Digital Signal Processing

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Lecture - 11
Discrete Fourier Transform (DFT) (Contd.)

Continuing, our discussion on discrete Fourier transform or DFT, okay.

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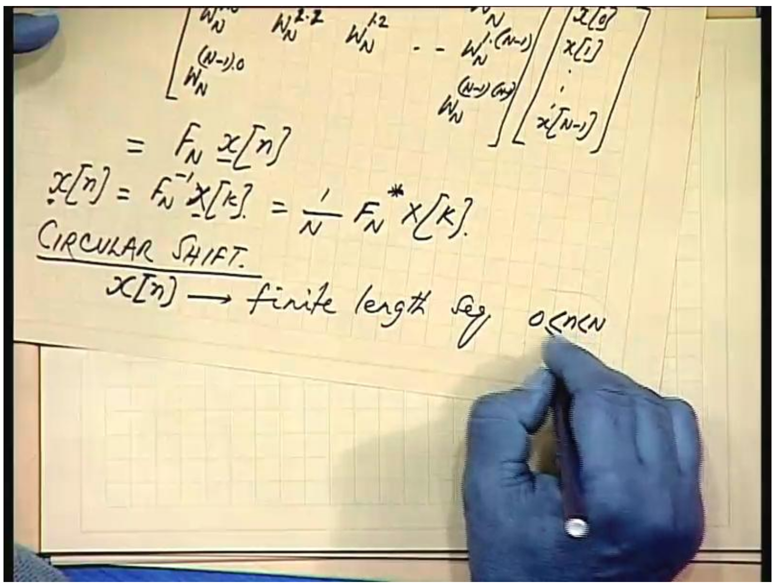

Now, last time we had written the expressions for the DFTs of a sequence which we can write in a matrix form, like this; if there is an N point sequence, we can write W N 1 into 0, W N 2 into sorry 1 into 2 1 into 1, W N 1 into 2, similarly, W N 1 into N minus 1 and so on, lastly W N N minus 1 to 0 and W N N minus 1 N minus 1, okay multiplied by x at 0, x at 1 and so on except N minus 1. That is if, I multiply these and then add them, I will get the corresponding components.

In short, we write this as a Fourier matrix in the discrete domain, it is written like this all right. X is a vector, X K is a vector, K equal to that is the values of x at different instance, we write in a vector form. We can realise, x n equal to F N inverse X K which is if we take the inverse of this matrix we will find it will be 1 by N F N star, that means you take conjugate of these complex quantities and then divide by a normalizing factor 1 by N, and we will get back the then we multiply by X K, you get back the sequence x n, all right.

Now, we will discuss about CIRCULAR SHIFT. The other day, we discussed a little about circular shift of a sequence. See, x n if it is a finite sequence finite length sequence, if it is a finite length sequence then say between 0 and n, okay.

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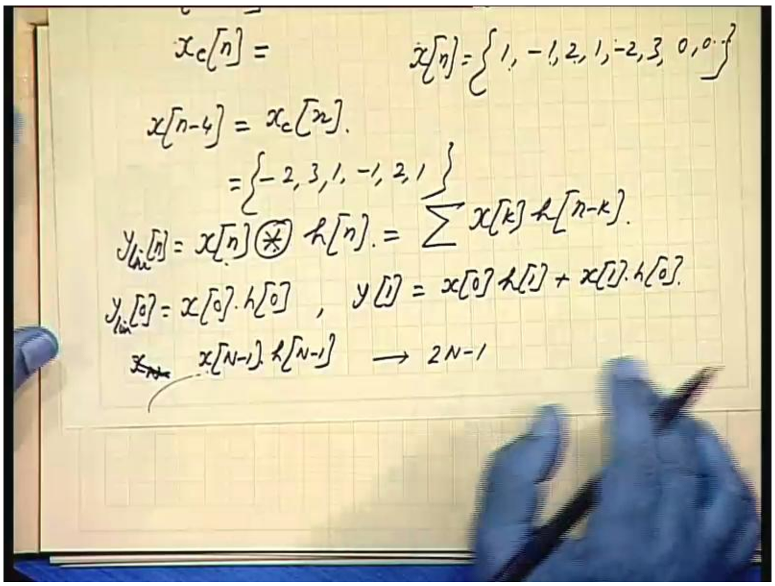

Then if I shift this sequence by n 0 steps, that is I call it as a shift n sequence; this sequence is not defined in the range 0 to N, 0 to N okay because it keeps on shifting, so some of the part, some part of the of the sequence will be lost. So it is not well defined here, so you define a new kind of a sequence; say x c that is circularly shifted which is let me take an example, that will be better.

Say, x n is a sequence 1, minus 1, 2, 1, minus 2, 3, 0, 0 etcetera, so this is the finite sequence. Then x c n if I give a shift by say, one two three four steps; so x n minus 4 is my shifted sequence x c n, what will it be like? It will start from minus 2, minus 2, 3 then it will wrap around this sequence of 1 minus 1, 2, 1, this is the circularly shifted sequence. It is a little different from a normal shifted sequence, okay.

Let us see, if I have the values, if I have the values written on this; okay x 0, x 1, x 2, x 3, x 4 and so on and after that it is all 0, we are not considering the zeros, the finite length up to, in this case it was up to 3, okay. So, if I wrap around then the sequence is the same values 1, minus 1, 2, 1 minus 2, 3 but then if you just give a shift of the first data; shift by four steps, you will find from the back side from the back side the values at the end will be reappearing, okay so this is a circular shift.

