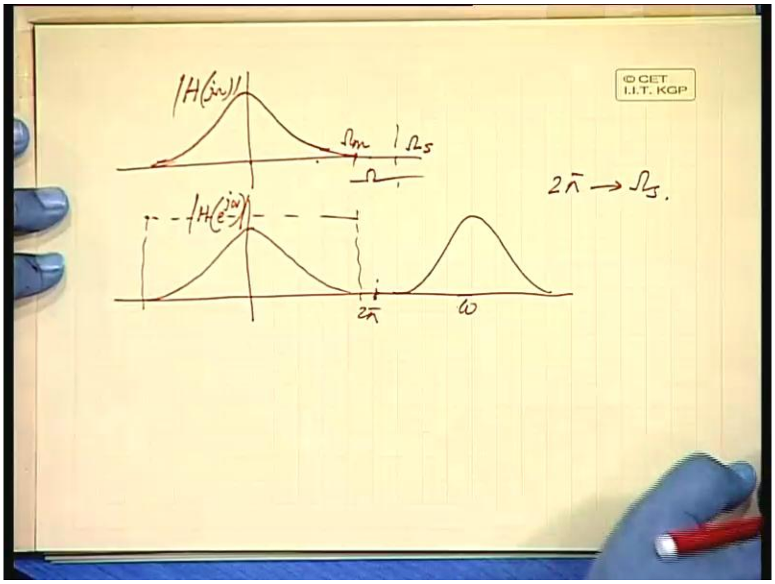
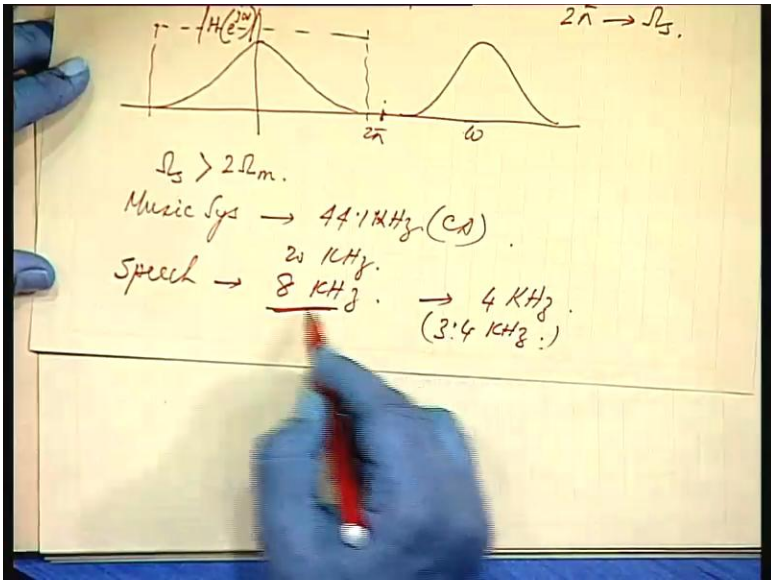
**Digital Signal Processing**
**Prof. T. K. Basu**
**Department of Electrical Engineering**
**Indian Institute of Technology, Kharagpur**

**Lecture - 10**
**Discrete Fourier Transform**

Last time we discussed about, the relationship between the two types of frequency transforms; the one in the continuous domain, the other one in the discrete domain of time.

(Refer Slide Time: 00:57)





So, we saw that if we know the analogue domain frequency response, analogue domain frequency response say; something like this if it is band limited then if we can maintain a sampling frequency which is much higher than the maximum limit of the frequency in the band, then we can have the frequency domain response of the system for discrete time systems as repetition of the same characteristics with a periodicity of 2 p i, where 2 p i corresponds to omega s the sampling frequency, okay.

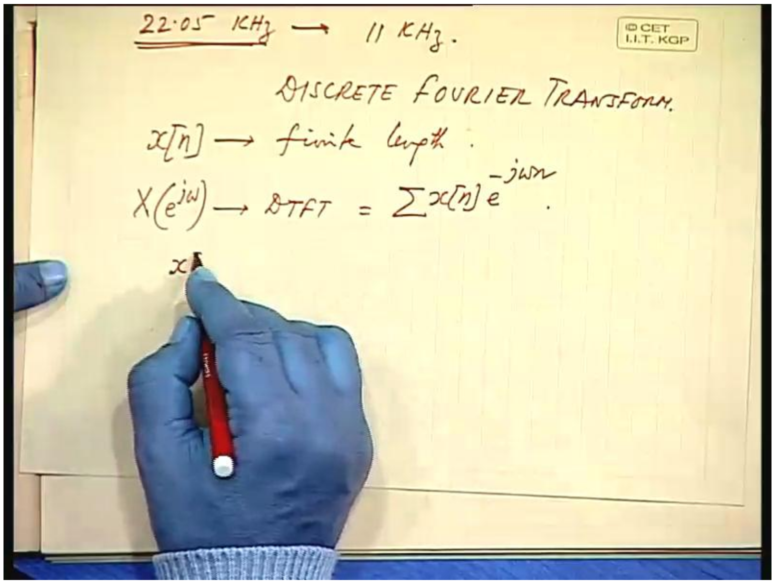
So, it is in terms of radian frequency, this is in terms of radian per second. You can always convert this to radian per second, this is analogue frequency. And hence; from a given characteristic like this given frequency domain response in the discrete domain of time, we can always get back to the analogue domain response by filtering this in the base band and then taking inverse transform. So, Nike's frequency criteria suggest, that the sampling frequency should be at least 2 times, the maximum frequency that is present in the signal.

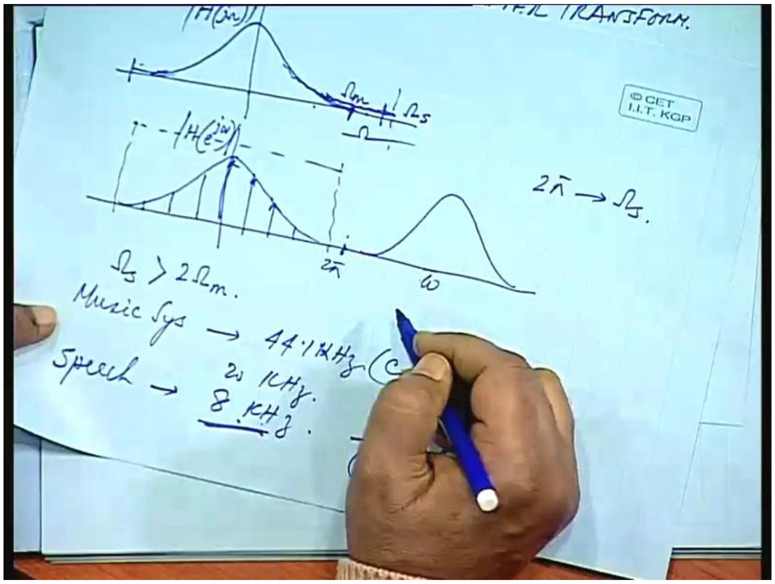
Now, for example for music system 44.1 KHz is the frequency of sampling that we use for CD, compact disc. Then what do you want to really capture is a signal up to 20 KHz or so 20 KHz or so. Whereas, for speech it is 8 KHz sampling frequency, that means we want to capture within 4 kilo hertz; the frequencies of speech that will be, within 4 KHz, this is for telephonic purpose, all right, for telephonic we require 8 kilo hertz of sampling.

