Digital Signal Processing
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Lecture - 13
State Space Representation

In the last class, we started discussions on state space representation of discrete systems; if you remember, we took up a system like this.

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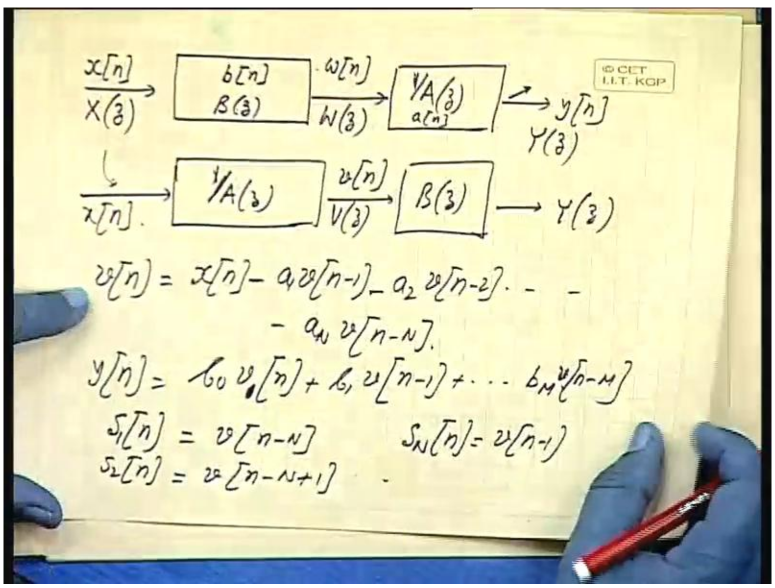

B z by A z is a transfer function, you broke up into two parts; the numerator and the denominator they were segregated, it can be either this way or this way, okay. We also wrote the discrete domain equations for these intermediate variables, either in this form or in this form. And the output Y n, we wrote in terms of this intermediate variable v n in this form. We selected some functions, time driven functions like this, so we will continue from here.

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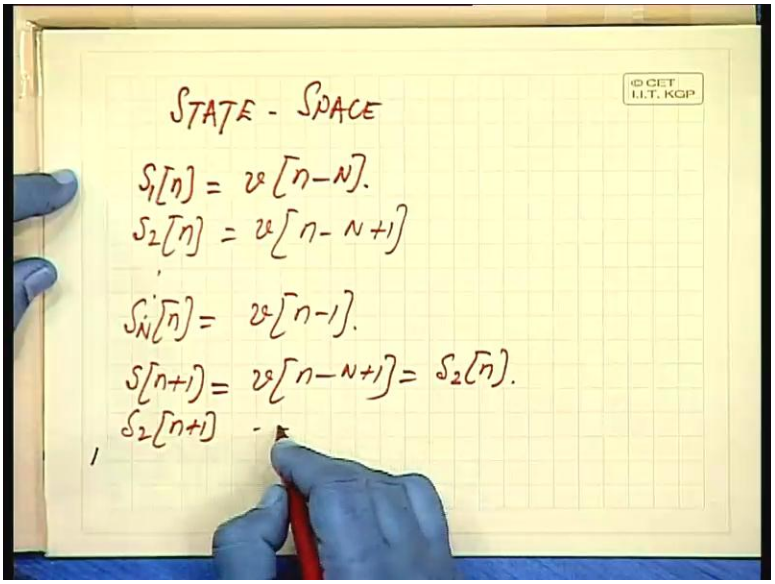

So, the discrete functions that you defined S 1 n was v n minus N, S 2 n is v n minus N plus 1, and so on. S N n was v n minus 1, okay. Therefore, you can write S n plus 1 equal to v n minus N plus 1 which is nothing but S 2 n. Similarly, S 2 n plus 1 will be equal to S 3 n and so on, okay.

Only the last one, S N n plus 1 will become v n which is see; v n minus 1 plus 1 so that will become v n, which we derived as x n minus a 1 v n minus 1 and so on. So, I can write this as x n minus a 1 into S N n minus a 2 into S N minus 1 n minus a N S 1 n, correct me if am wrong, is that all right?

Now, therefore we can get all these, S 1 n plus 1, S 2 n plus 1 and so on, S N n plus 1, okay.

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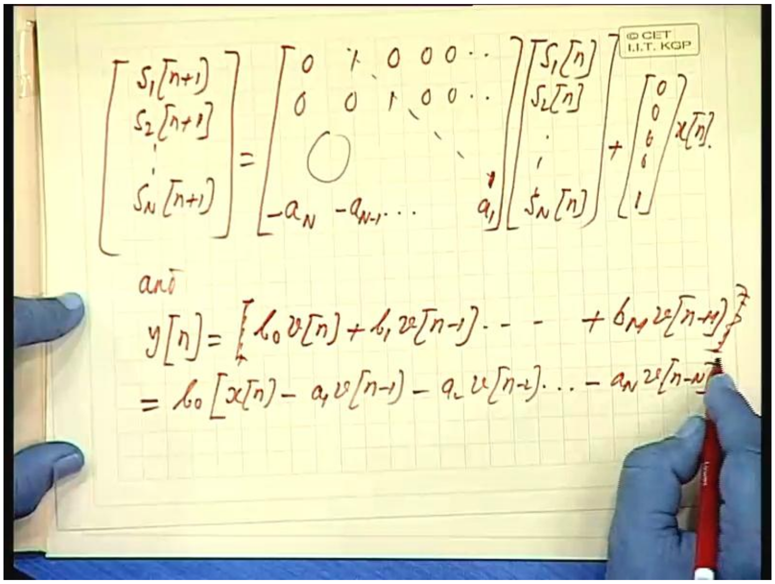

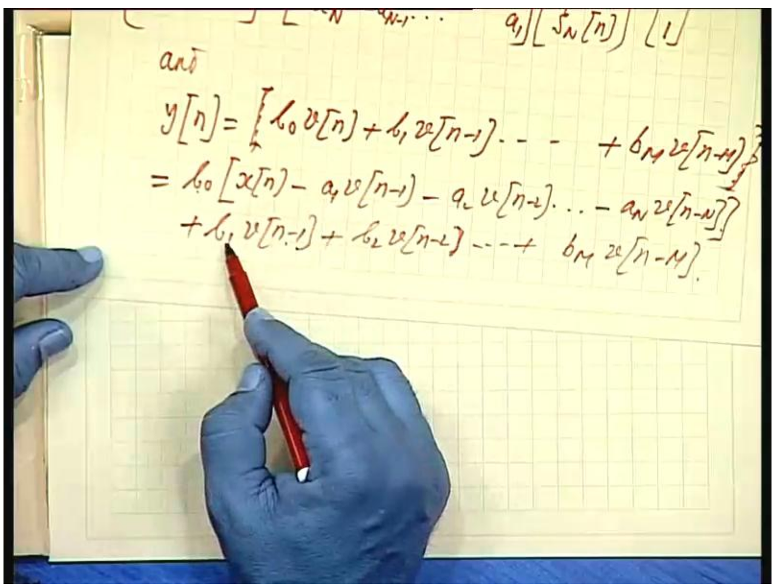

You watch it here; there is a matrix which is a diagonal in this sub-diagonal form and S 1 n S 2 n and S N n, last one will be minus it ends here, these are all zeros. a N minus a N minus 1 and so on, up to a 1, plus I can write 0, 0, 0, 0, only last 1 into X n, okay. This is a very familiar form that you have studied in control systems, okay, in the continuous domain.

And Y n is what was Y n? If you remember; b 0 into v n, b 1 into v n minus 1, and all these v n(s) are basically the state S 1, S 2. So, I can write this as b 0 v n plus b 1 v n minus 1, this summation plus b M v n minus M, I need not put a bracket these are all sum, anyway. I can write this as b 0 into v n was x n minus a 1 v n minus 1 okay minus a 2 v n minus 2 and so on, up to minus a N v n minus n, okay. Then this plus b 1 into this between two the next term and so on.

