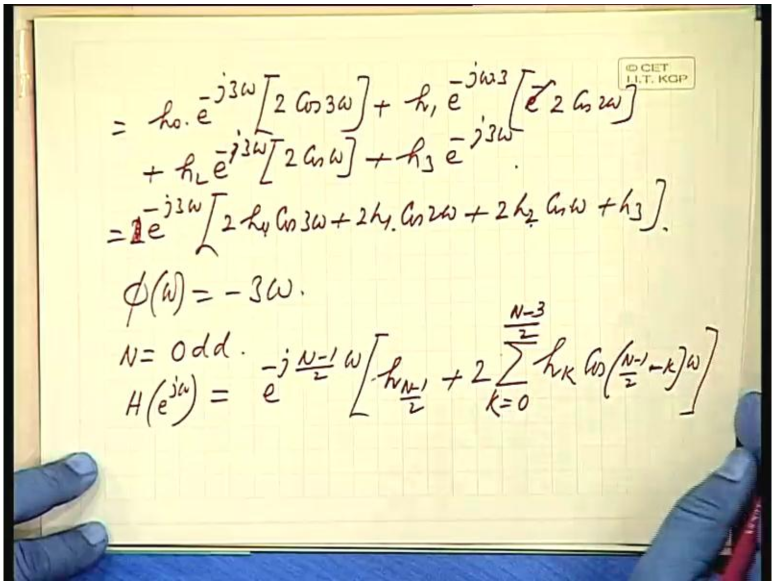

Inside, I will be left with e to the power plus j 3 omega e to the power minus j 3 omega, okay. So, it will be twice cosine 3 omega, is that okay. Similarly, second term could you suggest how much it will be? e to the power minus j omega if I take common, j 3 omega is it not? Because, it is a average of 5 and 1, that I am taking out, okay. So, I will get e to the power minus j 2 omega and plus omega, so that will be twice cosine 2 omega.

Similarly, h 2 e to the power minus j 3 omega into twice cosine omega plus h 3 e to the power minus j 3 omega, okay. If I take e to the power minus j 3 omega common; I find inside it will be, even 2 can be taken out okay, 2 need not be taken out. Twice h naught cosine 3 omega plus twice h 1 cosine 2 omega plus twice h 2 cosine omega plus h 3, okay.

So, these are all real quantities. There is an e to the power j theta term, so directly we get the phase as 3 omega. Phase is varying linearly with omega, is that all right? So, phase is minus 3 omega, magnitude is changing. Can we generalize this? If N is; you have taken odd therefore, H e to the power j omega summation will be e to the power minus j N odd, we had seven points is that so. So, N minus 1 by 2 it was half that angle, half that value, half of 6, is it not?

So, N minus 1 by 2 into omega, is that all right. Then H 3 is h n minus 1 by 2 that is a free floating term, plus 2 times sigma I can write h K; if this is K then this is 3 omega, if this is 1 this is 2, this is 2 this is 1. So, coefficient of that will be cosine of N minus 1 by 2 into minus K into omega, is that all right? K varying from 0 to, this will be less than 1; so N minus 3 by 2 is that all right? Let us now, go for N even.

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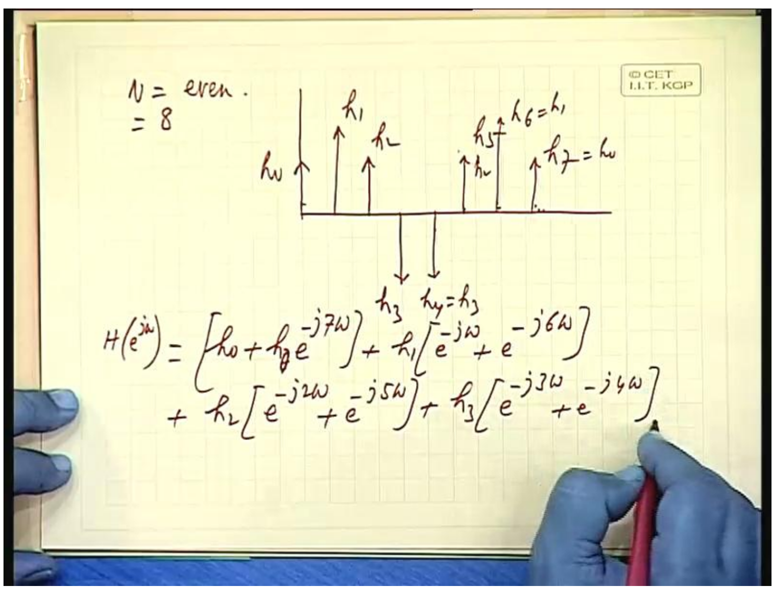

