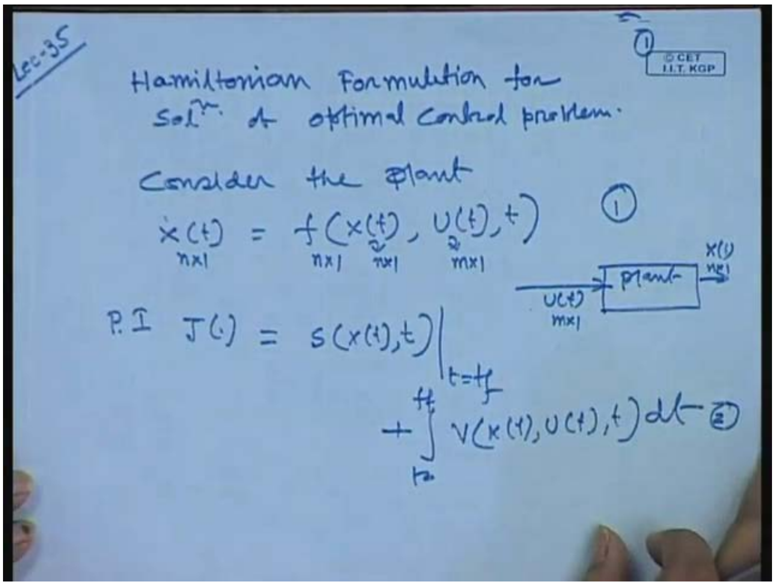
Optimal Control
Dr. Goshaidas Ray
Department of Electrical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 35
Hamiltonian Formulation for Solution of Optimal Control Problem and Numerical Example

We discuss the Hamiltonian formulation for the solution of optimal control problem, and then we will consider some numerical example.

(Refer Slide Time: 00:35)

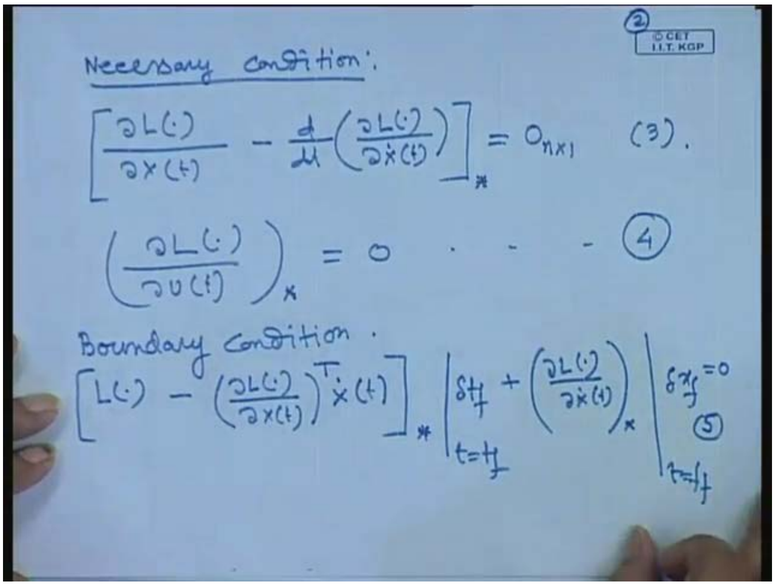


So, this formulation is Hamiltonian Formulation for solution of optimal control problem, of course using calculus of variation. So, if we recollect why we were going for a Hamiltonian formulation, suppose the system dynamics or system plant is described in state space form, means any system can be described with a n-dimensional vector. So, n-dimensional what is going any system can be described with n first order differential equation. If you represent the system in state space form, it is convenient to deal with Hamiltonian function rather than the Lagrangian function.

So, let us recall our earlier problem. Consider the plant or system x t is equal to f of x t U of t and t and this dimension is n cross 1, this is n cross 1 and this number of states is n cross 1 and number of inputs to the plant is m cross 1. You can think of it we have a plant or system this one input is U of t, which dimension is m cross 1 and the state you can say x of t, which dimension is n cross 1. So, this plant is described as in general we have a dynamic system is there, we can describe with a n th order differential equation, which can be converted into a nth first order differential equation.