So, plus b 1 into v n minus 1 plus b 2 into v n minus 2 and so on up to the last term; b M v n minus M, okay let me change the pen, is it all right. So, you can club the terms v n minus 1 together the coefficients of this, minus a 1 b 0 plus b 1. So, it will look like this if you permit me to put them together, it will be b 1 minus b 0 into a 1 into v n plus b 2 minus b 0 into a 2 into v n minus 1 and so on.

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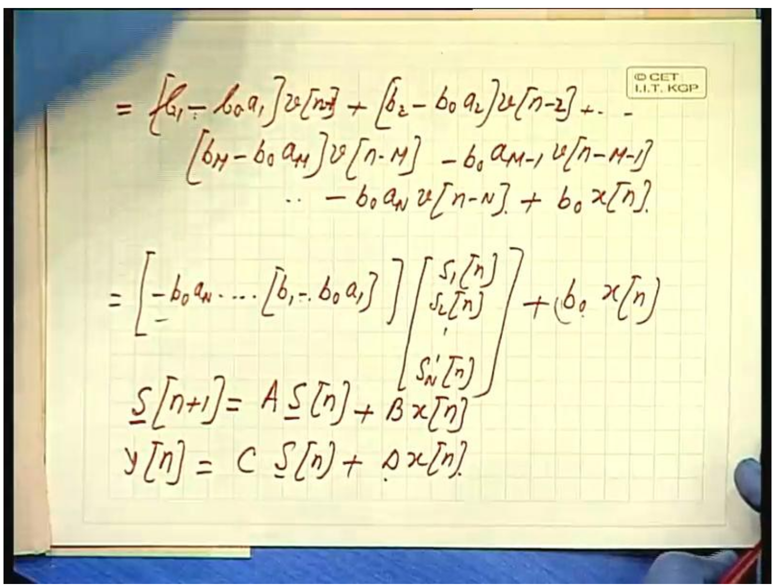

You can write, b M minus b 0 into a M v n minus M minus b 0 a M minus 1 v n minus M minus 1 and so on up to minus b 0 a N v n minus N okay. It will come as a difference of these two terms only up to M th term, all right. After that it will be a single coefficient, because the numerator is of M th order denominator is of higher order N th order

So, I can write this as minus b 0 a n then this is a row matrix okay; b 1, minus b 0, a 1 whole thing into S 1 n into S 2 n up to S N n plus b 0 x n, any slip anywhere, plus b 0 x n, this terms was missed out, okay. It was b 0 into x n the first term, so when we club them that that was dropped out by slip okay, is this all right, it is not correct, okay.

Let us see, what it say minus b 0 a n, b 0 a n into v n minus N, what is it? That is S 1, so it should have been reversed, okay. It should have been the other way…It will start with; no this is the row the first element is… previous step b 1 minus b of n minus 1? v, we are replacing v n thank you, v n minus 1, n minus 2 and so on okay, otherwise this is all right, thank you very much.

So, I can write this as in a compact form all right. A into S n where this is a vector of all the S S 1, S 2, S 3 and so on, okay plus B into x n where B is basically this all right; that column that I got here, B is this, A is this okay. And Y n is equal to C into S n plus D into x n. So, D is this single element okay because this is Y, this is Y n all right and C is this row.

So you get, what would be the solution for this?

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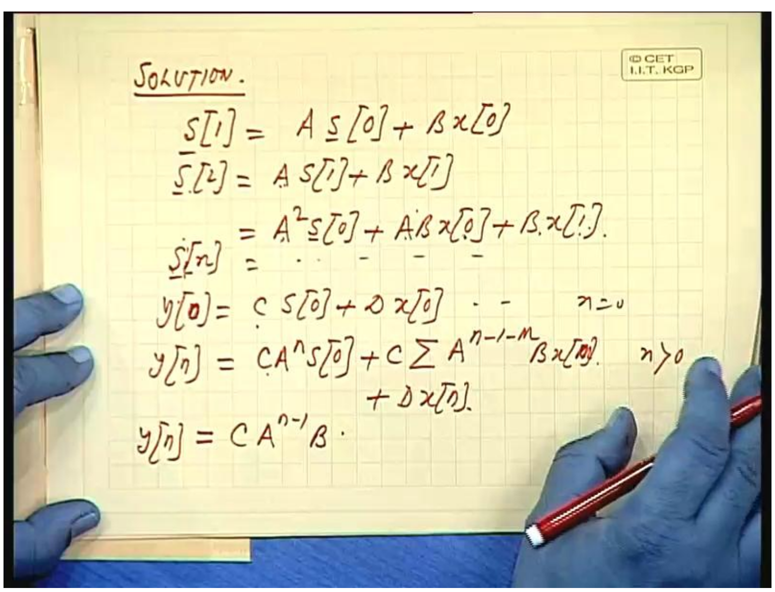

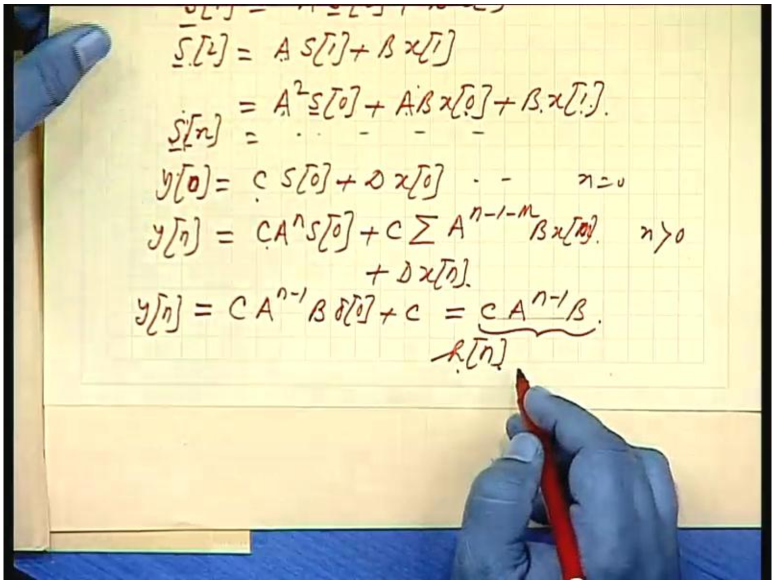

Suppose it is initially relaxed, we will take up take up that, if it is not relaxed then we will have to find out what is its initial condition, initial state. This S we call the state of the system, it is a general health of the system; it can be velocity, acceleration, you can have any parameter of the system, any variable of the system as a state, okay.

Now, the choice of states is not unique, it can be anything; any linear combination of the states that you take, I can choose as a different set of states, okay. So, under some choice suppose we have selected the states, S 1 I can write as A into s naught plus B into, okay. S 2 will be A into s 1 plus B into x 1. And S 1 is this much, if I substitute for S 1 this quantity, I will get A into A; A square S naught plus this whole thing multiplied by A, A B into x naught plus B into x 1, is that all right?

That means, the effect of the input at the previous instant will be felt even at, the present instant all right, it will leave a trail. So, you have like this and Y n output will be C into S naught; if I want to compute it for naught plus D into x naught okay, this is for n is equal to 0 okay. And if I compute for other values of n y n all right, I will have to keep on taking S 1, S 2 and all the terms.

So, C A n S naught plus C sigma A n minus 1 minus M B x K, I should write K okay M for n greater than 0, plus the D term is left out D x n. This you can see by substitutions I have not written, S at other higher values S n, what will be this? You will get S square, cubed, S 4 and so on, A to the power n all right, similarly A to the power n minus 1 and so on, okay.

B will not be square any time, B will appear as it is; it is A which will come start appearing here and next terms and so on, so these all these can be put in a summation form. And C times x, so C times S, so C times that at S n whatever is this; if I multiply by C, so that summation term leaving aside the 0 th term, so it will be like this, okay.