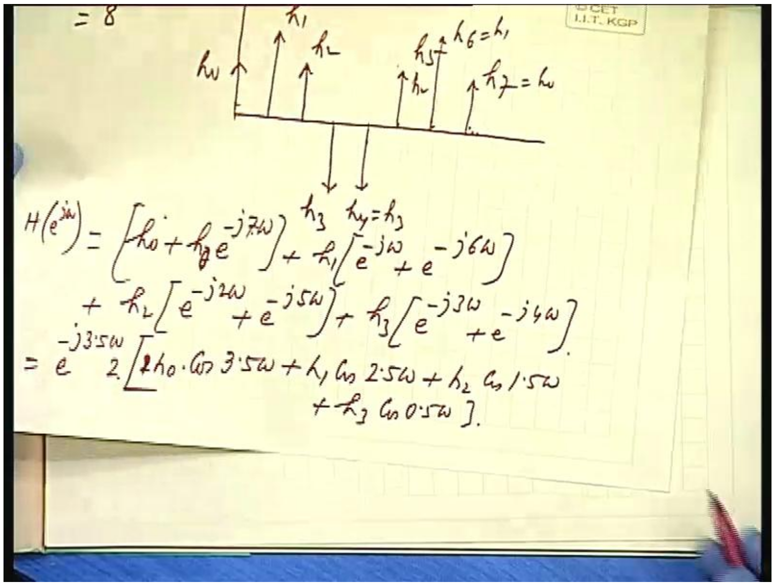

Then, we have say N is equal to 8, so h 0, h 1, okay h 2, h 3, h 4, h 5, h 6 and h 7. So, h 1 is equal to h 7, h 1 is equal to h 6, h 0 is equal to h 7, h 1 is equal to h 6, h 2 is equal to h 5, h 3 is equal to h 4, all right. So, this is equal to h 3, this is equal to h 2, this is equal to h 1 and this is equal to h 0, then what will be H e to the power j omega, like?

This and this if you pair, will be h 0 plus h 7 e to the power minus j 7 omega; and h 7 is nothing but h 0, all right. Plus h 1 into e to the power minus j omega plus e to the power minus j 6 omega, okay plus h 2 into e to the power minus j 2 omega plus e to the power minus j 5 omega plus h 3 into e to the power minus j 3 omega plus e to the power minus j 4 omega, is that all right.

Now, average of these two parts 7 and 0; so 3 point 2, 3 point 5, 3 and half? So, I I take 3.5 omega, e to the power minus j 3.5 omega common. So, e to the power minus j 3.5 omega, I should write 7 by 2, no, okay; see that, that would have been better. Now, I will get twice h naught cosine 3.5 omega plus.

Now, I can take 2 outside no, all of them are paired. So, h 1 cosine 2.5 omega plus h 2 cosine 1.5 omega plus h 3 cosine 0.5 omega. So, in a more general form if I write, it will be e to the power minus j; I have taken N even so 8 minus 1, 7 by 2.