So, we will find a very interesting situation in case of discrete Fourier transform with the help of, circular shift. In a linear convolution, say I have a sequence x n, it is linearly convolved with another sequence h n, all right. So, y linear is convolved with these that mean, it is a summation of x K h n minus K, is it all right.

So the first term, in this will be x 0 into h 0, y linear first element will be like this. Zeroth value will be x 0 into h 0. y 1 will be x 0 into h 1 plus x 1 into h 0 and so on. And the last term will be, x N minus 1 into h N minus 1, all right. How many terms will be there? Suppose, both of them are of the same length N, both of them are of the length N then what will be the total length? 2 N minus 1 will be the total length, okay. Now, let us see what happens in case of a circular shift.

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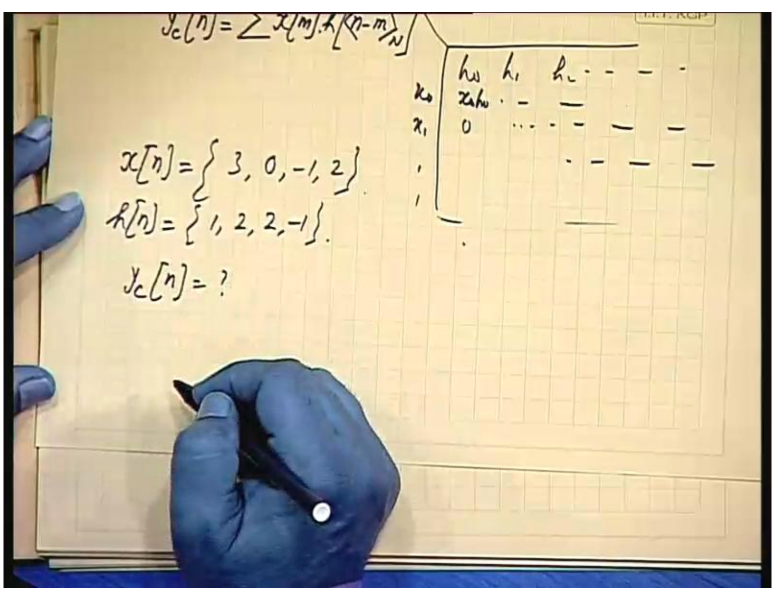

y circular, I have once again same number of points all right; say six points here 0 to 5, x 0, x 1 up to x 5, similarly h also I have got six points okay, it is like this. And if you remember in linear convolution, we reverse the direction of h or x either of them and then bring them closely, is it not bring them in steps, we move them.

So, here also you have the sequence written in a reverse mode for one of them, all right. As if this was your x 0, x 1, x 2 etcetera and this was h N minus 1, h N minus 2 up to x 0 I just reversed it, okay and then you keep on moving it okay, either this way or you can start from this side, its one and the same thing. So, if I keep on moving it, instead of this and now wrapping around both of them; and then keeping one of them fixed just moving the other one all right.

So, every time you will find some term will be touching the other one; some term of the other one, there is no zero all right, unlike the linear convolution where if you remember, we had written h 0, h 1, h 2 and so on, x 0, x 1 etcetera. So, this was x 0, h 0 and so on and then there was a 0 here then second term starts from here, third term starts from here. So, there were many zeros. So when we are summing up; it was only x 0, h 0 but now there will be other terms, what are the other terms?

So, y c n is x m h n minus m modulo N. Okay, let us let us see some example it will be clear. I have x n equal to 3, 0, minus 1, 2 and h n as 1, 2, 2 say minus 1, okay. I have got these two sequences, what will be y circular n, what will be this? So, what I do, I will take a little bigger space, okay.

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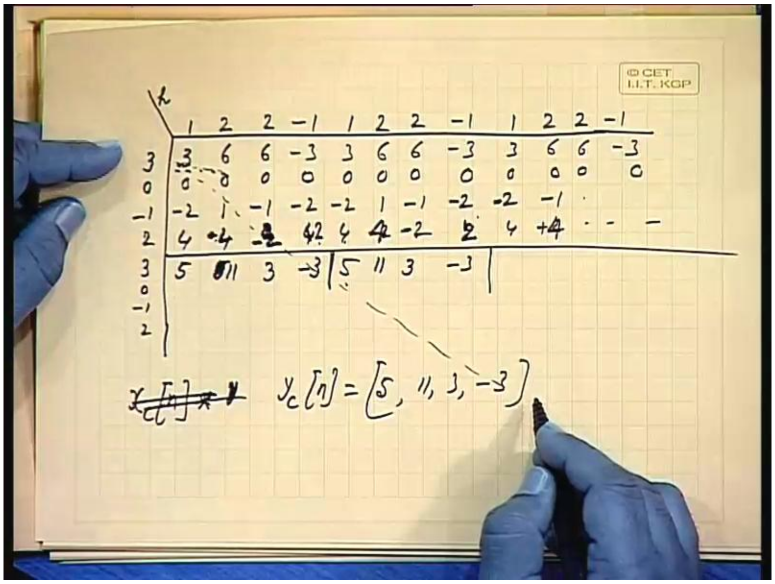

I will write these 1, 2, this is h; 1, 2, 2, minus 1, 1, 2, 2, minus 1, 1, 2, 2, minus 1 and so on. I keep on repeating it, because I can keep on moving it, see the effect all right, I keep on moving it. And then this one is 3, 0, minus 1, 2; I may repeat 3, 0, minus 1, 2 and so on, so this gives me three ones zaa 3, 6, 6, minus 3, 3, 6, 6, minus 3, 3, 6, 6, minus 3 and so on, I can keep on repeating it. Then next are all 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 since it is going on, so it can go on this side also, okay. So, this side also I can fill up with zeros.