So, once you convert into a state space form then our problem is here and the corresponding performing index j. If we recollect this one we have written x of the terminal cost is a function of x t and at t is equal to t at final time plus integration of t zero to t f v x of t u of t and t d t. So, our problem is to find out the control of U such that this performing index is minimize not only that subject to the constraint that equation. Let us call this is the equation number 1 that is the equation number 2. So, our problem is to find out the control of U of t, which in turn to find out the optimal trajectory x of t. Such that this performing index is minimized subject to this constraint equation number 1 that is what we discussed last class.

(Refer Slide Time: 03:56)

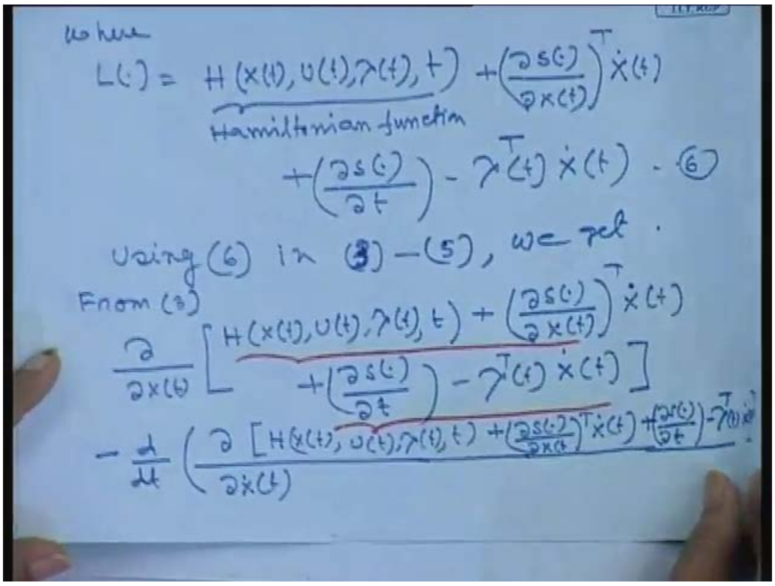


And we got what is called the necessary condition, if we say our necessary condition along with the boundary condition, you got it that del l this x of t minus d of d t del l dot function of l is a Lagrangian function, which is a function of x t u t lambda t and comma t. So, that differentiate with respect to x dot of t whole that you solve this one along, you have to find the star means, you have to solve this one whatever the solution you will get it. That is the optimal trajectory, or optimal contour law from which in turn it will give you the optimal trajectory x star of t.

So, that equation you have to solve, so you have a in turn we have a n cross 1 differential equation is there, so that differential equation in general it is a non-linear differential equation. So, that is let us call equation number 3 and not only this another x condition you got del u del t this one is equal to 0. So, this one is equation number 4 corresponding to the our this problem.

And we have a boundary condition that boundary condition if you see this one that we obtain boundary condition; we obtain l Lagrangian function minus del l of with respect to x of t. Since x is a vector, so l is a scalar quantity that will be a column vector. So, you have to take transpose of that one and then multiplied by x dot of t. So, this you along the trajectory t star, then t is equal to t f, this is the boundary condition we got it when we have plus then del l dot, del x dot of t whole star t is equal to t f and delta x f is equal to 0, so let us call this equation is 5.

(Refer Slide Time: 06:31)



We know what the l is, the l expression we have written in terms of what is called Hamiltonian functions where, l dot which is a function of x t, u t, lambda t, t. So, this is the function of that Lagrangian function is split up into a, what is called Hamiltonian function and then Hamiltonian function free from x dot. So this is the Hamiltonian function and that function is called Hamiltonian function.