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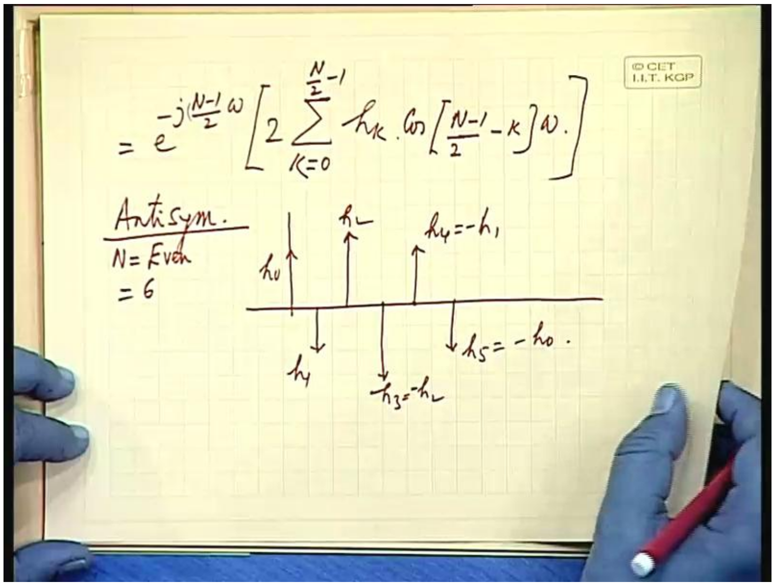

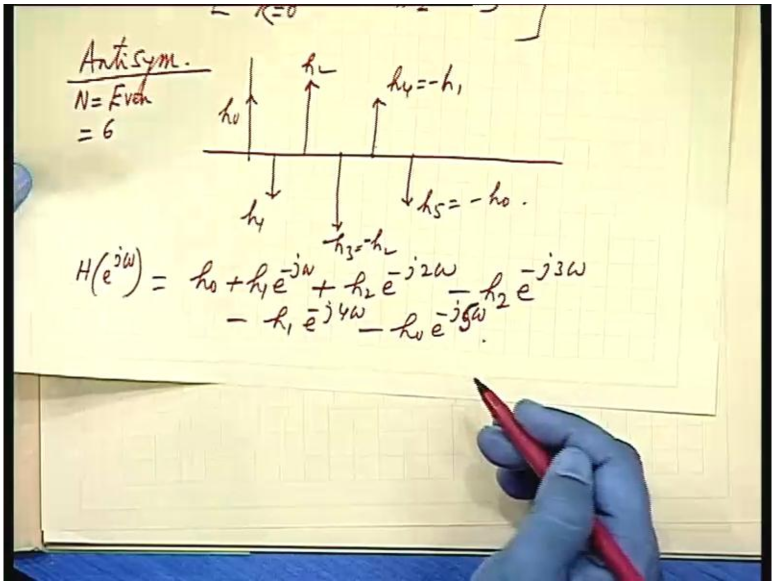

So, it is N minus 1 by 2 omega common, okay. Then 2 into sigma h K, if I write K varying from 0 to 3 that means; N by 2 minus 1, is that all right. N by 2, 8 by 4, 8 by 2, 4 minus 1 3, h K cosine, how much is it? 3.5 means, N by N minus 1 by 2 by 2 minus K into omega, is it all right? So, this is for N even. So, N even and N odd, there will be differing in the general expression; not in this, this is same, phase is same, N minus 1 by 2.

Now, it will be 3.5 omega, earlier it was 3 omega for seven point sequence, for eight point sequence, it is 3.5 omega. And inside this term was free, these terms were like this. Now, here it is identical. You see h K cosine and N minus 1 by 2 minus K omega; only this term was earlier present, now it is missing, all right.

Now, let us have another symmetry; this is anti-symmetry, an anti-symmetric sequence is like this. Once again let us take N is equal to even first, okay, even. So, this is h 0. So, this is h 1, h 2. Let me take only three points, three points on this side, three points on that side; say 6 then we have h 2, h 3 is minus h 2 then h 4 is minus of h 1 and h 5 is minus of h 0, all right.

Therefore, H z or H e to the power j omega will appear like this, h 0 plus h 1 e to the power minus j omega. I have arbitrarily taken plus and minus symbols, for h 1, h 2 but whatever is a value of h 1; will be negative of that in case of h 4. Whatever is a value of h 2, it will be negative in case of h 3. So, h 2 e to the power minus j 2 omega, minus h 2 e to the power minus j 3 omega minus h 1 e to the power minus j 4 omega minus h 0 e to the power minus j 6 omega, is it all right, 5 omega, 5 omega. Now, if you pair them what do you get?