So, one may write for n greater than 0; it will be this is the general term, this is y n for n greater than 0 be finally C A to the power n minus 1. If you have an initially relaxed condition that means initially S 0 is 0, S 0 is 0, all right; that means in an electrical circuit, there is no charge there is no stored magnetic field in the inductor and so on.

If you have an initially relaxed condition then S 0 will be zero, everything will come to a steady state all right, that we assume a state. So, this will be vanishing, it will start from the second term onwards B okay; into delta 0 plus C, that is equal to C A n minus 1 B. Suppose, basically what I am trying to find out is, initially relaxed and I given impulse all right; that is at delta equal to delta n, that is at n is equal to 0, I give an impulse what is the response? Okay.

Otherwise there is no response, if it is initially relaxed, let is relax, is it not, in a given input, so normally tested with an impulse signal. So, if I give an impulse input, this will be the kind of response, okay. So, this I can call as h n, normally the impulse response is denoted as h n of the system, okay. Now from the difference equation, we can write a state equation all right, that is what we started of, is it not.

If you remember the very first day; we could write the difference equation, this is a difference equation.

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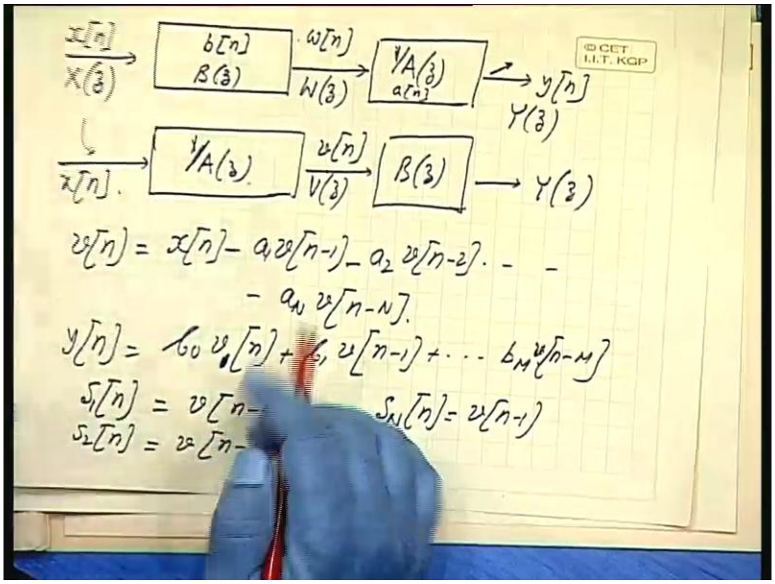

It could have been either in this form or in the other form, all right. B z by A z is given, you can always write the difference equation form; that is they all appear in the in terms of small a’s and small b’s, these are the difference equation constants, is it not. Now from the state space equation, can you get the difference equation? So, now we will just take up, we will take up an example.

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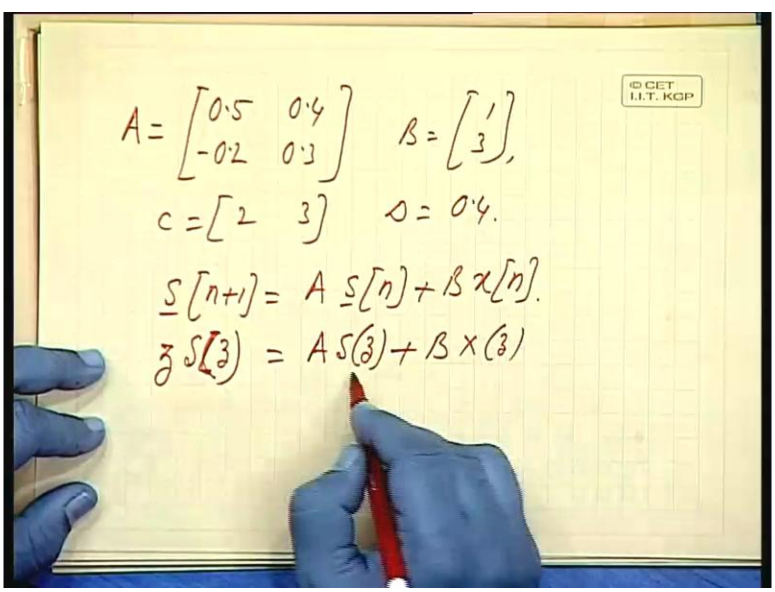

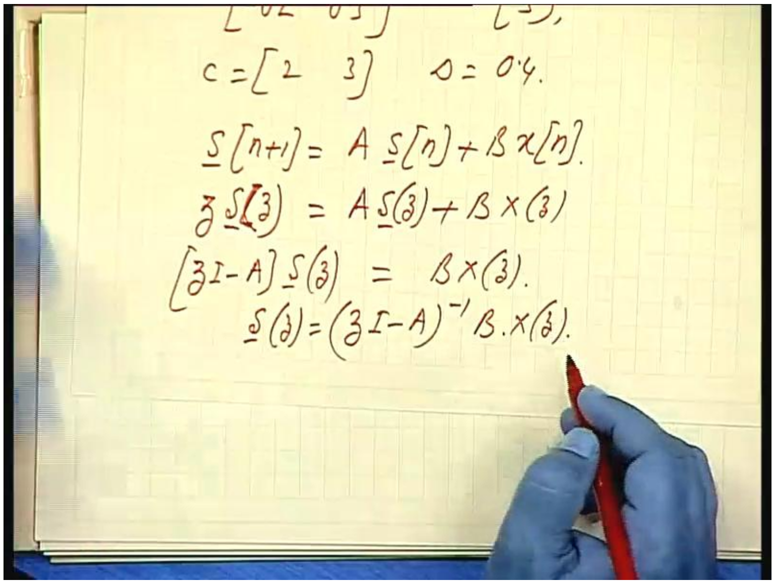

Given A as 0.5, 0.4, minus 0.2, 0.3, B as 1, 3, C as 2, 3 and D as 0.4, what will be the difference equation? Okay. Now, before that before we start of, let us get back to the state equation once again. If you take Z transform of this, what do you get Z into S z okay, this side you get A into S z plus B into X z; S z is a vector because S 1, S 2 these are all vector, elements of the vector. So, Z I minus A into S z is equal to B into X z, you all agree? Is this okay? Therefore, S z is Z I minus A inverse into B into X z, all right.

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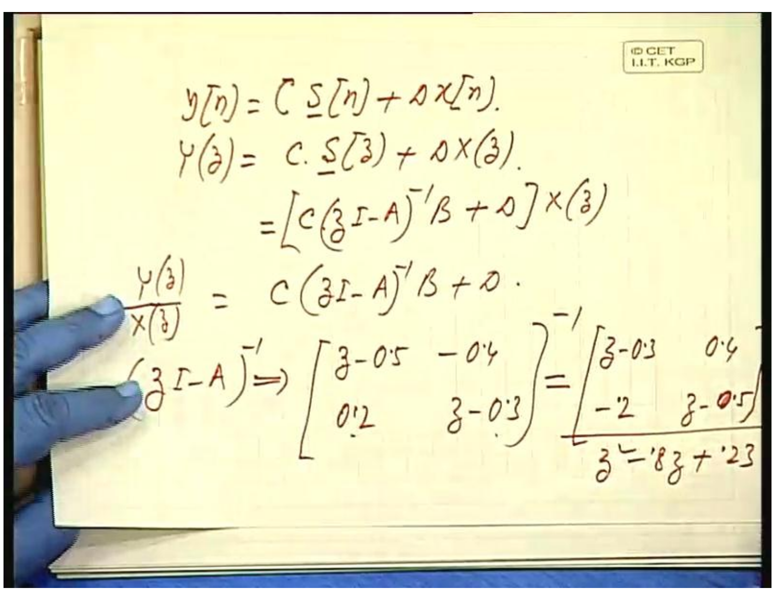

And then we have Y z see, what was y n? C into S n plus D into X n. So, Y z mind you, these are not vectors, finally the matrix product, S only will be the vector others are all scalars. So, after taking the products, we will get only a scalar quantity. So this will be C into S z plus D into X z okay and what is S z? This much. So, if I substitute C into z I minus A inverse into B into X z plus D into X z, so can I write like this? Okay.