Then minus 1 will be, say minus 1, minus 2, minus 2 plus 1 minus, 1 plus 2 plus 2 minus, 1 and so on, okay. So, this will be minus 1, this will be plus 1, this be minus 2, is it not. Then no these two, these I am maintaining sorry; earlier, this was all zero, so if you wrap around then this will also be filled up, is it not. One second, with 2 it will be sorry 2, 4, 4, minus 2, 2, 4, 4, minus 2 and so on with, is that all right. Then you can fill up the gap 2, 4, 4, minus 2, minus 2, 4, 4. So, sorry, minus 2 and 4 okay.

I need not carry on beyond one step because it will be repeating; these three is getting repeated here. So, if you add up what you get? 5, tell me; five, okay then…… this? Last line should be shifted, yes it should have been here, know, correct. So 2, 4 thank you; 2, 4 then minus 2 then plus 2, 4, 4 and so on. So, this should have been 2, 4, 4, minus 2, so this should have been minus 2 then 4, okay is that all right. 5, so this is thank you very much, 11 then 3 then minus 3, 5, 11, 3, minus 3 and so on.

We will find the actual sequence, if you would not have filled up these zeros; then you will find a periodic sequence that starts appearing is 5, so sorry minus 3, 11, 5, 11, 3, minus 3 and that keeps on repeating. So, x c n into y sorry, y c n; I will write as 5, 11, 3, minus 3, okay. Now, see how do you write the sequences, yes please, sorry!...Third row minus 2, 1 this one? 2, 2, this is minus 2, is it so, okay, okay, minus 2. So, y c becomes 5, 11, 3, minus 3, okay.

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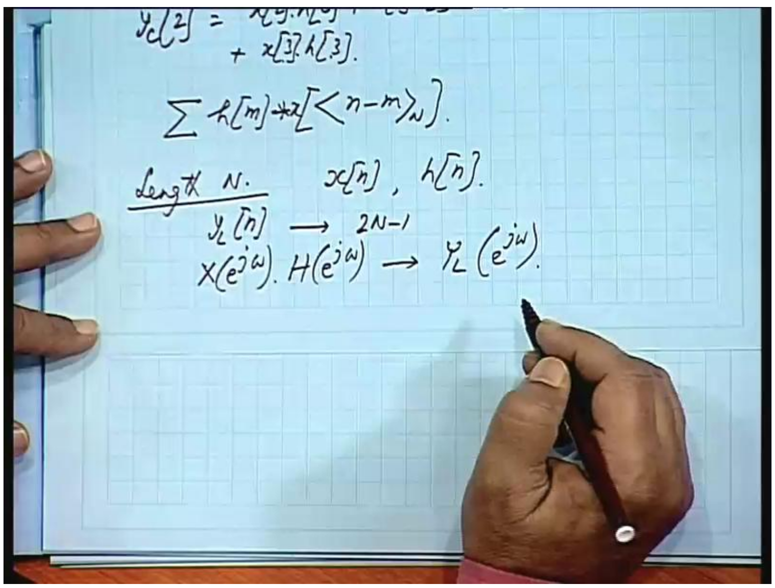

So, if you write in this form; with different values of n, it becomes x 0, h 0 plus x 1, h 3 plus x 2, h 2 plus x 3, h 1 okay, unlike the previous one. You see the some of the terms is 0 plus 0, 0, 3 plus 1, 4 modulo 4 is 0. So 2 plus 2, 4. So, the total sum should be either 0 or n, when it is n, when it is 0 then it can be 0 or 0 plus n, okay, you can add always n; y c 1 should be therefor, the total sum should be 1, x 1 h 0 plus x 0 h 1, total sum is 1. And then x 2 h 3 that will make it 5 modulo 4 will give you 1.

Then x 3 h 2, is there any other possible combination? I have already taken 2, 3, 0, 1, 2, 3 you see all the values of x; there will be only 4 elements y c, 2 will be x 2, h h 0 plus x 0 h 2 plus x 1 h 1 plus x 3, h 3, 3 plus 3, 6, modulo four will give you 2 just subtract 4, that will give you 2. So, the total sum in the argument, inside the bracket that should be 2, okay. Similarly, you can write y c 3 and so on.

You may also write in this form, sigma h m x n minus m N because while moving it; I can move either this one or this one, any one of them, okay it its one and the same thing. So, that is why we just write this modulo for one of them, the other one you just keep static, I will evaluate at x 0, x 1, x 2, x 3, four points all right. And the other one I can move or keeping this one, I move the other one.

Now, there are two sequences of length N all right. Length N x n and h n; what will be their linear convolution? Two sequences linear convolution, we have seen just now; it will have okay a length of 2 N minus 1, and if I take Fourier transform, discrete time Fourier transform, see convolution of these two series, these two sequences in the frequency domain will be in the product form, okay. So, it will be X, okay.