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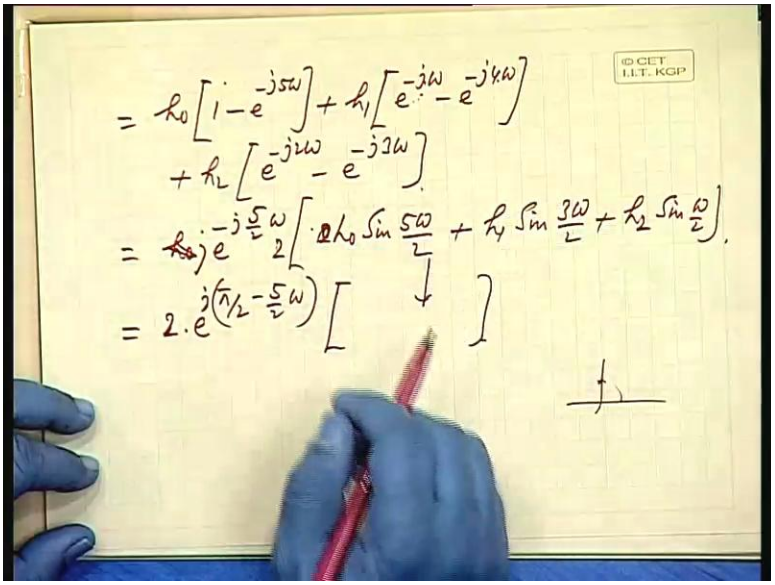

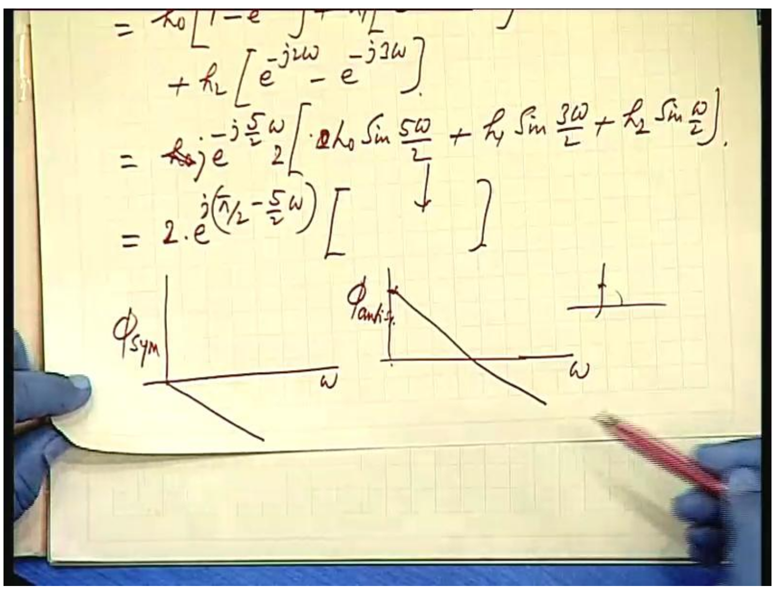

Equal to h 0, okay into 1 minus e to the power minus j 5 omega plus h 1 into e to the power minus j omega minus e to the power minus j 4 omega; correct me if I am wrong, h 2 e to the power minus j 2 omega minus e to the power minus j 3 omega, okay. So, once again you take the averages. This is 5, this is 0, this is 4, this is 1, so 5 by 2 is a common.

h 0 okay, I will take e to the power minus j 5 by 2 omega common. Then what do you get inside? You get 2 h naught sin 5 omega by 2 with a j term, everywhere there will be a j term because it is minus. So, I will put j term also they are paired, so 2. I can write like this, 2 j outside. So, I get h naught sin 5 omega by 2 plus h 1 sin 3 omega by 2 plus h 2, how much? Sin omega by 2, is it all right?