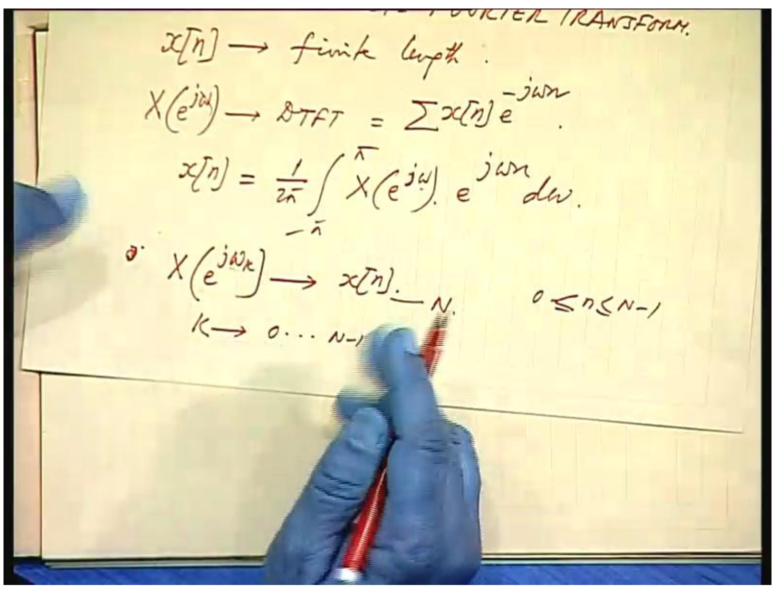
If you want, normally the frequency band is approximately up to 3.4 KHz, that you want to capture. So, it is little above twice this frequency, so this is good enough. If you want to identify a person, if you want to identify a person very accurately then we go a little above this. Quite often you must have observed, you must have experienced when you receive a telephone call, you are not able to distinguish the voices of the speaker at the other end all right, unless he introduces himself quite often, we make mistakes, okay.

So, if you want to also recognise the voice then we go for22.05 KHz, that means if you want to go for a little higher band; as I told you yesterday actually it is not suddenly coming to a drastic end at a particular frequency, we are truncating it by pre filtering, others it will be going tapering of like this to a larger value.

(Refer Slide Time: 04:39)






So, where you are cutting? Cutting this off will be ensuring the quality of the speech, that you recovering, all right. So, if you want to identify a person's voice; that means there are certain high frequency components, if you miss out you will not be able to identify. That is why for speech recognition, speaker recognition we go for a little higher frequency of sampling, okay. So, this will be less than little less than 11 KHz.

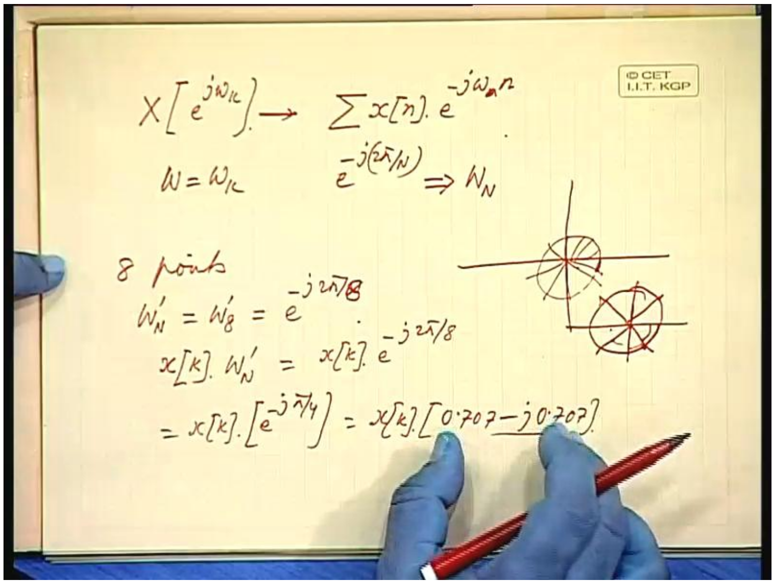
Now, today we shall be taking up Discrete Fourier transform. So, in discrete Fourier transform, what exactly we want to find out? When a sequence is a finite length, x n is a finite length then the DTFT, which we defined as summation x n e to the power minus j omega n. And from this DTFT, we can recover the signal x n; which means what should we write, 1 by 2 p i, all right e to the power plus j omega n d omega, so we have to perform this integration.

You have to know this in the continuous domain of frequency, all right. So, as I had mentioned this x may be of this type, okay. So, you have to take the inverse transform of this to go from frequency domain to time domain. Since it is periodic, so in the time domain, it will be discrete lines which will be x n, okay. Now, instead of computing this from this integral, can we can we realise this from some discrete values of this frequency domain representation?

That is we shall not take a continuous set of values but we can take some discrete values of this. Can we compute x n from these discrete lines, if x n is a finite sequence? Yes, then it is possible. If, x n is finite then x n can be recovered from only specific values of omega K, okay, we can realise x n. If x n is having a length of N, say 0, 2 N minus 1; x 0 x 1 up to x n minus 1 then we can take those many values, that is n number of values K varying from 0 to N minus 1.

We can take only N number of frequencies and evaluate the function at those N numbers of frequencies, all right. And that is sufficient to define the function x n. You can always retrieve x n from, not from the continuous function but only from those n number of frequency domain representation, so this we defined.