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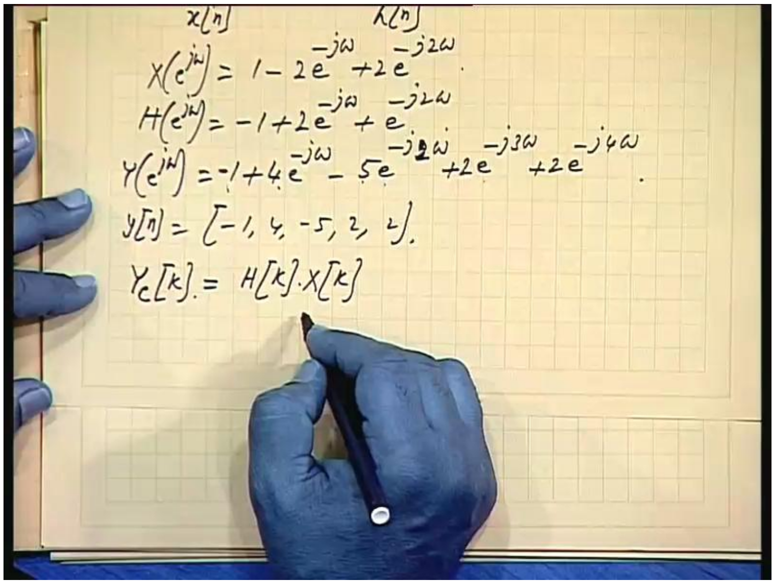

Suppose we have a sequence, 1, minus 2 and 2, one sequence the other one is h n is minus 1, 2, 1; this is h n, this is x n. So, what is X e to the power j omega? It will be 1 minus 2 into e to the power minus j omega plus 2 into e to the power minus j 2 omega. H e to the power j omega will be minus 1 plus 2 into e to the power minus j omega plus e to the power minus j 2 omega, okay. If you take the product, it will give you 1 plus 4 e to the power minus j omega, any question?

It should be minus 1, okay, minus 5 into e to the power minus j 3, 2 omega plus 2 into e to the power minus j 3 omega plus 2 into e to the power minus j 4 omega. So, 1, 2, 3, 4, 5 three points' sequences, three plus three, six minus one so that is the total length, okay. Now, you will find what will be y n, convolution will give you these values only, is it not.

If you take circular convolution of these two sequences, what will be Y c K? Now, for a four point sequence, for a four point sequence if you take DFTs, four point DFTs, I will get four point DFT and four point DFT, okay. Now, Y c K is very interesting. H K into X K same result. In the frequency domain, they appear in a product form but then four point sequences with four point sequences; will give you four point sequences, term by term you are multiplying, four elements you are multiplying term by term.

Unlike in the continuous domain in DTFT, when you take a general expression, all right; there are infinite number of possible values of omega, that you can substitute, is it not? So, basically you are taking a general expression of three terms, with three terms which will give you in a polynomial form of omega, which will give you five terms here all right. Unlike this, here you are having just four points sequence; a four point sequence on the other side, if you take DFTs not DTFT, DFTs then you will get again a four point sequence, all right.

So, sequence length remains same, you can just take product of term by term. Let us take an example, it will be clear and from DFT you can always reconstruct DTFT, all right. So, let us once again go back to the earlier one.

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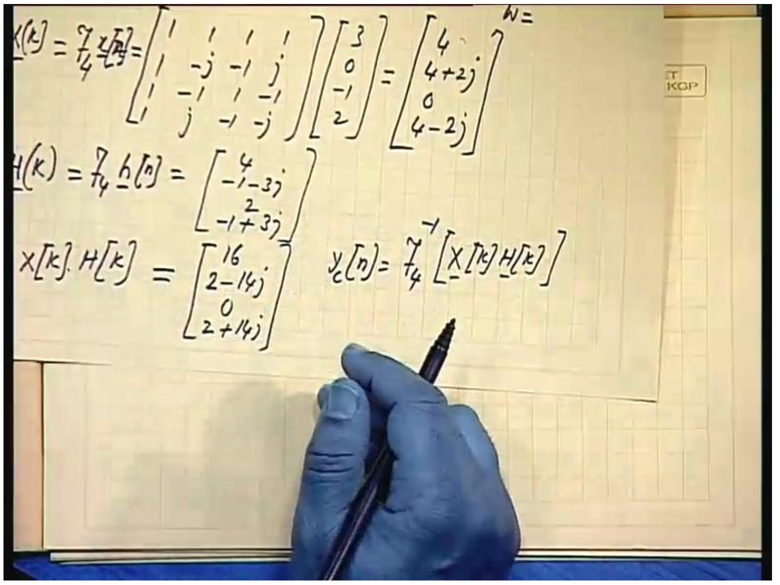

x n was 3, 0, minus 1, 2, the same example that we have taken earlier 1, 2, 2, minus 1. And we get a circularly convolved sequence, all right; this x, this was the general x n and h n, so we have made x c n out of this and N point sequence, four point sequence. So, what will be Fourier transfer matrix, F 4 n is equal to four, what will be the element value of w? Four means, ninety degrees two p i divided by four, means ninety degrees, so, minus 90 degrees. So, it will be minus j okay.

So, 1, 1, 1, 1, 1, 1, 1 then first one will be minus j, is that okay. Then minus j square means minus 1 then plus j, then this one will be j square; so minus 1 plus 1, minus 1, okay and then this one will be 1 plus j minus 1 minus j, is that okay? So, multiplied by x that, means 3, 0 minus 1, 2 so that will give me X K vector okay, that is x 0, x 1, x 2, x 3. So, by this multiplication we can simplify this and we will get these values, I am just writing the final value, okay.

Similarly, h 0, h 1, h 2, h 3 can be written as, F 4 into vector h okay, here I should have written n okay. And that gives me that is this multiplied by h 1, 2, 2 minus 1 that gives me 4, minus 1 minus 3 j, 2, minus 1 plus 3 j, okay. Therefore, if we take the product X K into H K term by term, it will be 4 into 4, 16, minus 1 minus 3 j and 4 plus 2 j; that gives me minus 4 then 3, two zaa 6, so 2, okay. So 2, then minus 1 plus 2 j, minus 2 j, minus 3 j plus 4, 12, so minus 14 j, is it okay.