Therefore, Y by X in the Z domain is a transfer function; is C into z I minus A whole inverse B plus D, okay. Now, our aim is to obtain from this set of given matrices A B C D, compute this. So, in the Z domain you get the transfer function; once you know the transfer function, you know the difference equation. So, let us compute this from this now, I hope this is all right.

So for the given problem, z I minus A, so this will these terms will be all neglected, subtracted from z I. So, z I minus A inverse will be, how much is it? z minus 0.5, minus 0.4, this will minus, this will minus, this will be plus, this will minus. So, this will be 0.2 and Z minus 0.3, is that all right to the power minus 1, okay.

So this comes out as, check if I have made any mistake. z minus 0.3 and then minus 0.4 will become plus 0.4, this will become minus 0.2 and this would become z minus point, sorry 0.5 and that divided by this plus this. So that gives me, z square 0.5 plus 0.8. So, minus 0.8, 0.8 and 0.5 and 0.3, and then 5, 3za 15 and 8, so plus 0.23, correct me if I am wrong, is that all right, okay.

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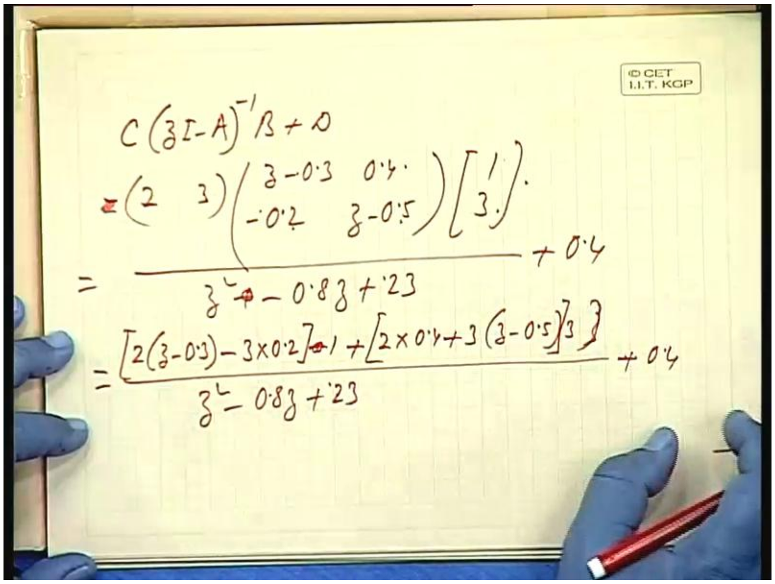

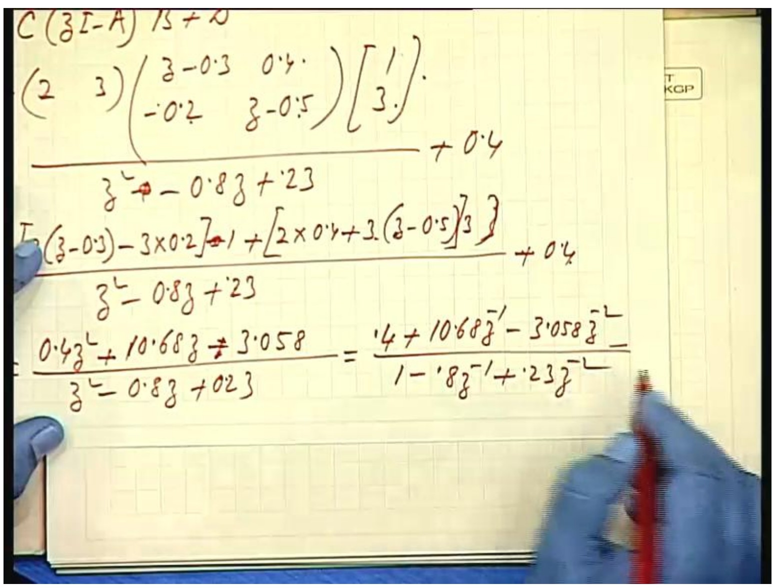

So, C into z I minus A inverse B plus D turns out to be; 2, 3 then you have z minus 0.3, 0.4, minus 0.2, z minus 0.5 into B that is 1, 3 okay. Whole thing divided by z square, minus 0.8 z plus 0.23 plus 0.4, okay. If you simplify 2 into this plus 3 into this; whole thing into 1 plus 2 into this plus 3 into this whole thing into 3, that will be the numerator.

So, I write 2 into z minus 0.3 minus 3 into 0.2 whole thing into 1 plus 2 into 0.4 plus 3 into z minus 0.5 whole thing into 3, divided by z square minus 0.8 z plus 0.23 plus 0.4. So, 0.4 times these you can add in the numerator. So, finally you get if I add these 0.4 z square plus 11; see 2, 3, 3za are 9, so 9 plus 2, 11 minus 0.4 into 0.8, so minus 0.23. So that gives me, 10.68 z add you let me know, what final value you get here, I have got this much; I may be wrong, z square minus 0.8 z plus 0.23, okay.

So divide throughout by z square, so that gives me 0.4 plus 10.68 z to the power minus 1, minus 3.0 this is minus I am getting minus, minus 3.058 z to the power minus 2 divided by 1 minus 0.8 z inverse plus 0.23 z to the power minus 2, okay. So, the difference equation from here comes out as, this is equal to Y by X, is it not. Y by X, so Y multiplied by this will be x multiplied by this.

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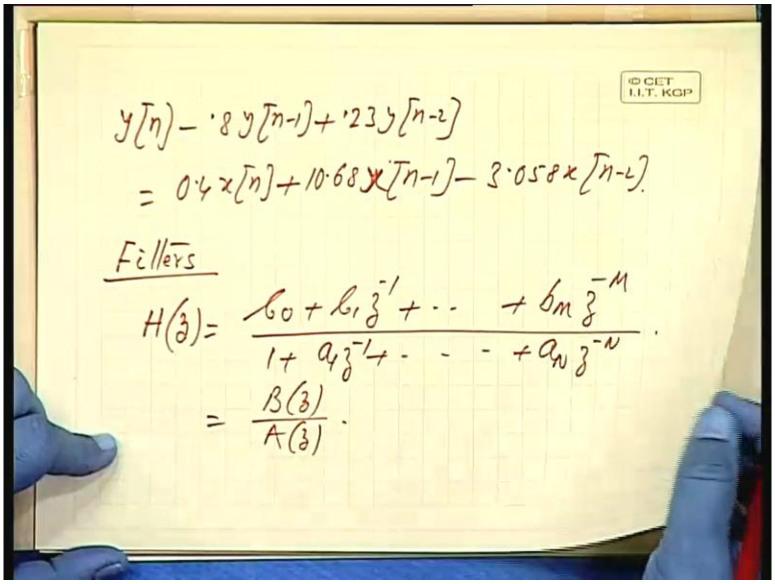

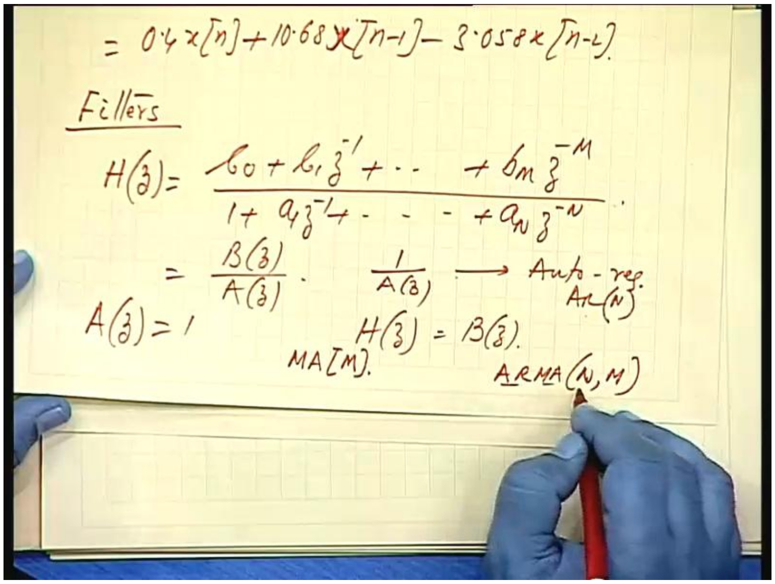

So, Y n minus 0.8 y n minus 1 plus 0.23 y n minus 2 is equal to 0.4 x n plus 10.68 y, sorry x n minus 1, minus 3.058 x n minus 2, is that all right. So this is the difference equation you get, after taking Z transform and then breaking it up.