(Refer Slide Time: 09:11)

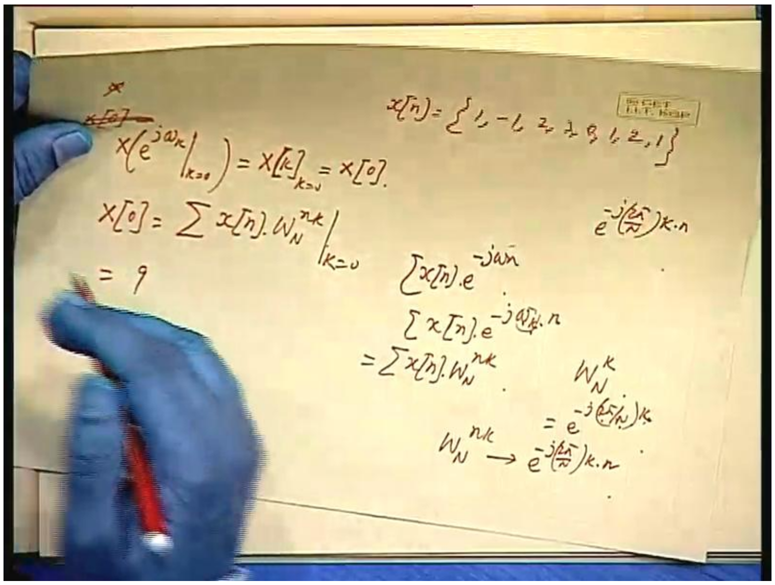



This is also very interesting, this will be the same expression; x n e to the power minus j omega n, this we are sorry omega into n, this this we are evaluating, earlier. We shall be taking only some specific values of K, omega equal to omega K. We take this as, e to the power minus j 2 p i by N, we subdivide the entire angle of 2 p i into n number of equi-space points, all right. We call this basic operation of an angular shift as W N; this capital W stands for a unit, basic unit of the angular shift. See, e to the power minus j 2 p i by N is, if this is 2 p i you divide into n equal parts, okay like this.

So, each of these will be corresponding to an angular shift of WN. So, if I have 8 points then WN will be; WN 1, that is one unit will be e to the power minus j 2 p i by N, that is multiplying any quantity if I, say multiply x K by W N 1, it will mean x K multiplied by e to the power minus j, this N is now 8, 2 p i by 8. How much is it, 45 degree shift.

So, it is x K into e to the power minus j p i by 4; which means x K multiplied by sorry, 0.7 o 7 minus j 0.7 o 7, 1 by root 2. That means the entire circle has been divided into 8 parts, all right. So, this is 45, 90, 135, 180 and so on. So, cosine of minus 45 degrees and sin of minus 45 degrees, so that give me this; so it is basically a complex quantity that you are multiplying with the real term x K, all right.

(Refer Slide Time: 12:18)



So, X suppose you are given some values x n; so let us take a very simple set of values 1 minus 1, 2, 3, 0, 1, 2, one two three four five six seven; let us have one more value 1, as for simplicity I have taken very simple values; 1, 0, 2. So, what will be capital X 0? We define, okay before we go through that; X we are evaluating at e to the power j omega K, at K equal to 0. We call this as capital X K at K equal to 0.

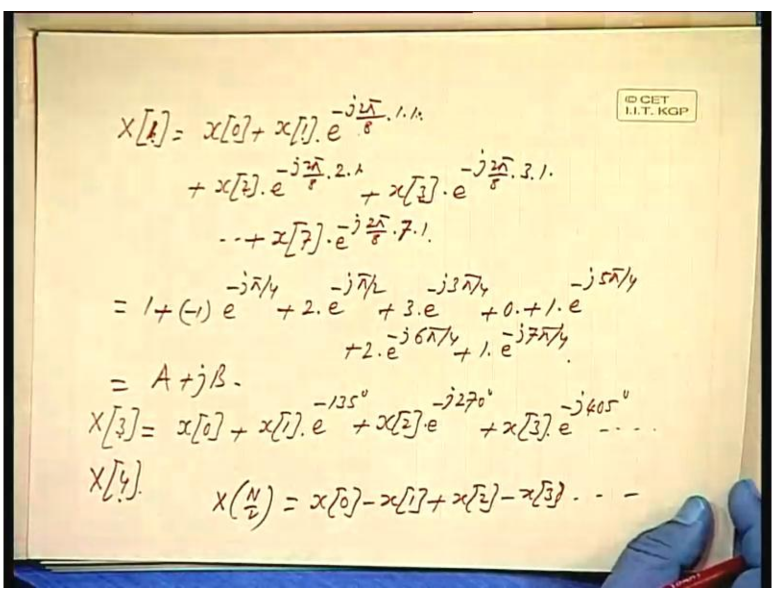
So, that is capital X 0, okay. Hence forth, in the transform domain we shall be denoting this because there are only 8 for an 8 point sequence; there are only 8 frequency domain representations, so I will call them as, capital X 0, X 1, X 2, all right. They are very similar to your Fourier harmonics, in the continuous domain you talk about harmonic terms all right; they are also exactly similar to harmonics.

So, X 0, how much will be that, X 0? It will be x n, WN; see this was e to the power minus j 2 p i by N into K into n, is it not, if you look at it we were multiplying by e to the power minus j omega n into x n, is it not summation like this? And what is it, x n e to the power minus j omega K into n? And what is omega K? Which means, e to the power minus j 2 p i by capital N into K, is it not? K equal to 1 corresponds to this, K equal to 2, K equal to 0, K equal to 3, 4, 5, 6, 7, so there are eight values. So, you are taking different values of K between 0 and 7, okay there there are eight points.

So, if you substitute here, so W N into n K which means; e to the power minus j 2 p i by N into K into N. So, basically this reduces to W K into n and W K is nothing but, 2 p i by N into e to the power minus j 2 p i by N into K, okay this small w. So, this is x n, W N, n K is that okay? And this is to be evaluated at K equal to 0, to get these value, okay.

So, what is it? If I put K equal to 0, this will be always 1; e to the power minus j 2 p i by N multiplied by 0, is always 1, so it will be just summation of x n, sum of these terms all right. So, it will be 1 minus 1, 0, 2 plus 3, 5 plus 1, 6, 7, 9 is that okay? So, let us compute one or two more then we I will go to the details of this. What will be X 1 for the same sequence, what will be X 1?

(Refer Slide Time: 16:46)



It will be, x 0 plus x 1 into e to the power minus j 2 p i by 8 into 1 into 1 n into K, all right. n is this, K is this. So when I am going to compute for K, I multiply by this plus x 2 into e to the power minus j 2 p i by 8 into 2 into 1 plus x 3 into e to the power minus j 2 p i by 8 into 3 into 1, is that all right, and so on plus x 7 into e to the power minus j 2 p i by 8, 7 into 1, okay.

So, 1 comes every time because I am evaluating it, for K equal to 1 that is corresponding to K. And this 7, 3, 2 etcetera, they will be coming along with the term x n, all right because that n is varying. So, it is summated over n. So, how does it look like? 1 plus first one is 1 then minus 1 next value is minus 1 into e to the power minus j; now, there is a common multiplier that you can see, e to the power minus j 2 p i by 8 into 1. So long as this is 1, e to the power minus j 2 p i by 8 into 1, 2 p i by 8 into 1, so e to the power minus j 2 p i by 8 is a basic element of rotation minus 45 degrees.