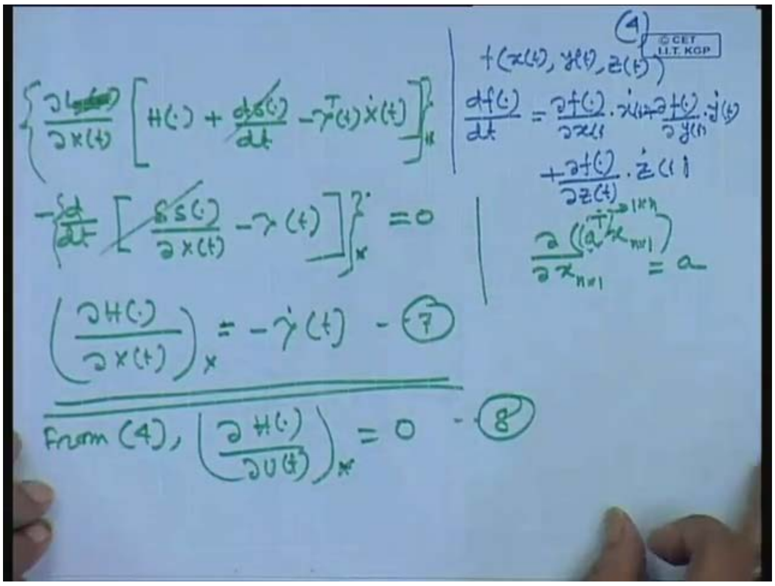
So, Lagrangian function this Hamiltonian function is nothing but if you say what we have considered this, that will write it in this plus del s dot plus del x of t whole transpose x dot of t plus del s of t del t minus lambda transpose of t x dot of t. So, this is the function of what is called v, v is the that function, if you see integrant part of this integral 1, v plus Hamiltonian function plus lambda transpose of f that means, this two terms if you see this term and this term is free from x dot and that function we consider as a Hamiltonian function.

Because why we have expressed that necessary condition what we got it here necessary condition equation number 3 4 and boundary condition 5. If you express this thing into Hamiltonian function then we will see the system is described in a state space form. If we express this in place of Lagrangian function, if we replace by Hamiltonian function a set of equation what we will get it in a simpler form and it will be easier to solve if the description is state space form, so that we will see later of this one. So, this is our equation number let us call 6.

Now, I was replacing, you can say using 6 in equation 3, we get what, let us see. So, our first from 3, we can write from 3 is del l del x, so in place of del l I will just write it in terms of h, that whole the expression I will write it, so del l del x. In place of l, I am writing h x of t U of t lambda of t comma t this one plus delta s dot of del x of t whole transpose the dot of t plus del s, this del t minus lambda transpose of t x dot of t. So, if you just say, the whole thing from here and including this one is nothing but our l Lagrangian function.

So, I have just written the first term of our equation this is that one and the second term if you see this one, I am writing is minus d of d t then del l again this whole quantity differentiate this with respect to x dot. So, what is the del l, I am just writing h x of t u of t lambda of t comma this plus del s dot del x of t whole transpose x dot of t plus del s dot del t this one minus lambda transpose of t into x dot of t. So, that is why we have written l this whole thing just say as this one same as this one, we have writ10 it here, then this is differentiating with respect to x dot.

(Refer Slide Time: 12:13)



Now, look at this expression that we have used frequently that change whole is like this way, suppose we have a function x is the function of x t, t is the parameter and y t and z of t, this is the function of this. Now, if we want to find out the differentiation of x with respect to d t, this nothing but a partial differentiation of f with respect to x then x dot then del f del y. Then another variable del y of t then y dot plus del f del z of t equal to z dot.

So, what is this, if you have a function x of t, y of t function of x y t, which is a function of time each then I can write differentiation of f with respect to time t is nothing but partial differentiation of f with respect to x multiplied by x dot of t. Similarly, partial differentiation of f with respect to y t is y dot of t and z dot of t. So, this thing we will write it here. Now, see this one s is a function of what and this I can write it s is the function of x t and t. Now, I am writing it this one that, this term and this term combinely I can write it, look by using the change rule this term and this term combinely I can write it that del d s of d t that is what we can write it this and this combinely.

So, rewrite this equation what we write it this one just say del l del x of t then h, I am not writing h dot means this is the function of x t, u t, lambda t and t, this h plus d s dot d t. And what is left this and this d s or d t and lambda transpose minus lambda transpose of t x dot of t. This is the first part we have simplified and the second part of the Lagrangian equation d of d t. See this one that whole thing, this is this quantity, this quantity you are differentiating with respect to x dot h. This h is not a function of h dot, so partial differentiation if you do with respect to x dot, this term will not be there.

The first term will vanish then s is a function of x t and t, so this is a function of x dot. So, we can take this is a constant because it is not a function of x dot t, so if you do the differentiation of this one del s partial differentiate of s with respect to x t we will get it that one. And this is not a function of x dot t, so this will not come into the picture. And now it is a function of x t, so only this term and this term will be remain in the differentiation of the Lagrangian function with respect to x dot, when Lagrangian function x we see in Hamiltonian form.