So, one may write 2 j, j can always put as e to the power j p i by 2, is it not. j I can always write as; e to the power j p i by 2, so 2 into e to the power j p i by 2 minus 5 by 2 omega, is it all right? And inside I will have this quantity. Again this is a real quantity, so this is the magnitude and this is a phase. So, how much is the phase? p i by 2 minus 5 by 2 omega?

So in the earlier case, I was getting a phase shift like this, all right. Now, I am getting this was phase shifted by p i by 2 but again linearly falling. This was in the previous case; when there is symmetry and this is when there is anti-symmetry. And there are sin terms, in that case it was cosine terms. So, this is also linear characteristics, all right. If you have some phase, fixed phase removed then this is as good as this. What will be the general term for this?

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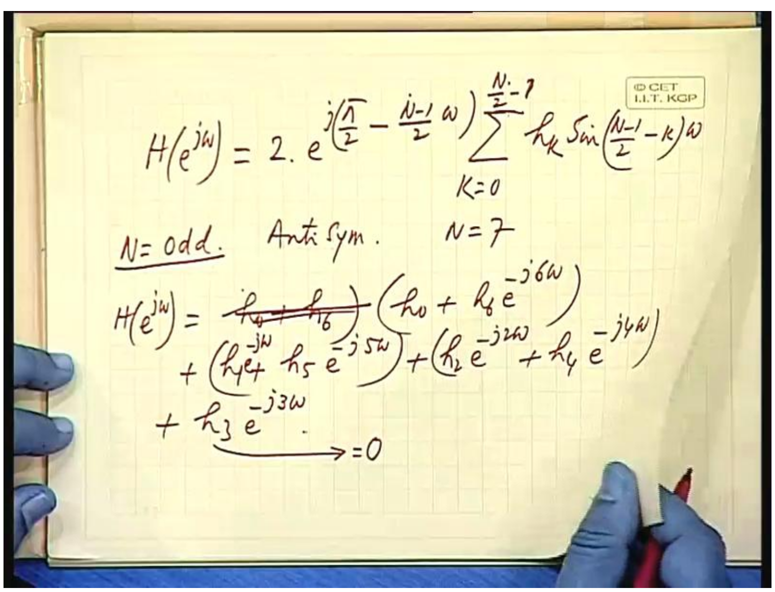

H e to the power j omega will be when N is even; it is 2 e to the power j p i by 2 minus, this is 5 by 2 that means, six points were there so N minus 1 by 2 all right into omega, summation. Now h 0, h K and then sin this is decreasing gradually; so this is N minus 1 by 2 minus K into omega all right. What is the range of K? K is equal to 0 to 0, 1, 2. So, N by 2 minus 2, okay minus 1 all right, so this will be the general expression.

What about the anti-symmetric sequence, when N is odd? Anti-symmetric, so let us have again H z, okay, H e to the power j omega I straight away write this form, say h 0. Let us take five terms or may be seven terms. So, h 0 plus h; if I take N odd, it will come at say 6 including h 0, there there are seven terms, say N is equal to 7.

Then h 0 plus h 6 into okay, h 0 plus h 6 e to the power minus j 6 omega; h 6 is minus h 0 that I substitute later plus h 1 plus h 5 e to the power minus j 5 omega, e to the power minus j omega, thank you. Then h 2 e to the power minus j 2 omega plus h 4 e to the power minus j 4 omega plus h 3 e to the power minus j, then what will be h 3? It could be zero anti-symmetric, h 3 cannot be equal to minus h 3, h 3 has to be necessarily 0. So, this is what I was expecting you to answer h 3, there is a mid-point which is 0, what will it look like? The sequence looks like this.