And it is to be multiplied by 1, 2, 3 depending on the sequence element, x 2 x 3 and so on. So, the first element, it will be e to the power minus j p i by 4 okay then 2 into e to the power minus j, 2 times this plus 3 into e to the power minus j three times this, this is the basic angle plus 0 plus 1 into e to the power minus j 5 p i by 4 plus 2 into e to the power minus j 6 p i by 4 plus 1 into e to the power minus j 7 p i by 4, is that all right. Whatever be that result, sum A plus j B, you can write all right.

If I go for say, I want to compute X 3, what should I do? I am arbitrarily taking some value other than one, to make my point clear. x 0 plus; what should be the multiplication of x 1, now the basic angle will be three times p i by four that is minus forty-five, minus ninety and then minus 135 degrees, okay. It is minus 135 degrees from your side, minus 135 degree should be the basic angle and multiples of minus 135 degrees, is that all right.

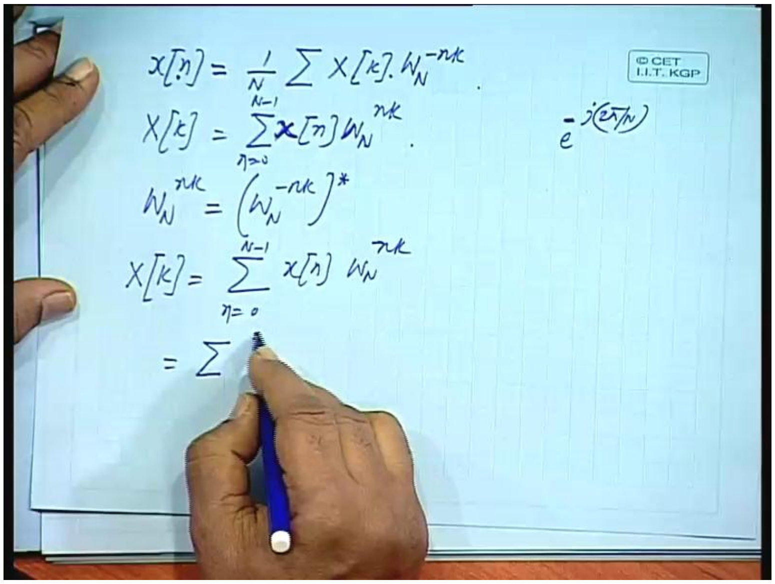
So, x 1 into, if you permit me to write in terms of degrees which you are probably all convenient with; minus 135 degrees I will write like this, only for is of understanding then x 2 into e to the power minus j, 270 degree all right plus x 3 into e to the power minus j 405 degrees and so on, is that okay. And you compute the values for minus 45 degrees, it is plus 0.7 o 7, minus j point 7 o

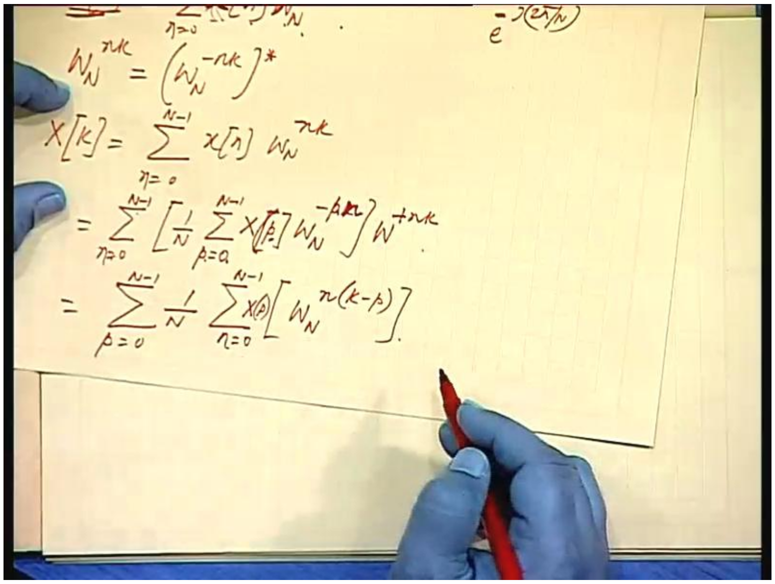
7. For 90 degrees, it will be just j. 135 degrees it is in the third quadrant, both will be negative. All the time, it will be 1 by root 2.

Then if I take x 4, what will be the basic angel, four times 45 degrees. So 180 on; so it will start with minus 1, 2 times 180 plus 1, 3 times 180 minus 1 again four times 180 plus 1 and so on. When we compute X 4; X 4 which is half of capital N that is 8 point sequence, if you take half of it, the basic angle will be 180 degrees all right. So, alternately then science will be changing okay.

So, if I ask you to compute X 4; blindly you can write 1 minus of this, which is plus 1 minus then plus of 2, minus of three plus of 0, minus of one plus of 2, minus of 1 that will be the net result. So, whenever you are going to compute X at N by 2, X at N by 2 the sequence will be just alternately appearing with the negative sign and so on, is that okay. The reverse relation, that is to get x n from x K is also very simple, it is 1 by N, X K, W N minus n K.

(Refer Slide Time: 23:07)





See, the forward sequence was X K sigma x small x n, W N n K; mind you W has associated with it negative power here, all right minus j 2 p i by N. So this means, plus j 2 p i by N, okay. If you are given this set of X K; 8 discrete Fourier transforms, if I give you, you can always get back x n. Earlier, you for the continuous domain you are integrating from minus p i to plus p i, here you do not have to do that; you just take summation of those eight points and you have to multiply by just complex conjugate.

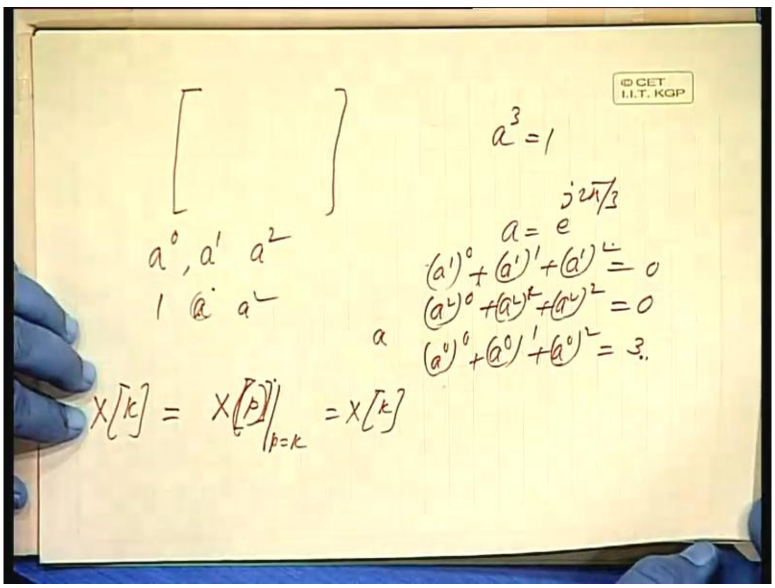
You see, earlier it was minus p i by 4, now if I put a power of plus one, it will be plus p i by 4, all right. So, it will be just complex conjugate. So, W N, n K is nothing but W N minus N K conjugate, all right. It is a just a complex operator. Now, how do you get these? Where is the proof? So, let us go for the proof. So, X K is n varies from 0 to N minus 1. I can write x n, W N, n K, n vary from 0 to N minus 1. Now, this x n we have written as, 1 by N X say, p you can write X p W N minus p K, okay; p varying from this is n equal to 0 to N minus 1, I should write p N W minus n K okay, plus n K, is that all right.