Now, we shall take up the general case of filters okay. Sometime earlier, probably I had mentioned, I do not recollect about the types of filters that you have in digital systems. We have two types of filters; we define them as what their impulse response IIR and FIR okay. Before we go for the design structure, I mean a design and a structure of filters; we would like to know the general properties of filters.

See H z, suppose H z is b 0 plus b 1 z inverse, we will talk in terms of the transfer function; that would be very convenient divided by okay and that is equal to B z by A z, this is what we mentioned earlier. If A z is equal to 1 then it is having only the numerator polynomial, okay. So, H z is only B z. So, this is known as finite impulse response filter. If I give an impulse after sometime, the output becomes zero. So, it continues only up to a finite length.

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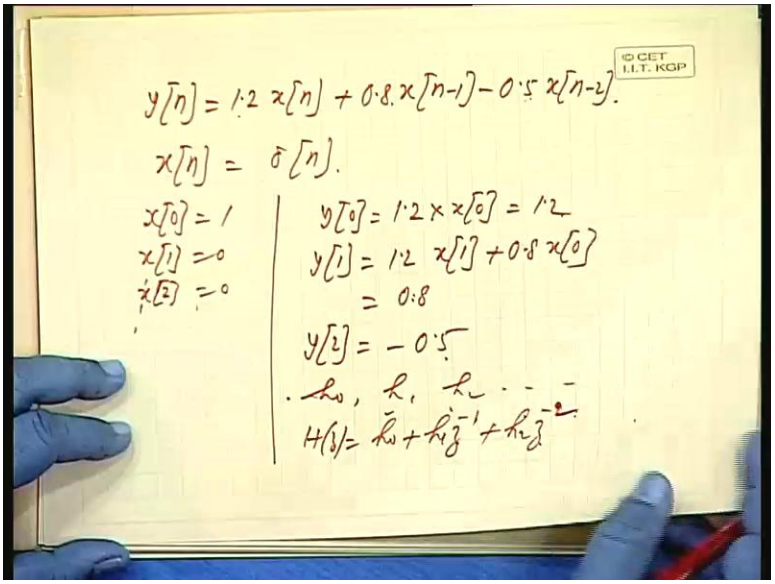

So, if say y n is 1.2 x n, I just taken an example n minus 1, minus 0.5 x n minus 2. If I consider an initially relaxed condition and if you give input x n is equal to delta n, that means what is x 0? 1. x 1? 0. And afterwards all of them are 0's, x 2, x 3 etcetera all are zeros. So, what will be the output, y 0? It will be 1.2 x n, so 1.2 into x 0 which is 1.2.

What will be y 1? It will be 1.2 into x at 1 plus 0.8 into x at 0 which will be; this will be 0 but this will be 0.8. y 2 will be similarly, 1.2 into 0.8 into 0 and minus 0.5 into 1. So for such a filter, these coefficients are nothing but the sequence of impulse response, 1.2, 0.8 minus 0.5, okay. So, basically this will be the coefficients of the impulse response; that is what we have also seen, H z if it is given only the numerator form. So, H z will be h 0, h 1, h 2 and so on so.

So, H z is h 0 plus h 1 z inverse plus h 2 z to the power minus 2. So in the filter coefficient, that is the coefficients of z to the power minus 1, z to the power minus 2, you get the impulse response terms all right. These terms denote basically, the response at different instance for a impulse input. Suppose, you have on the other hand y n Y n is equal to x n minus a 1 y n minus 1, minus a 2 y n minus 2, and so on, okay.

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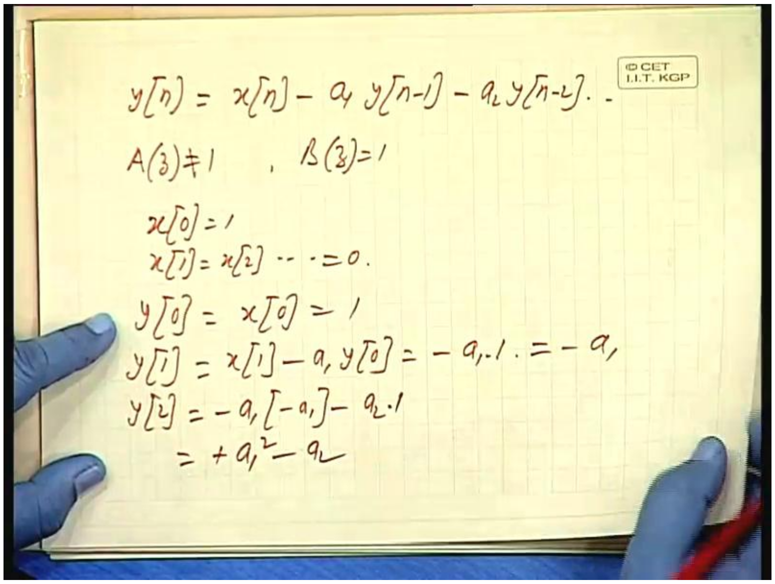

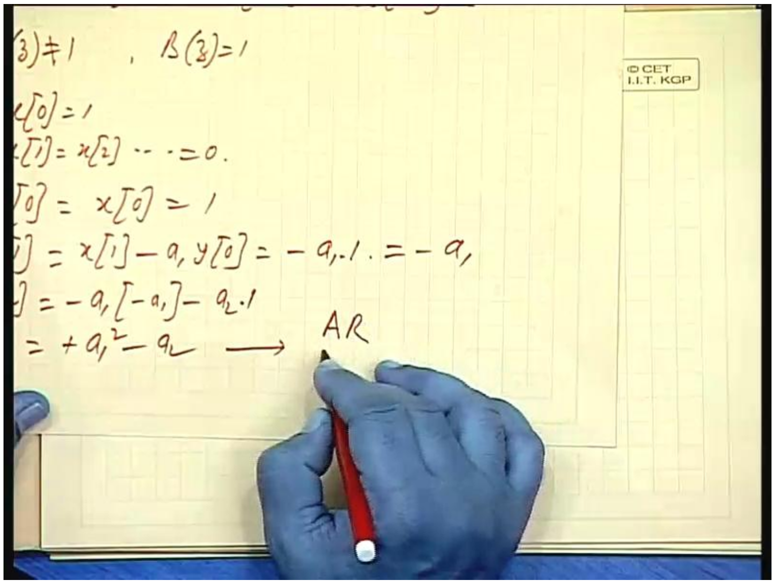

This means, A z is not not 1, this is finite but B z is 1, is it not. If B z is 1 then only it will be; if this is 1, so x gets multiplied only by the numerator and y gets multiplied by the denominator and then, if we shift these terms to the other side you get such an expression, okay. So, here you will find, even if I give a delta input x, 0 is 1, x 1 equal to x 2 they are zeros etcetera equal to 0, then what will be y 0?