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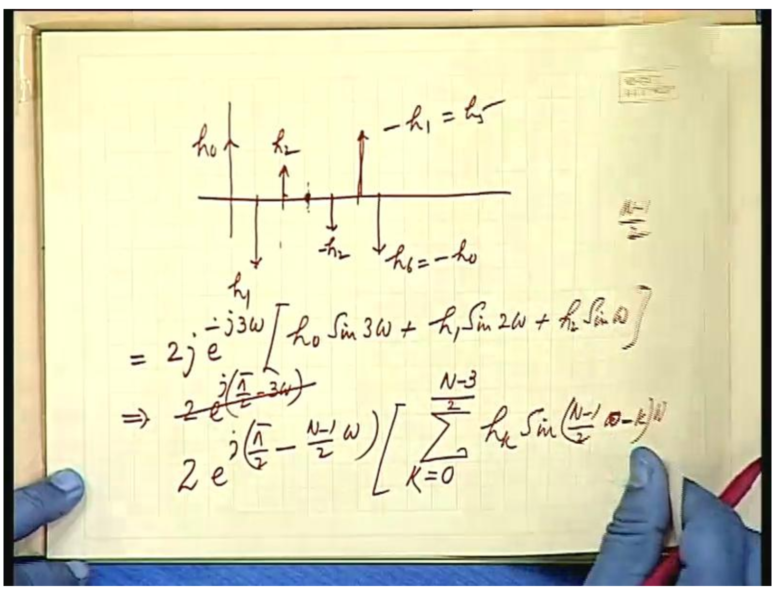

Suppose, this is h 0, one two three four five six all right, so at h 6 it will be all right. Suppose, h 1 is like this then h 5 will be like this minus of h 1. And then this one say, only this one h 2, this one will be h 4 is equal to minus h 2, okay. Had it been some value here? Then from the other side also it would have been the negative of it; so it it cannot be anything other than 0, all right. So, here it is 0. So, what will be the sequence sum now?

h 1 and this is minus h 1, so I can take h 1 common and again it is minus, that will generate sin terms. So, if you permit me to write; it will be twice j e to the power minus j, now if N is odd 1 and 5, six by two, so that will be three. So, minus 3 omega, it will be h 1, h 0, sin 3 omega plus h 1, sin 2 omega plus h 2 sin omega, all right.

So, the general term will be, the general term will be 2; again j you can push as p i by 2, so e to the power j p i by 2 minus three omega. So 3, I will write as N minus 1 by 2, okay. So, 2 into e to the power j p i by 2 minus N minus 1 by 2 omega okay, is it all right? Inside we will have a summation, h K sin N minus 1 by 2 omega minus K into omega; K varying from 0 to N minus 3 by 2, why N minus 3 by 2? h 3 is zero, all right.

So, we have seen different types of filter functions; one is with a pole with a 0 either on the real axis, negative real axis or positive real axis or if both poles and zeros are present. And then we have also seen, we have also seen if there are complex roots; complex poles and complex zeros then the nature of the function can identified either as a low pass, high pass, band pass and so on.

Now, we have seen symmetric distribution, different types of symmetric distribution and anti-symmetric distribution; if the length is odd or even, so there are four possible cases, N is equal to odd, N is equal to even, symmetric and anti-symmetric.

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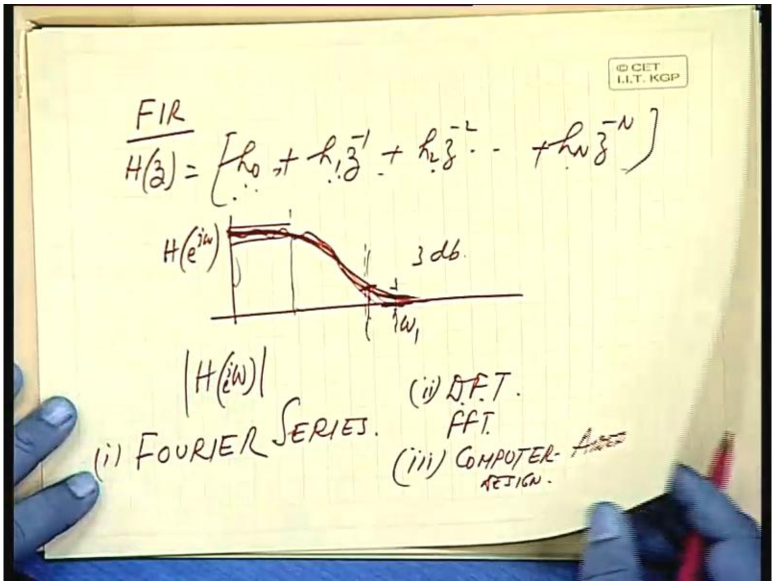