Then 0, 0 into 2, 0 and then 4 minus 2 j and minus 1 plus 3 j; so that will give you 2 plus 14 j okay. Now, if I take the Fourier inverse of this, I should get y c n all right. So, y c n will be Fourier inverse okay of this product, X K H K; this whole thing term by term product, it is not a matrix product, it is a term by term product.

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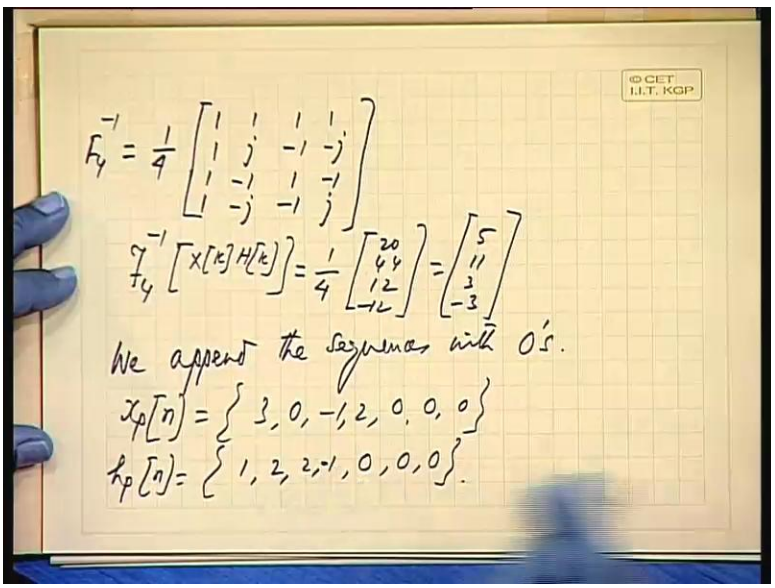

So that gives me, if you take Fourier inverse it will be 1 by 4, what will be the inverse matrix; instead of minus j the element value will be plus j okay, so it will be one 1, 1, 1, 1, 1, 1 then j minus 1, minus j, minus 1, plus 1, minus 1, then minus j, minus 1, plus j is that all right. For a four by four matrixes, that is n is equal to four here you just interchange, second and fourth row; we will get the inverse matrix, okay.

So, if I take F 4 inverse, this is F 4 inverse; say F 4 inverse into that product X K H K, so you just multiply here by the values that we got 16, 2 minus 14 j etcetera, I will get 1 by 4, 20, 44, 12 and minus 12 and that gives me, 5, 11, 3 and minus 3 which we got earlier, we got these values earlier by circular convolution. So, circular convolution can also be performed by taking DFTs of the two sequences, take their products term by term products and then take the inverse all right.

Now, so this is different from this is different from DFT of the convolved product, x n star h n, if you take the DTFT; is different from the circularly convolved sequences, those four points sequences if you take in a circularly convolved manner and if you take the product then it will be just a four point sequence and it is different. Can you see, whether we can get by the help of DFT, because DFT is a very powerful technique.

When you have a very large number of points, all right then if you want to take a convolution product; it is better to take the DFT algorithm because F 4 and F 4 inverse, they are almost similar, you just replace w by w star in your algorithm. So, if you want to do that, can you get from the DFT, can you get DTFT, okay?

Let us see, these four point sequences, we pad them. So, we append with zeros, it is also known as appending, appending or padding. Append the sequences with zeros, how many zeros should we put? You can put but then for DFT computation, you cannot have any infinite number of points, so you can have minimum number of points. So, you try to make it 2 n minus 1, all right. So, four point sequences you put three more zeros, so it will be a seven point sequence. Both of them will be seven point sequences, okay.

So, the product will be if you take a linear convolution the product will be 13, but if you take but out of 13 there will be many zeros, effective ones will be seven non-zero elements. Now, how will you get that? So, you will find in circular convolution of seven point sequence, you get those seven points. There are two seven points sequences; if you take the circular convolution, you get seven point product and these are the seven non-zero values.

So, you pad them with number of zeros which will make the length 2 n minus 1, so let us see the sequence. Suppose, x p n the padded sequence is written as 3, 0, minus 1, 2 then 0, 0, 0. And h p n is 1, 2, 2, 1, 0, 0, 0, minus 1, okay. If you take, let us see the convolution itself. The linear convolution of this, y linear convolution, I call that also of the padded sequence.

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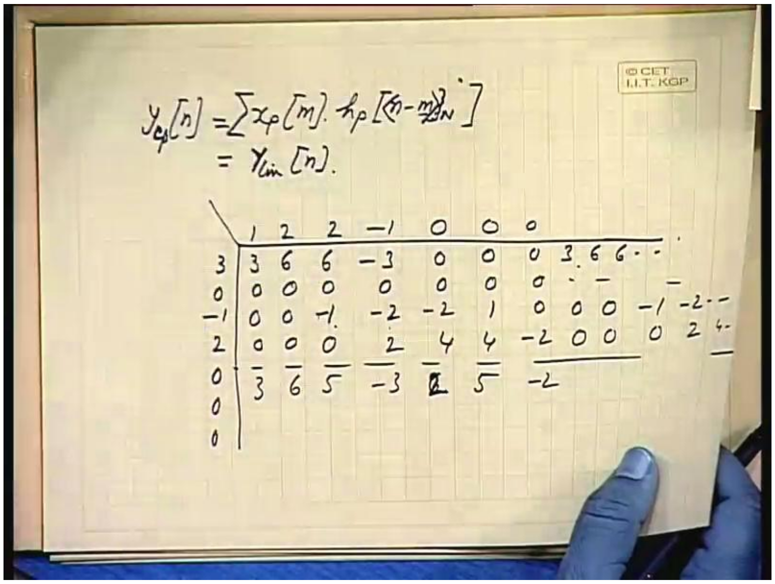