These are all initially 0, initially relaxed. So, there all zero initially, so it will be x 0 which is 1. What will be y 1? It will be x 1, which is 0 minus a 1 times, y at 0 which is minus a 1 into 1 that is x 0. y 2, similarly will be how much? This will be 0, this will be minus a 1 times y at the previous instant which is minus a 1, minus a 2 into y n minus 2 which was at y 0 which is 1.

So, it will be minus and minus will make it plus a 1 square, minus a 2 and it will continue like this. You will find because there was, an output in the previous instant. So, it will be affecting the present output. So, it will continue till an infinite time. So, this second type of filter we call it, an AR process, auto regressive process; whenever you have a situation then B z is 1 and A z is finite, we call it auto regressive process or AR process of order N.

If, there are it continues up to N th term then AR of order N and the previous one, when A z is 1, B z is finite then that is known as moving average of order M, okay. When both are present then it is AR and MA, you call it ARMA, order N A R is of order N, M A is of order M. So, ARMA means N and M, okay. Now, in digital filter AR or ARMA, we call them IIR filters.

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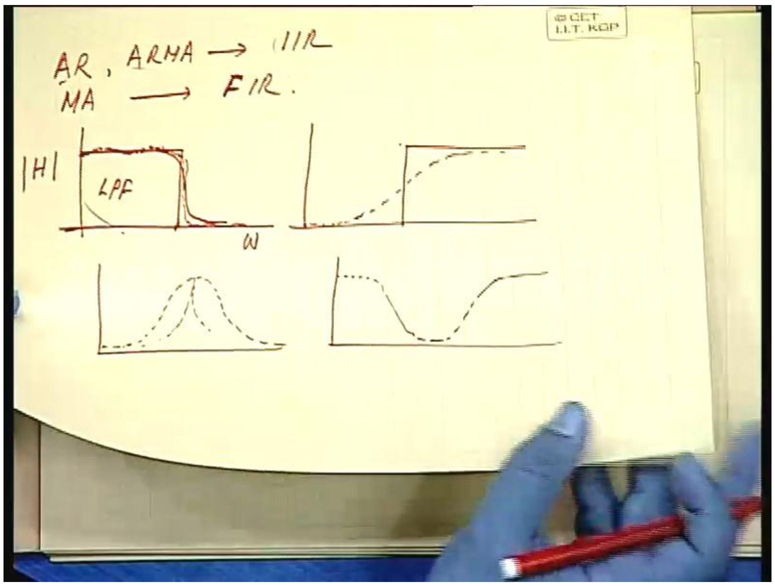

So, long as there is an AR part it will continue up to infinity; whether this this part is one or some finite length it is immaterial, because of the presence of the denominator, there will be always an infinite sequence. And MA filters are known as FIR, only numerator is present. Now, what are the advantages of this, what are the advantages of this; that we will come to know very soon.

Now, there are different types of filters, if I take in the discrete frequency domain this is a gain. An ideal low pass filter is like this, real filter cannot maintain such stringent requirements. It will be somewhat deviating from the ideal characteristics all right; there will be a tolerance within which we will be restricting the filter gains, at the designed specifications will be given in terms of tolerance and other parameters like cut off, frequency and so on, you have to design a filter.

So, this will this will be the type of the low pass filter. Similarly, for a high pass filter, ideal high pass filter, an ideal high pass filter will be like this and again; here I will consider this also as an as a high pass filter. It is not an ideal one, but this also ideal high pass filter. Similarly, a band pass filter will be like this. And a band reject filter will be like this. It all depends on the thickness of this band; that how sharp this change is taking place, we term them as notch filter or inverted notch filter.

Now let us see, so far we are not really discussing about the phase. We will discuss in details, at a later stage. Just one characteristic, I would like to mention here.

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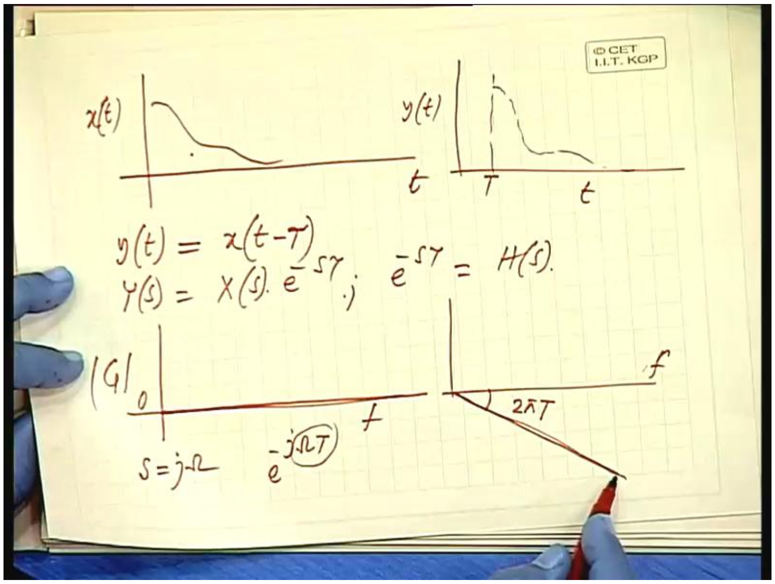

In a continuous system, in a continuous system suppose, you are having an input like this; if you want a faithful reproduction of this as an output, obviously the system will not respond immediately. This system cannot respond immediately, so there can be a delay but after that I want the same output. That means, I want y t to be x shifted by sometime ‘T’. If I take a Laplace transform, if it is a linear system then X S into e to the power minus s term, this we studied earlier in signals.

So, e to the power minus s tau will be the transfer function of the system, in the continuous domain. What will be the phase characteristics of this? What is a Bode plot? What is the Bode plot for this? Gain plot? e to the power minus S T. If I put s is equal to j omega; so it is magnitude is always one okay, if we take twenty log of one, it is always zero.

So, Bode plot against frequency is always 0 degree, okay. What about phase? e to the power minus j omega T, so omega T is the phase. So with respect to frequency, it will be negative minus omega T. So, phase is S equal to j omega we are putting; so it is e to the power minus j omega T, that means the phase is omega into T and omega is 2 p i f. So, it is basically in terms of f, it is having a slope of 2 p i into T, this is a slope okay.

So, you get a linear characteristic that means; if I have a filter ideally of gain 1 gain 1, and a linear phase characteristics then I will have the exact reproduction of the signal, okay with some delay, constant delay. So, linear characteristic is very important. For example, when we transmit speech frequencies, speech signals for different frequencies; if the gain is varying, if it is a filter say a low pass filter like this, the gain is constant.

If my frequency is limited to this band and if I can maintain the gain, more or less constant and I have to design a filter, where phase is linearly changing with frequency then only my voice will be faithfully represented, otherwise there will be distortion. Now, this is known as linear phase characteristics.

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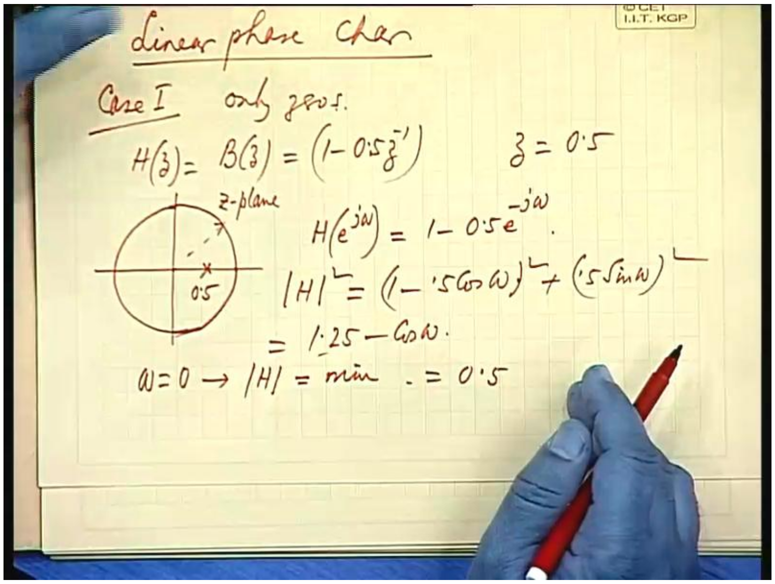