So, this term ultimately if you see, it is coming like this way, if you just do it then it will come. And if you recollect once again that if you have a function, if you are differentiating that function with respect to x, x is a vector. And you are getting a transpose x, a is a row vector of dimension n cross 1 and so it is a scalar quantity, so that it differentiate it with respect to a vector that results is a. So, now if you just use that one, the differentiation of this is you can concise this scalar multiplied by a, what is called differentiation this thing by a, what is called a vector, so that results will be a row vector.

And you are differentiating with respect to this, so results will come del differentiation of this with respect to x dot that term. So, ultimately I will get differentiation of s with respect to x of t and this term. Similarly, this is minus lambda transpose x dot differentially x dot is here one and this will be lambda of t minus lambda of t. So, this whole thing if you see this one, you can put it that whole thing is in bracket of that, if you just do it here this is not l, this is of that one star you can put it in star that one.

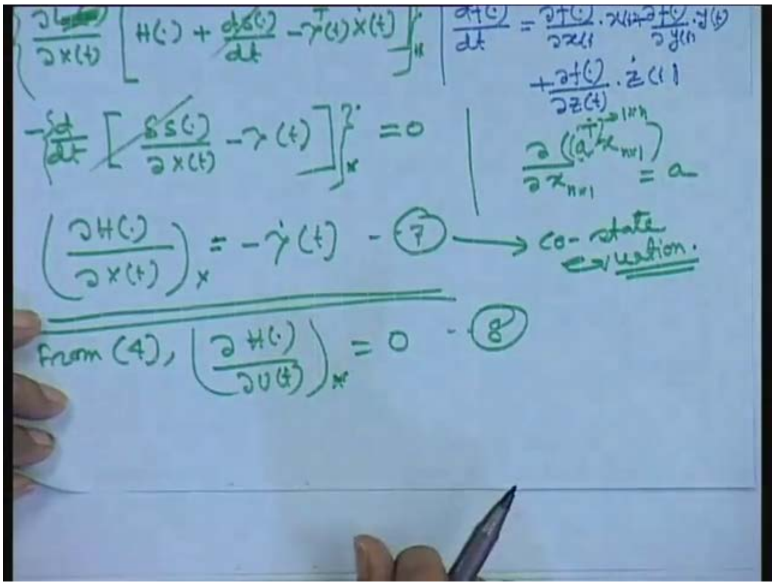
Now, this equal to 0 because what formula you are getting that first equation of from equation 3 this equal to 0. So, right hand side of this one equal to 0, so from 3 I am writing from 3 this is 0 and I am getting that one now. Look at this expression partial differentiation of that quantity with respect to x of t and this order of differentiation you can change it. So, if you change it this one, this is plus, this is minus so this is cancelled only left over is this one del h because that is the function of you have to differentiate with respect to x t.

So, that will be vanish that this is lambda t lambda transpose t x dot of t so you have to differentiate with respect to x. Ultimately it is a del h, this del x of t whole star, this bracket this star is here this and this is also started from here to here. So, this equal to this plus this minus this minus plus equal to your term is what, differentiation with respect to t. So, it will coming minus, minus plus if you take right hand side it will be lambda dot of t. So, let us call this equation, you have used the equation number up to 6, let us call this is equation number 7.

So, now see this one, while you use the Lagrangian equation the del l Lagrangian function you partial differentiate with respect to x minus d of d t del l del x dot is equal to 0. So, this expression is boils down to what is called a simple Hamiltonian function form and this function is free from x dot is a function of x u t lambda t and t. So, next is this is the equation and what is this equation, another equation is that one from 4, what you can write from 4?