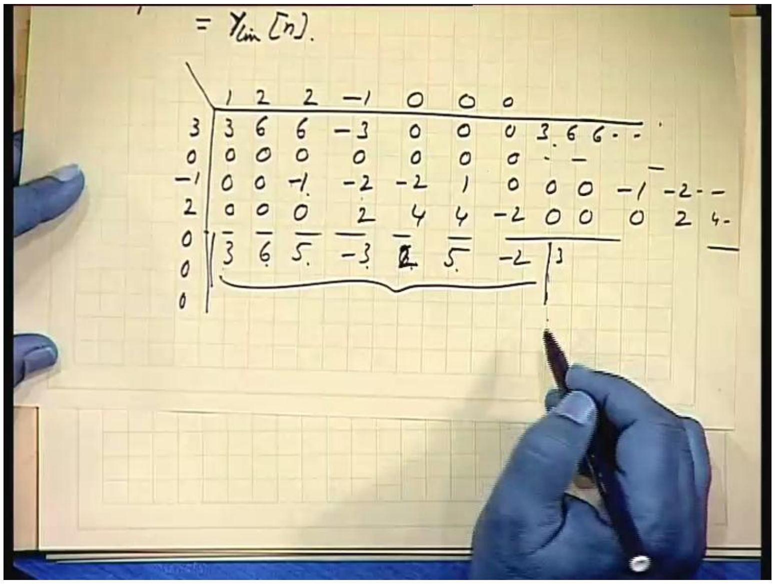

Why linear convolution of the padded sequence? That is x p m into h p n minus m, okay. Summation, what does it look like? If I, okay if I take minus N, N circular convolution of this okay, circular convolution of this is same as Y linear n okay original linear n, let us see why it is so. 1, 2, 2, minus 1, 0, 0, 0, okay. 3, 0, minus 1, 2, 0, 0, 0. So what do you get? 3, 6, 6, minus 3, 0, 0, 0, 3, 6, 6 and so on. Get again circular convolution, so I repeat, again here it will be 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.

Then one two three here, minus 1, minus 2, minus 2, this will be filled up with zeros minus 1, minus 2, minus 2, 1, 0, 0, 0 again minus 1, minus 2, and so on. So, a seven point sequence will keep on appearing but the last three terms will be zeros. Two, so it will be starting from here, 2, 4, 4, minus 2, 0, 0, 0 sorry 0, 0, 0 and then again 2, 4, etcetera; so before 2 there are three zeros, one two three zeros.

Now, if you add up what you get? You will get a seven point sequence, 3, 6, 5, minus 3, 6, check whether you get 2, 2, 5, minus 2, should it be then 0, 3, 6, 5, 3, minus 2, minus 3? I made a slip anywhere? 3, 6, 6 just check up all appears to be all right, okay anyway. So, if you keep on doing that, you will get the sequence. What was the result?

The linear convolution, out of that linear convolution did you not compute in the beginning? Okay, if you would have done that, then you will get from here from here a seven point sequence which repeats afterwards, okay. I should have taken a few more steps here then then that would have been better. So, here onward it will be again three, one two three four five six seven, okay, this will be the values, all right.

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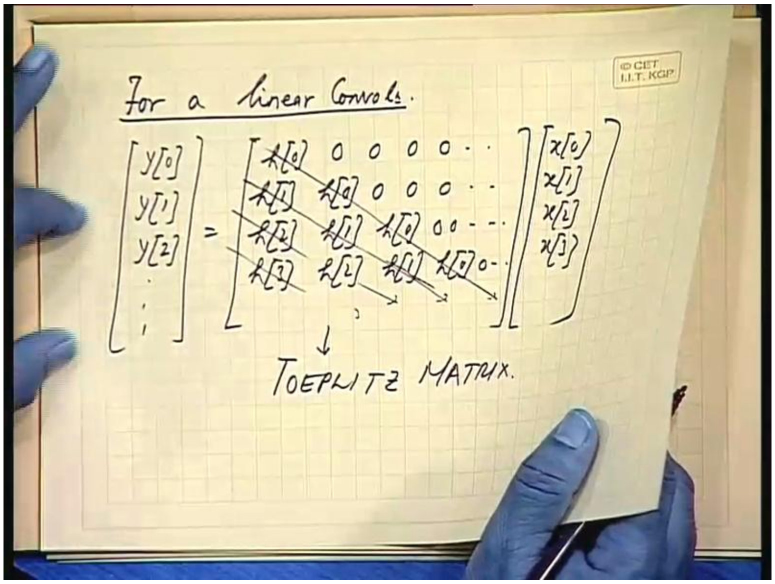

For a linear convolution, for a linear convolution we may write the result in a matrix form; like this h 0, 0, 0, 0, 0 and so on, depending on the number of x values that you have got, x 0, x 1, x 2, x 3 and so on. Next one will be h 1, h 0, h 2, h 1, h 0, h 3 and so on, h 2, h 1, h 0 and then 0, 0. We will find these elements are diagonally identical; the diagonal and sub diagonals are identical elements, this type of matrices if all such diagonals are identical, diagonal elements are identical, say p 1, p 2, p 3, p 4, p 5 these are the values, they are called TOEPLITE MATRIX.