Now, FIR filters are basically functions of this type, where you have h 0 plus h 1 z inverse plus h 2 z to the power minus 2 and so on, some h n z to the power minus N, say. This is the type of filter coefficients that you want to find out. Suppose, I ask you to design a filter having a linear phase characteristics then these coefficients will have symmetry. We are just now seen, if there is a symmetry there is a linear phase characteristics. And if I give you the gain function, you can calculate what will be the h coefficients.

Once you know h, h 0, h 1, h 2 all the h values; so it is an eleven point sequence I say, then you can design it from the specifications and the filter design is very simple. The filter is going to meet all the requirements. Now, what is the specification, nature of specification that you give? We normally give H j omega function.

Suppose, it is a low pass filter, there are various types of specifications; somebody may give a monotonically falling low pass function, somebody may define this alpha sorry, the 3 d b cut off point is so so much and roll off, how fast it should fall, the transition is given all right. Somebody may give within this frequency band, within this tolerance it should come at such and such frequency; at omega 1 it should come within a certain d b level all right. There can be also specification like this; there can be a tolerance in the pass band where we define a pass band and so on, there can be various types of specifications.

Now, if you know H omega, H e to the power j omega magnitude function, if you are asked to design a linear phase filter; then we will take up only the magnitude function first, then try to see how the linear phase can be obtained. There are various approaches. The first one is Fourier series approach, Fourier series all right. The next one is discrete Fourier transform, DFT approach for which we take the help of FFT algorithms which is very fast, all right.

And then there are many computer aided design, we will take up those also; computer aided design approach, there are many variations of this that we will see later on. So, I will just give a brief introduction to Fourier series approach for the design of an FIR filter.

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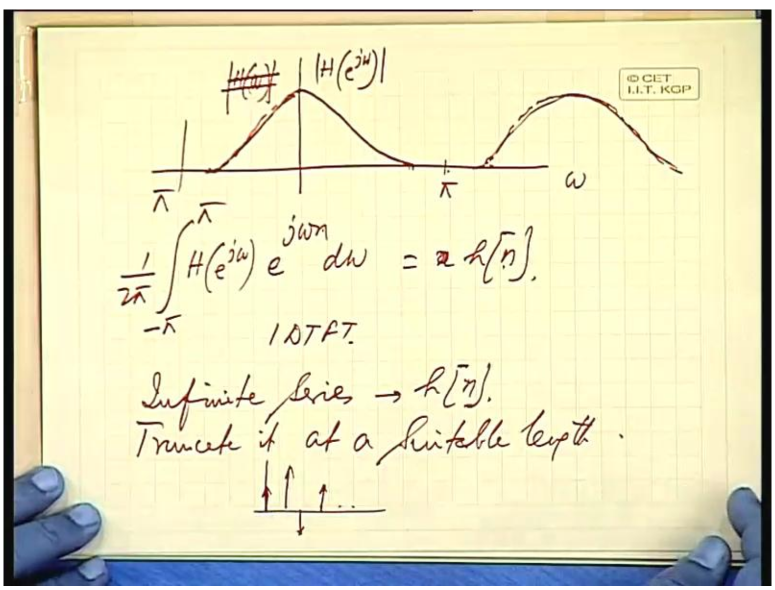

You know H omega that you are considering; I am in the habit of writing only omega, so quite often I slipping that, H should be always written e to the power j omega to distinguish to distinguish it from the analogue domain. Suppose, this is a filter function, this is our p i by 2 p i; this is kept normally well within the band is kept within p i, it will be repeating, is it not?