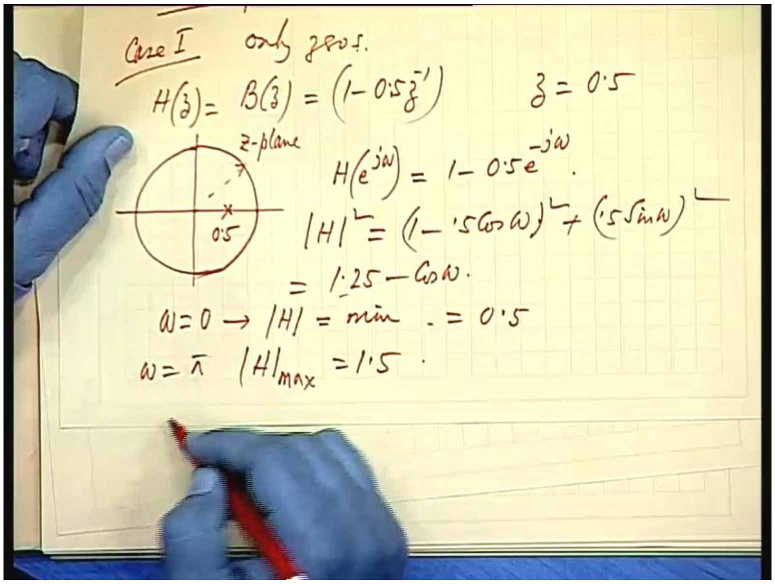

Now, we will see how to achieve linear phase very soon. Now, let us again study the characteristics of filter functions from a from the location of poles and zeros, all right. Let us take first case, when there are only zeros; that mean H z is having only B z, all right. Let us take a very simple function, say 1 minus 0.5 z inverse all right, that means basically z is equal to 0.5

Let us draw the unit circle z is equal to 0.5, is that okay. So, H e to the power j omega, this is Z plane complex Z plane, this magnitude is unity okay, this unity circle. Then H e to the power j omega will be how much? 1 minus 0.5 e to the power minus j omega, okay. So, H magnitude square, I am not putting e to the power j omega every time in the augment, H square will be 1 minus 0.5 cosine omega, whole thing square plus 0.5 sin omega square, correct me if I am wrong.

So that gives me, 1 plus 0.25 0.5 square, cos square omega, sin square omega; so 1.25 and then minus 2 into 0.5, so 1 into cosine omega all right. When is it minimum? When is it minimum, when cos omega is maximum; that is at omega is equal to 0, at omega equal to 0 H magnitude is minimum, is that okay.

So, that is equal to 0.5, 0.25 under under root, this is A square; so 1.25 minus 1 under root of that, so that become 0.5. And when is it maximum? When is it maximum, when this becomes negative; so this become 2.25, so H maximum is under root of 2.25, that is 1.5, okay. At what frequency when this becomes negative, so omega equal to p i, okay. So, let us see what it looks like.

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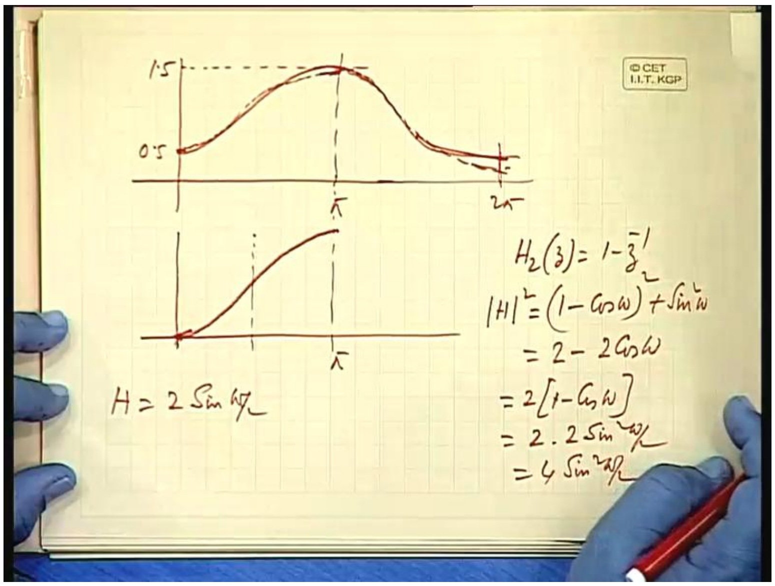

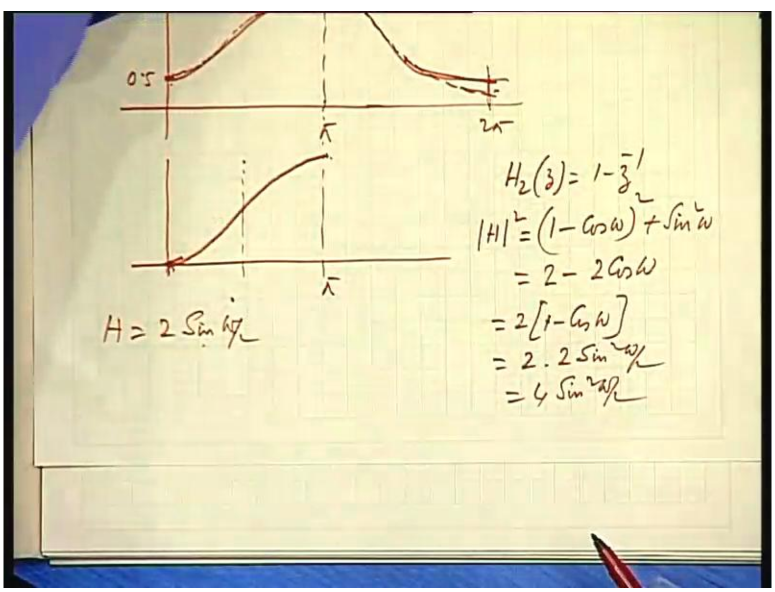

It starts from 0.5 and something like this; I am not making an exact sketch, 1.5, is that all right.

And then basically in this, you are putting z is equal to e to the power j omega, is it not. z is equal to e to the power j omega means, you are going along this, you are taking omega at different points on this circle. So, that is what it is, e to the power j omega; omega is this, all right this angle.

When it is 2 p i, it is omega full circle magnitude is 1, it is 1 into e to the power minus j omega. This is the substitution that you are making for z inverse. So, I am going along this, how much is this distance? I will put a 0 here, because it is in the numerator to be more precise the root is shown as a 0. So, if I go along this, how much is this distance that is given by this B z is it not?

So, I am going along this, when is this distance minimum? Here. When is it maximum? When it is coming to this side; when it is p i, again it is making 2 p i so it is coming down. So, it will be maximum at p i and then again it will be just mirror image 2 p i and it repeats sorry, it is like this, okay. So, is it a low pass filter or high pass filter? It is a high pass filter.

Now, we will talk about only the frequency range up to p i, because it will be replica of that. All our filter functions should be restricted to p i only, 0 to p i; if you remember that Shannon's criteria that I will follow, all filter must be with a band limited signals. Next, suppose we have it as okay, let us take another function H 2 z equal 1 minus z inverse, what is it, 1 minus z inverse?

How much is H square? 1 minus cosine omega square plus sin square omega; that gives me 2 minus 2 cosine omega, okay. So, what is it like; is equal to 2 into 1 minus cos omega and what is 1 minus cos omega? 2 sin square omega by 2. So, 4 sin square omega by 2. So, H is square root of this, twice sin of omega by 2 okay. When omega is equal to p i, this becomes p i by 2; so it becomes 2.