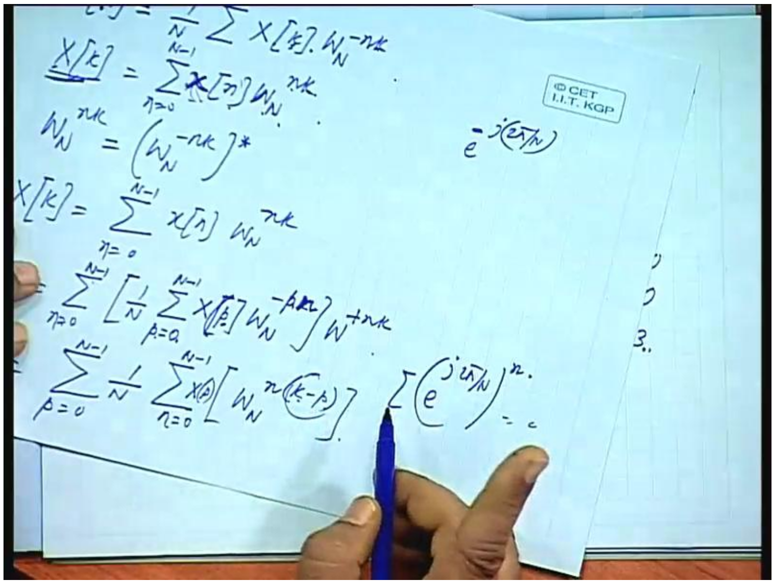
See, x n X K, I started of this X K is equal to x n W N n K. And x n, I can write in terms of this. I am not using the same variable K, I am taking I can have any running variable K. So, I have taken p okay; X p W N p K, it should be p n, all right and W n K okay, equal to 0 two N minus 1 without loss of generality.

P is the running variable, so summated from this range 0 to N minus 1, okay. K is not a running variable, K is at a particular value K we are evaluating and over this again; I am having a running variable of n, I can interchange p and n summation n equal to 0 to N minus 1, okay. I can put X p outside W N n into K minus p okay, n is varying.

Now, most of you have studied in three phase power systems, balanced and unbalanced voltages, all right. Any unbalanced system, we can always represent in terms of sequence components of three sets of balanced voltages, all right. So, if you remember the matrix of transformation that we used, was having an angular shift of 120 degrees, okay.

(Refer Slide Time: 28:16)





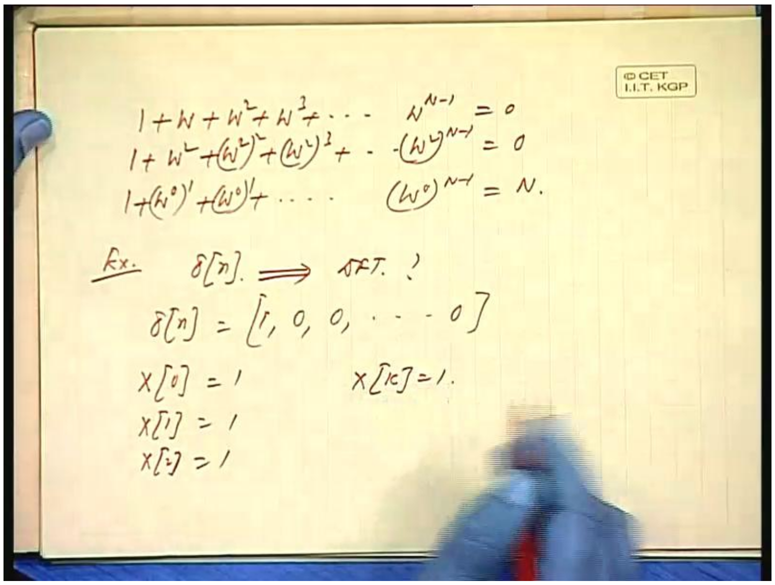
We are writing that as an operator a, okay.  So, we had a to the power 0, a to the power 1 and a to the power 2; where a cube was making 1, all right. So a was basically, e to the power j 2 p i by 3, 2 p i by 3, hundred and twenty degree shift, we are multiplying in the positive reaction. So, a 0 means 1, this was a, this was a square, all right.

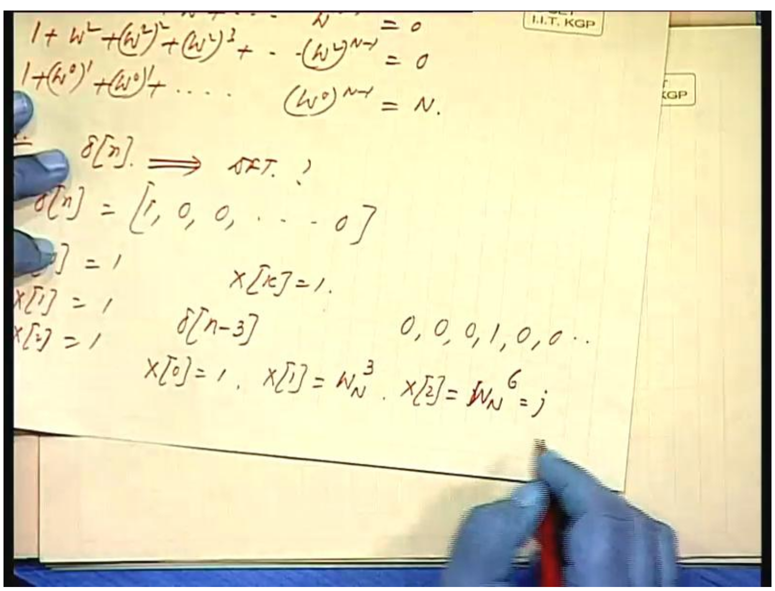
Now, you take any one of them, a to the power any of them okay and if you sum them together; that is a, suppose I take a basic element of a 1 then a 1 to the power 0 plus a 1 to the power 1 plus a 1 to the power 2, will be equal to 0, a 2 to the power 0 plus a 2 to the power 2 plus a 2 to the power, sorry 1 plus a 2 to the power 2; so by the extent, that same logic here, we will find W N is a basic element all right.