Now, can you write a similar matrix multiplication form for circular convolution, what will it be like?

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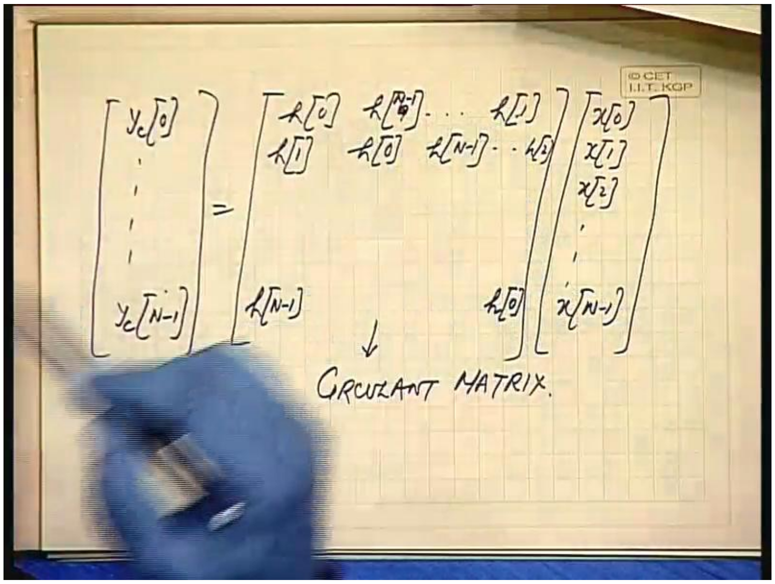

Y c 0 and say, Y c N minus 1, suppose it it is an N point sequence then what will be these values? h 0, h, x 0, x 1 sorry a; I am taking N minus 1 because I am going back to the four point sequence, actually it is an N point sequence. So this will be this will be, N minus 1, okay? x 2 and so on x N minus 1. So, this will be h, last one will be h 1 then this one will be h 1, h 0, h N minus 1 and so on, h 2 ending at h 2.

So, this one will be growing up to h N minus 1, okay. And this one will be h1, h 2 up to h 0; h 1, h 2, h N minus 1 next again h 0, these are known as CIRCULANT MATRIX, that is first row first row if you just turn it by one step, if you rotate it by one step you get the second row. See h 1 it is a circular one, all right, the elements are placed circularly, you shift it by one step.

So each one goes to the first position, h 0 comes to the second position and then they all shift by one step. Again one step, if you give one shift shift of one step then again you get the second row and so on. That is a set of properties of DFTs all right like DTFTs, you go through them, some of them are very interesting.

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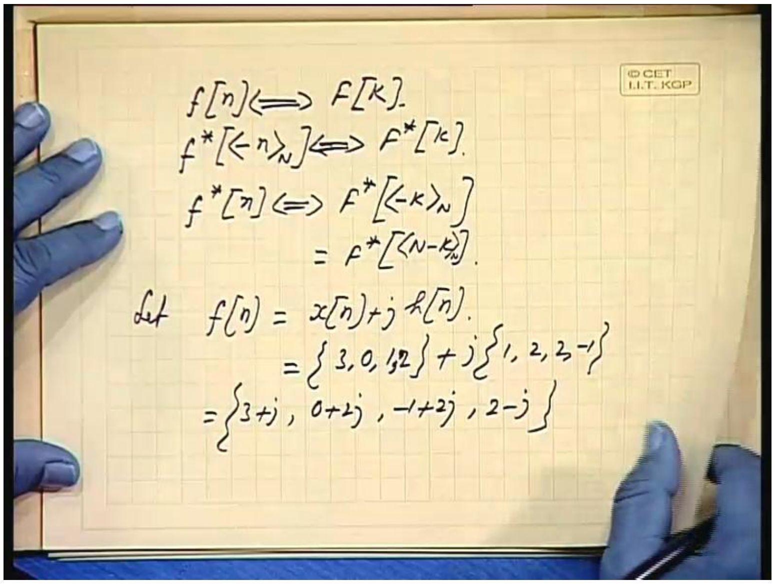

Suppose, you are given a sequence f n; whose N point sequence say, DFT is F K, if it is a complex sequence, if it is a complex sequence then f star minus n will, be F star K this is a very important relation. Another important relation is f star n; is giving you F star minus K N, minus K N means basically, you can add N or subtract N. Modulo N means, we can always add N or minus N. So, that is as good as F star N minus K, is it not? I can put N minus K and modulo N, wherever required I will add N or subtract N.

So, let let us have a sequence f n is equal to x n plus j times h n, okay. What was the sequence that we had earlier? Taken; 3, 0, 1, 2 plus j times 1, 2, 2, minus 1, is that all right. So, this can be written as 3 plus j, 0 plus 2 j, minus 1 plus 2 j and 2 minus j, this is a complex sequence, is that all right. Now, if you take the Fourier transform, discrete Fourier transform of this sequence multiplied by F K, this vector that gives me, F K I have chosen wrong notations; because capital F if I take and that should not get confused with, Fourier transform matrix F okay, all right.