So, not exactly this is p i, suppose I define this function, I give you some analytical form all right; then H omega whatever be that function, if this is given multiplied by e to the power j omega n d omega from minus p i to plus p i, all right will give me into 1 by 2 p i, will give me h n, is it all right.

n will correspond to this n, is it not? So, this is basically reverse transform. We are taking I DTFT, because it is periodic; so you will get a sequence h n which is of infinite length. So, you will get an infinite series for h n then truncate it in a suitable place. Next step will be truncate it at a suitable length.

So, suppose by doing this you get a sequence of h 0, h 1, h 2 obvious; h n by integrating from minus n to plus n, it will be, will it be symmetric? You cannot say, okay. Let us see, what it will be like.

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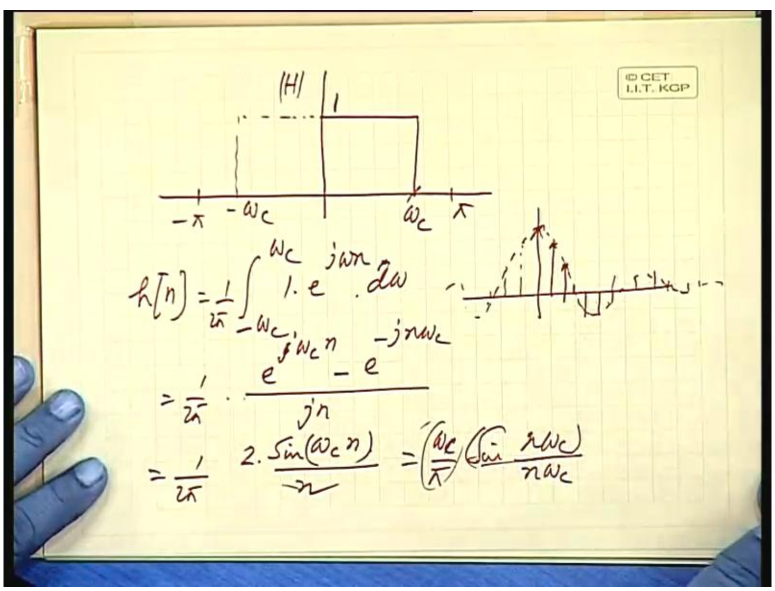

Suppose, you take a very simple function like this; it is already defined here also because it will be a mirror image. So, minus p i to plus p i and this is the cut off omega c, minus omega c. I am taking H equal to 1, normally valued and a very simple low pass filter. What will be h n? It will be 1 by 2 p i minus; this is 0, so I can take minus omega c to plus omega c 1 into e to the power j omega n d omega.

How much is that? 1 by 2 p i, e to the power j omega n; so it will be e to the power j omega c n minus e to the power minus j n omega c divided by j n, is equal to 1 by 2 p i. This minus will give me, 2 j sin omega c. So, j will get cancelled. So, 2 sin omega c into n divided by n, okay. So how much is it? 1 by p i sin n omega c by n, I can include omega c sin n omega c by n omega c and, I can multiply by omega c these at multiplier.

So, what is this? It is a sink function. So, h n will be these values, a sink function looks like this these values. If I would have considered h minus n then I would have get these values, okay. So, if I now truncate say; at the tenth term is h 0, h 1, h 2, h 3, h 4, h 5 and tenth term. I take h minus 1 as h plus 1, h minus 2 as h plus 2 and so on, okay. And then give it a shift by ten steps, then what do I get?

At twenty-one point sequence, first middle one is h 0 and then there are ten terms after this; there are ten terms before these, I am considering them and then give it a shift, what is it look like? Symmetric function, which will ensure a linear phase, is it all right. So, what you need to determine is, only the first eleven coefficients including h 0, another ten coefficients all right.

And then you can generate a sequence with a shifting, I will call it h 0 dashed which is equal to minus h 10, h 1 dashed which is equal to minus h 9 and so on. So, I will get a sequence of twenty-one points. So, it is a twenty-one point sequence filter, which will ensure that this is more or less approaching this.

If you have a larger length then it will be coming closer to this that means it will be less, okay. Okay, I think we will stop here for today. Thank you very much, so we will take up from this in the next class. If you have any questions, please bring them in the next class all right.