If you take any difference K minus p other than 0, now if I take a 0 to the power 0 plus a 0 to the power 1 plus a 0 to the power 2 then I will get 3, all right. So, it is basically the same thing that is how, we get the 0 sequence component, is it not. W N is e to the power j 2 p i by N, okay. Now, if I multiply by n and summit over n of a quantity other than 0; if K minus p is other than 0, 1, 2, 3, 4, whatever you take if you summit, it will be always equal to 0.

But at K equal to p, it will be equal to n; it has come to 3, is it not. So, it will be equal to n. So, this n will get cancelled with these and this will stay only for K equal to p, is that all right. So, it becomes only X p, after summation it becomes only 1, 1 into N, N gets cancelled. So, we get X K is equal to X p where p is equal to K; this this is valid only for p equal to K, okay.

(Refer Slide Time: 31:18)



So, basically 1 plus W plus W square plus W cube and so on, W N minus 1 is always 0; sum of these will be always 0. If we take 1 plus W square plus W square squared W squared cubed W square to the power N minus 1 will be also equal to 0. You take any power of W, as a basic element and then you add up these terms, it will be always equal to 0, except for W equal to W to the power 0, okay. Now, there are certain interesting functions, that we shall discuss now, delta n all right.

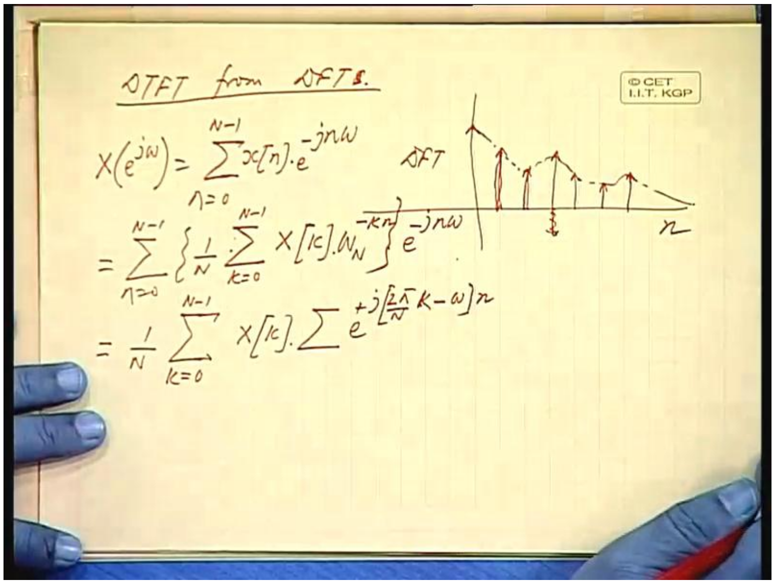
What will be its DFT, what will be its DFT? Could someone suggest, what will be the DFT of delta n, n? Delta n is basically, a sequence like suppose we have an eight point sequence; we take only eight points n 1, 0, 0, there is seven 0, all right. So, if we have an eight point DFT, what will be this? Okay, what is X 0? 1. X 1? 1. X 2? All will be 1, okay. So, X K equal to 1, this is what we have observed also in DTFT, will you remember?

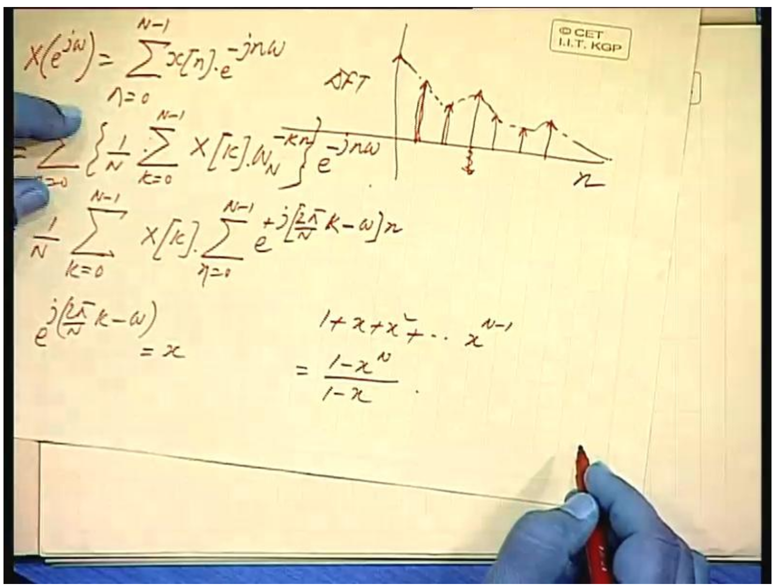
What will be delta; say n minus 3, what will be delta n minus 3? Okay, delta n minus 3 is a sequence, like 0, 0, 0, 1 then again all zeros, okay. So, what will be capital X 0? 1. X 1? W N 3, do you all agree? What will be X 2? What is W N 3? Minus 1 by root 2, minus j 1 by root 2; third quadrant 135 degrees, is it not? If n is equal to eight, you are considering an eight point sequence then it will be 135 degrees.

And what will be X 2? Two times 135 degrees, so 270 degrees, so minus 270 degrees, remind you. Minus 270 is plus 90 degree, so that will be plus j, okay. I will write W N 6, that is nothing but plus j, we have taken n is equal to; you should put 8 then only then will be clear. Yes, we are considering only eight point sequence for ease of understanding, you change the value of N; obviously these values will be changing, okay.

So, this is how you can obtain the DFT's. There are now, can you compute DTFT from DFT's that is what we are trying to achieve is; if you are given say the DFT's of course their complex, there is an angle associated with that.

(Refer Slide Time: 35:39)




I will show only the magnitudes on this side and so on. Suppose this this is a DFT, what we are trying to find out is; what is that continuous function whose discrete values are these, that means you are trying to reconstruct, you are trying to reconstruct the original DTFT whose sampled values are these DFT's.