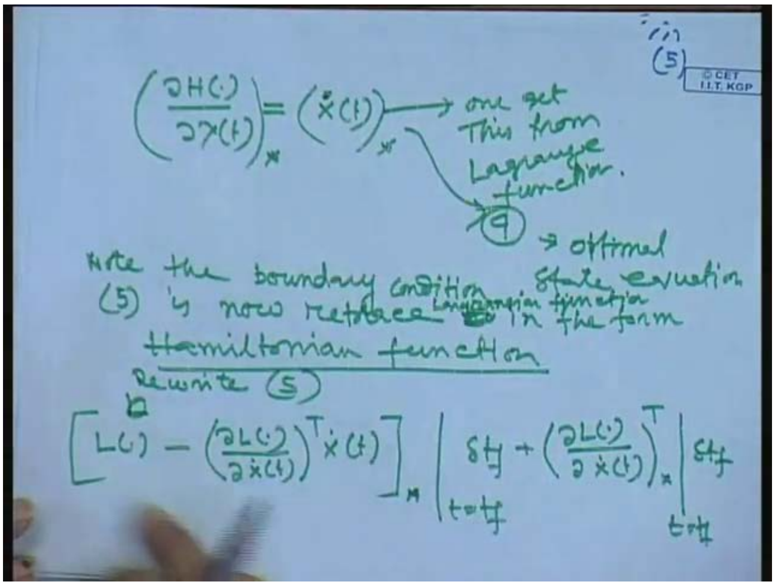
So, your h is l is expressed if you see l is del l del u if you do del l del u and l you express that one, so it is a function of you are differentiating partial differentiating with respect to u there is no u is here, so del h del u must be 0. So, from 4 we can write del h dot del u of t whole star is equal to 0, let us call this is equation number 8. Now, you see this one another expression we call this equation co state equation, number 7 is called co state equation.

(Refer Slide Time: 22:17)



So, this is the co state equation, so another equation one can derive like this way del h dot del lambda is equal to x dot of t.

(Refer Slide Time: 22:38)



Now, see our objective function, if you refer to your last class note we have written what is the Lagrangian function? We have written the objective function plus the lambda transpose constraint that is our Lagrangian function. So, del l del one of the what is called equality condition del l del lambda must be equal to 0. So, we can get this equation from Lagrange expression, one can get this expression get this expression from Lagrange function or expression. So, this is also if you write you can write star, so let us call this equation is equation number 9.

So, this you called is optimal state equation and you see the co-state equation dimension because lambda dimension is if you say n cross 1 same dimension of this state vector, so that is why it is called co state equation. Nature of this equation and this equation is they are co-state, one of co state of another, so this is called co state of a state equation vector.

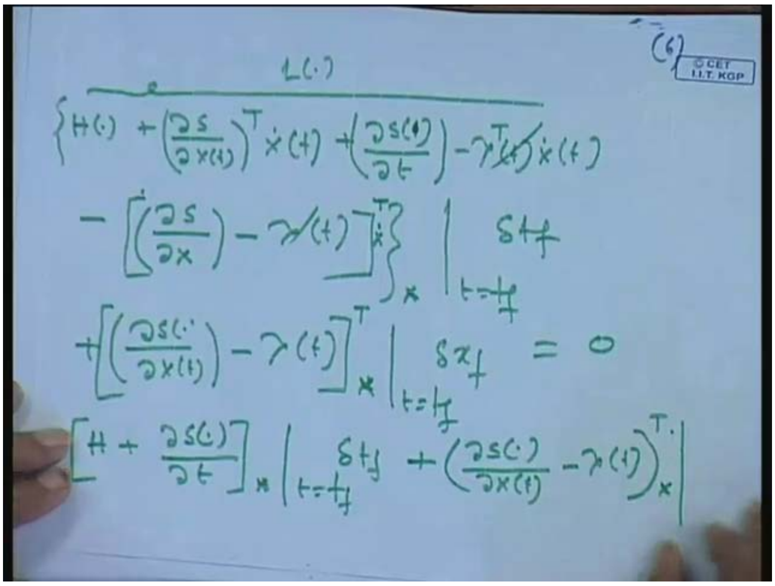
So, now we have that this is the equation number 3 is now when you express Lagrangian function in terms of Hamiltonian matrix, this expression simplified form you are getting del h that corresponding expression that one, we get in a simpler form del h del x equal to minus lambda dot of t. Another expression you have to write in place of del l del u partial differentiation of Lagrangian function with respect to u, when l is expressed in terms of what is called that Hamiltonian matrix then you will get del h del u. And the state equation that what we have equation del h del lambda is equal to x dot of t.