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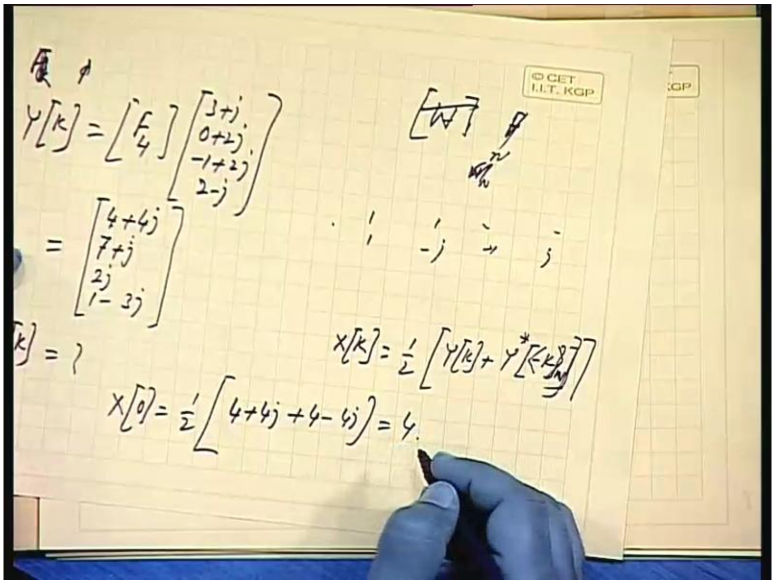

In many books, they write capital W as a matrix, that is Fourier matrix; whose elements are W N n, okay or K whatever it is, a very bold W, they write okay. In many text books, they write capital F, so either way I should have chosen. Say, let us write Y n okay, let us write Y n, these are general formulae, let us write Y n then what will be Y K? It will be Fourier matrix F 4 multiplied by this sequence, all right 3 plus j, 0 plus 2 j, minus 1 plus 2 j and 2 minus j, okay.

So, if you take the product you will find, this will give you 4 plus 4 j; it is actually if you look at it three plus j all these if you add up together 3 plus 2, five minus 1, four and 2, 2, four, 4 j, okay. First element is 1, 1, 1, 1 is it not, so add them together. Next one is 1 minus j, minus 1 plus j. So, if you multiply each element 1, 3 plus j then minus j into 2 j, so that gives me plus 2.

And if you add them together, that gives me 7 plus j. Similarly, this one will be 2 j; this will be 1, minus 3 j, okay. So, what will be X 0? What is X K? X K, X K now, if you are given a complex sequence; if you are given a complex sequence then if I take the Fourier transform Y K is X K plus j times H K, okay. How do calculate X K and H K from Y K? Like that, in the normal sequence you take real part and imaginary part, in the transform domain so X K will be half of Y K plus Y star minus K, okay.

So, here let us write X K, minus K modulo N minus K modulo N, capital N. Here it is four, so let us see the values that we have computed. What will be x 0? It will be 0.5 of 4 plus 4 j and y star zero modulo four, it will be again zero. So, 4 plus 4 j, complex term of that will be 4 minus four j; it will be 4, okay.

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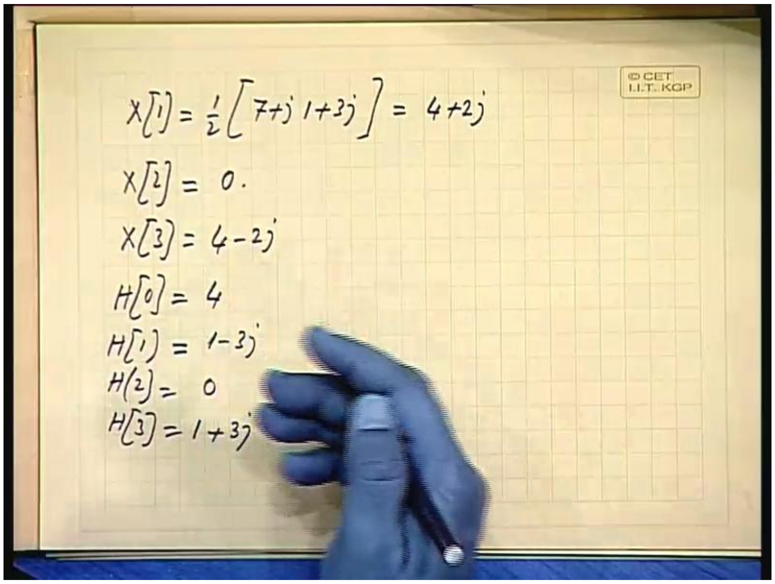

X 1 will be half of f 1 that is 7 plus j and then this one, 1 minus 3 j; star of that will be 1 plus 3 j, is that all right? It will be 4 plus 2 j, okay. X 2, similarly you see for yourself will be 2 j, x zero one two, x 2 and see, y 2 is 2 j and what about Y minus 2? Minus 2, if you take modulo 4; that mean it will be again plus 2 and complex conjugate of that will be minus 2 j, so it will give you 0, all right

And X 3, X 3 will be similarly 4 minus 2 j, okay. Similarly, H 0 the same logic H 0 will come out as 4 H 1 will come out as 1 minus 3 j, H 2 will be coming as 0 and H 3 will come out as 1 plus 3 j, okay. What was our earlier result, same; that means you are reducing the efforts of computing DFTs of two sequences in one go, you get my point?

There are two sequences; x 1 and x 2 say, x 1 and x 2 n of N points, you club them together as a single complex number x 1 plus j x 2, take that DFT, all right. Suppose, DFT is Y K all right and then take its negative values, complex conjugates with modulo N then add them together, divide by 2, you get X K, subtract one from the other, divide by 2, 2 j you get the other one, all right.

So, computationally this will be much more effective when you have large number of points; instead of taking two four D F, I mean four point DFTs, I have done it in one single operation all right, only thing we are taking DFTs of complex numbers, all right. Thank you very much, we will continue with these in the next class.