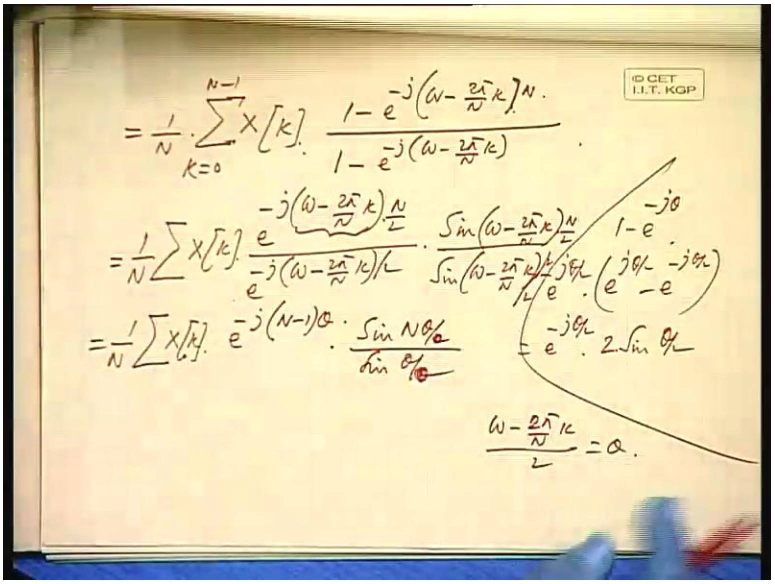
So from the sample values, we are trying to reconstruct the original DTFT, okay. So, we go by this, X e to the power j omega is; sigma x n e to the power minus j n omega, and what is x n? You can write in terms of the DTFT's, the DTFT; we can write in terms of DFT, it will be X K W N minus K n, all right. Then e to the power minus j n omega, this is for x n from the discrete domain, is it not.

I can take 1 by N outside, this is summated over K, I can interchange the summation operation and then X K summation. What is W N minus K n? We can write in terms of e to the power minus j 2 p i by N into K, is this all right. This is minus that means; here it will be plus minus n omega.

So, minus omega whole thing into n is that all right, is this okay? n varying from, so what you are adding up is; capital N number of summation of this exponential terms, all right for a particular value of omega. We are now trying to find out all continuous, the continuous domain representations of that frequency function.

So, here if I take e to the power j 2 p i by N K minus omega as a basic element x all right; so what is this? 1 plus x plus x square, n is equal to 0, 1, 2, 3 means up to x to the power N minus 1, okay. I can write this as, multiplied by 1 minus x divide by 1 minus x; so 1 minus x to the power N by 1 minus x, is it not?

(Refer Slide Time: 40:11)




So, you can write this as equal to 1 by N summation X K, then 1 minus e to the power minus j omega. I can always take omega on this side, so I put a minus; minus 2 p i by N K, okay whole to the power N divided by 1 minus e to the power minus j omega minus 2 p i by N into K, correct me if I am wrong, K equal to 0 to N minus 1, is that okay. I can write this, this divided by 2; if I take out then it will be sin function.

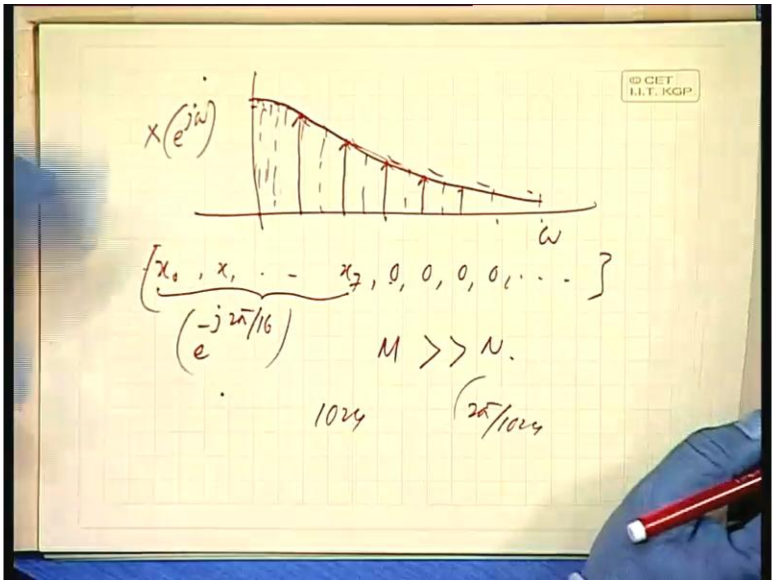
Similarly 1 minus e to the power minus j theta; I can take minus j theta by two common, so e to the power minus j theta by 2 into e to the power j theta by 2 minus e to the power minus j theta by 2, okay; which will give me, e to the power minus j theta by 2 into twice sin of theta by 2 into j, okay. Now, both of them will generate j, so they I will get cancelled. Both of them will generate 2, so they will also get cancel. So, 2 j will get cancelled from both.

So, I will take out 1 by N sigma X K e to the power minus j omega minus 2 p i by N K into N by 2, divided by e to the power minus j omega minus 2 p i by N K divided by 2. And this side, I will have sin of omega minus; okay if I call this as theta okay, 2 p i by N K into N by 2 divided by sin of omega minus 2 p i by N K divided by 2 okay, this angle divided by 2.

So, that gives you 1 by N, in a compact form; if I write X K e to the power minus j, this is this by 2 if I take as theta omega minus 2 p i by N K by 2 as theta. Then it is N theta and this is theta, so N minus 1 into theta, e to the power j theta 1 e to the power j theta 2, so this will be theta 1 minus theta 2; okay into sin of N theta by 2 by sin of theta by 2, where or this I have taken as theta. So, sin N theta by sin theta okay.

So, given the values of X K, you can always reconstruct capital X e to the power j omega. The other alternative is; this will be giving you an analytical expression, all right, so that anybody can put in any value of omega we can evaluate this, corresponding to omega you have to evaluate theta and then we can make a program. So, normally what we do, we do not make an analogue plot. We take large number of points, you know. Say, if I ask you to sketch X versus omega capital X versus omega, like this, what do you do?

(Refer Slide Time: 44:49)




You compute for omega equal to 0.001 radian, 0.002 radian and so on. Large number of points you compute and then you sketch it, is it not. So, if you have to do it for getting a continuous domain function, if you have to do it in a discrete way then why not do it with the help of DFT, so what do you do? Suppose, there are eight points given to you, so eight points will give you this kind of a characteristic; out of which you are trying to identify, with those eight points, if you take direct DFT, you get only eight points here, okay. Instead of eight points, I could have selected sixteen points, all right; sixteen points and I could have got sixteen such values. So, x 0, x 1 up to x 7, there are seven points given to you.