And when omega is 0, go to 0. So, when omega is p i, it starts from here. It is a sin square function, it is a sin function sorry, sin square is this, its sin function omega by 2, so it is like this, all right. So, is it a low pass or high pass? It is also a high pass. So, if the 0 is here or here, somewhere in the on the real axis here; it will be seen, see as you vary omega, this distance becomes minimum when it is exactly at this point, so at omega is equal to 0, it is minimum.

In case the 0 is on this then it becomes 0, it is start from 0; if it is little inside then it starts from a non-zero value, but in any case it reaches maximum at omega equal to p i okay, omega equal to p i means, here then the distance is maximum. From here the distance is so much, at any other location; it is maximum at this point okay, so this is a high pass filter. Next, let us take up a pair of roots.

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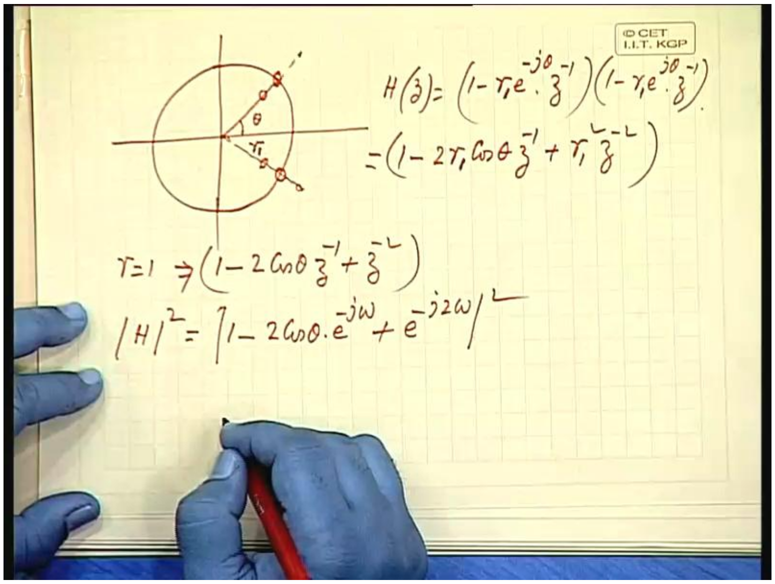

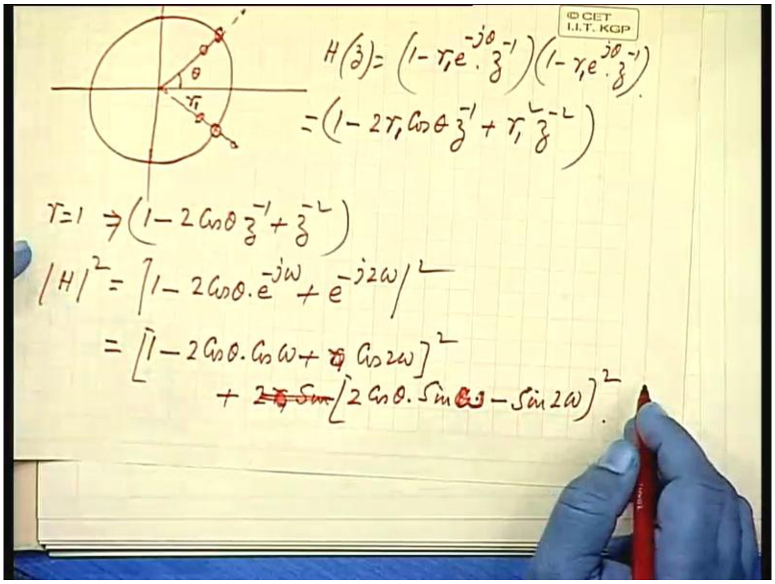

Suppose, we have two zeros here okay, you take this distance as r; it can be anywhere inside or on this or may be outside, so in general let the distance be r, all right. So, r into e to the power j theta, r into e to the power minus j theta these are the tools. So, the polynomial H z will be 1 minus r e to the power minus j theta into z inverse into 1 minus r e to the power plus j theta z inverse, is that all right.

So, that after multiplication gives me 1 minus 2 r if you take out the real parts; okay imaginary parts will get cancelled. We will find 2 r cosine theta, let me call it r 1, you may take r 1 is inside; we will take r 2 when it is outside, okay. So, 2 r 1 cos theta z inverse plus r 1 square z to the power minus 2 okay, correct me if I wrong.

Now, suppose r is equal to 1 r is equal to 1, then we get 1 minus 2 cosine theta z inverse plus z to the power minus 2 as a function all right. So, how much is H magnitude square? 1 minus twice cos theta e to the power minus j omega plus e to the power minus j 2 omega, this magnitude square okay, is it all right.

How much does it come to? 1 minus you already got this, 2 cos theta cosine omega, all right plus r 1 square is known; cosine 2 omega square that is all, is that all plus twice r 1 sin, r 1 is 1, cos theta twice cos theta sin omega okay, sin omega, sin omega. And is it minus or plus?. Any one of them will be minus okay and then? Sin 2 omega whole square, is it okay.

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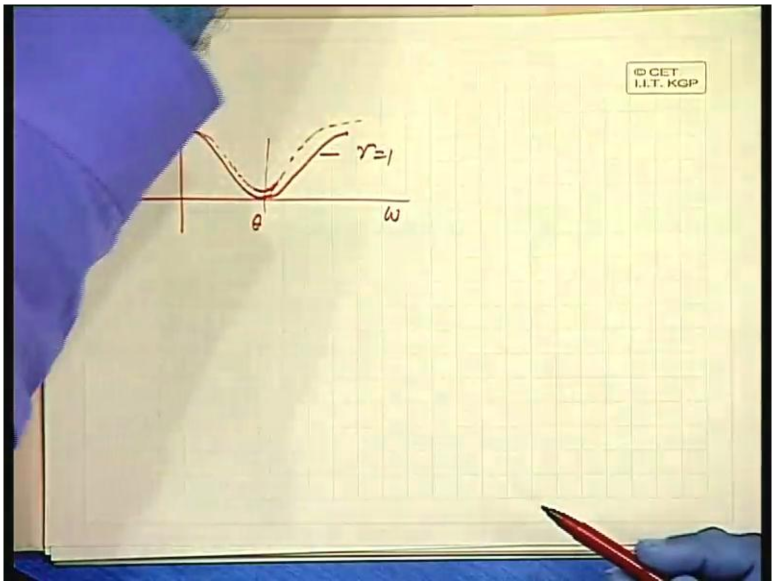

So, here we will find, if you simplify this okay I leave it to you. If you simplify this and then put different values of omega we will find; here you have got the zeros, you are varying omega like this. Now, at this point one of the factors is becoming very very less, here it is very small minimum somewhere around theta, we will find these values minimum, again it starts increasing all right.

So, it starts from some finite value, it reaches a minimum here and again it goes up, at omega equal to theta approximately; you get a figure like this, thank you. So, if I now keep on shifting this, along this line, this pair I maintain at the same theta, but I change the values of r 1. If, I put it on this then it will be 0; if it is inside, it will be non-zero. Again if it is here, it will non-zero, is it not.

So, it will become 0 for touch like this r is equal to 1, okay. Now, you will find whether you bring it inside or outside, the magnitude will remain identical; it will touch a minimum and then it will start increasing but the nature is identical. So for the same identical nature of functions, you can have two possible locations of zeros; either it it may be inside or it may be outside, having the same angle theta okay, minimum takes place at theta.

So there comes the question of phase, what would the phase? How do identify, I mean there is an ambiguity. If you give only the game plot, it can be a 0 here or it can be a 0 here, all right. So, we will see very soon, how the phase is affected by the position of zeros, the situation when these zeros are all inside the unit circle; we call it a minimum phase system.

There are poles, so far we are considering only zeros; there are poles also, we will take up that in the next class. So, when you have a pole-zero combination, the zeros need not be inside, zeros can be outside, the system performance is still stable. It is a poles which are to be necessarily inside the unit circle, but the zeros can be anywhere. So, we will take it up in the next class, we will have.