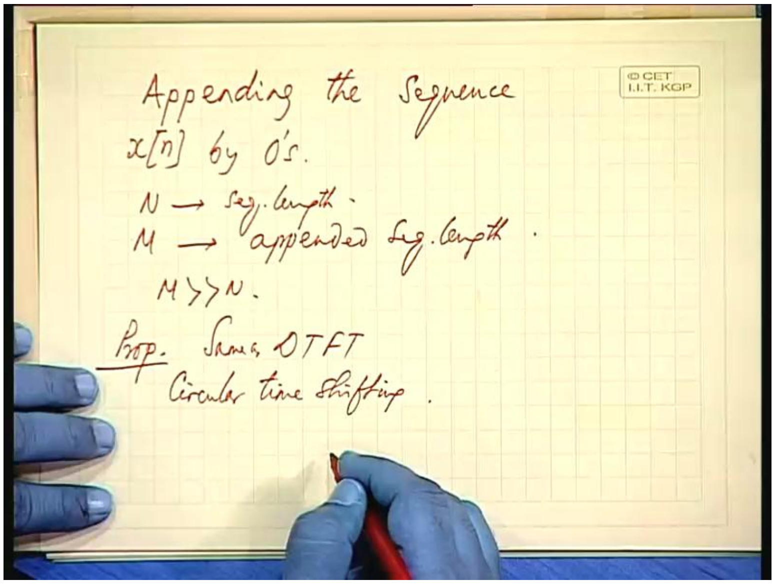
There is no harm, if I put some more zeros, eight more zeros. So that becomes a sixteen point sequence, okay. Now, my multiplier will be e to the power minus j 2 p i by 16, basic element will be 2 p i by 16, 22.5 degrees. I will keep on doing that and then I will get sixteen components, all right. If I appended with say, large number of zeros to make it a length of 256, 256 then I will get two-fifty six such points, all right. So, basically you append this by zeros to make it of a length M, which is much much greater than the given sequence length N.

Well, I am getting large number of points; I can get a very smooth curve. So, my idea is to get DTFT from DFT, one is using this relationship all right. But then this will remain in any case this will remain only on paper, as a as an analytic function, as an algebraic expression but when you want to compute it, for making a plot you take discrete points and compute, is it not. Compute these values and then you make a plot.

So, when you are going to take discrete points; that means what values of theta will you take? What values of omega will you take? Say, 0.1, 0.2, 0.3 so many radians. So, I suggest you take thousand such point, thousand twenty-four points, 2 p i divided by thousand twenty-four that will also give you of that order, 0.1, 0.2, is it not. So, if I can take 1024 points that means these eight points I append with zeros; to make it a length of 10, 24 then I will get those ten, twenty-four points by direct competition of DFT.

Now, if a faster algorithm exists for computation of DFT, for large number of points where number of points is 2 to the power N then the problem is very easy, simple, all right. So, that algorithm is known as FFT, so fast Fourier transform. Basically, the discrete Fourier transform is computed via very fast algorithm. So, fast Fourier transform is not something different, it is only an algorithm. So, whenever you are asked to compute the DFTs of a sequence, we normally pad it with zeros all right; we get large number of points and we try to take the advantage of FFT algorithm that is very simple.

(Refer Slide Time: 48:58)



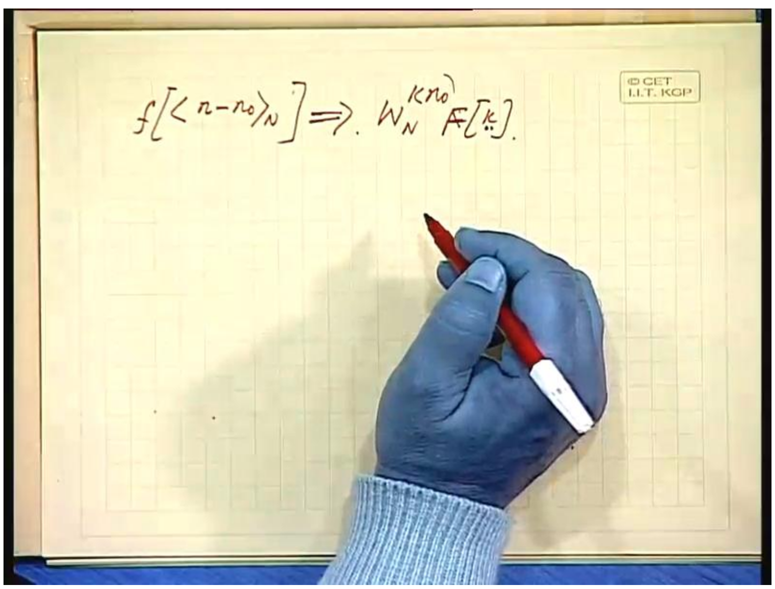

So, we go for appending, also known as padding. Appending the data of sequence x n by zeros. Sequence length was N and M is a appended sequence length, okay. So, M is much much greater than N. So, this is the practical way of computing DFT from DTFT. Now, there are certain important properties of DFT that we can just discuss. The properties are most of the properties are same as DTFT okay, most of the properties are similar to DTFT. There are some interesting features of DFT; circular time shifting.

Now while computing DFT, when we talk about DFTs, we introduce this terms circular sequence. Suppose, we have okay we have a sequence say zero, one, two, three, four like this; I mark it 0, 1, 2, 3, 4, 5, then. Suppose, zero, one, two, three, four, five this five point sequence; I have not shown you the values for which a particular variable, say x n exists, that is x 0 is a plus 3 okay, x 1 may be 4, x 2 may be 1 and so on.

There are some values at these points and after that, the function becomes zero, this is a normal linear sequence, okay. If we roll it that is next point is made 0; that is this zero is brought here, next to 5 then what will you see? If I keep on shifting it, it will be periodically appearing, is it not? If I keep on rotating it, we I will get zero, one, two, three, four, five, zero, one, two, three, four, four, five and so on.

So such a periodic sequence, can we generated from any finite length sequence, by just putting it like this, rolling it all right, winding it and then you get a sequence like this. Now, if I just move it, I will get the same sequence with different starting point, all right. So, this is known as a circular shift; I mean this kind of a sequence we shall be using, in circular shift property. So, circular time shifting property is, suppose there is a function f module N.

(Refer Slide Time: 52:57)




Module N means; basically one, two, three, four, five, six, this capital N is six. So, after every six points, it is repeated, okay. So, if I take n is equal to 3, n is equal to 3 and n 0 is any particular shift; so 3 minus a n 0 is 1, so 3 minus 1. And then I keep on 4 minus 1, 7 minus 1, 8 minus 1; if I want to compute 8 minus 1, if I take module, module means you subtract keep on subtracting six, all right. If it is in the negative region, then you keep on adding six.

So, x of minus two is same as, if I add six instead of four all right, same as x of ten. So, if if I evaluate this, the corresponding DFT will be W N K times n 0, F  F K. So, if I want to compute the K th element of that shifted sequence then it will be the original sequence value of F K multiplied by this W N K n 0, okay. We I will stop here for today, we will take it up in the next class.