So, now see the boundary conditions, if you see the boundary condition of that one then what we are getting it, we will just see the boundary condition now. So, the boundary condition what we will do, we will replace the Lagrangian function in terms of Hamiltonian function that is all, what is the final boundary condition we will get that we will see it. Now what is our problem, a boundary condition is the expression is equation 5, so boundary condition 5 is now replace by in the form of Hamiltonian function. Now, replace the Lagrangian function in the form of Hamiltonian function.

If you replace this one, now you see first what will write it for this one that equation number 4 if you see here. So, this is h you will get it that is l differentiate of this with respect to x whole transpose x dot putting t is equal to t f that one and then del l del x dot t is equal to t f del x t f.

So, let us see what we can write for this one, so we can write it for this one, if I just put it here that let us call rewrite equation 5. If I rewrite this one l dot minus l dot of this x, x dot of t whole transpose x dot of t then this star t is equal to t f delta t f plus delta l dot del x dot whole transpose then your star t is equal to delta t f. Now, see this one our expression for that one, what we can this l, l is our expression Lagrangian function expression.

(Refer Slide Time: 28:25)



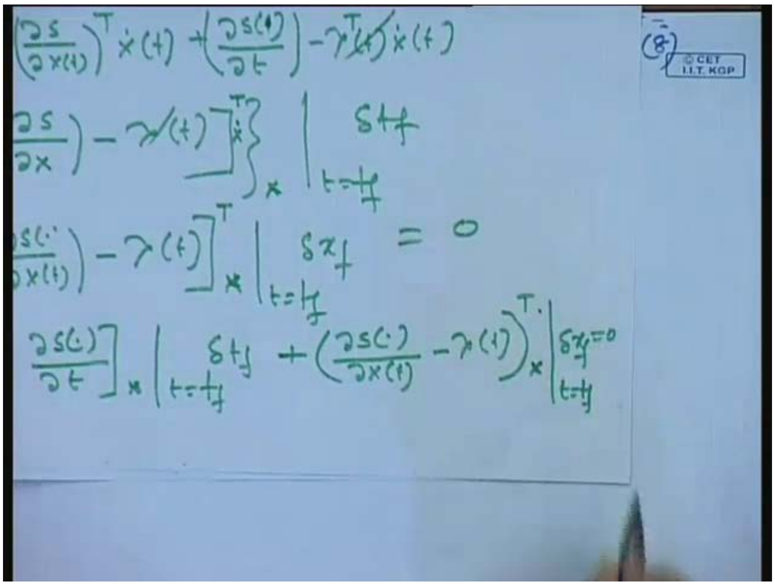
So, if I write it this one you just see what we are writing l, l is nothing but h dot del s del x of t whole transpose x dot of t plus del s of t del t, this minus lambda transpose this is lambda transpose of t x dot of t, so this I have written in place of l. Now, we are differentiating l with respect to x, so if you do this one then what will we get it? This we are differentiating with respect to x dot then it will be a del x, the whole thing we are differentiate with x dot then it will differentiate with respect to x dot l. Here this is x, see this one; I am differentiating this with respect to x dot, so you have a minus x. So, this will be a del s del x whole minus there is a x dot here, this is l I am differentiating this thing with respect to x dot.

So, then what will be this one, this will be a del s del x t minus lambda t and that thing will be bracket because it is a minus sign is here. So, the whole thing is a bracket, so I have written if you just make this one l del l del x dot transpose x dot what is this value that I have written it that quantity. Then this star t is equal to delta t f plus again this term, this equal to zero because our boundary conditions from 5, this is equal to 0.

So, next I am writing similarly, that one will be delta s delta x of t there is differentiation with respect to x dot. If I differentiating with respect to x dot that is x delta of x t this minus delta of t plus term and that you are making star t is equal to t f and delta x f is equal to 0. Now see what is it, this is l and this is the delta l you are doing that differentiation of that one with respect to x dot, we got it that one then again it is a differentiation with x dot, we will got it get it this one.

Now, see this one what simplification we can do it here because now delta s and that I missed it here that your what is called x dot. You see this expression that this is the x dot is there, that x dot is missed here. So, delta s this transpose also this delta x dot this x dot this cancelled. Now this is plus this is minus, so this term this term cancelled, this term this term cancelled, what is left that is h plus delta s dot delta t. This term whole star t is equal to delta t f plus what is left here, delta s dot and delta x of t minus lambda of t, this is the transpose, that lambda transpose this that means star t is equal to t f and l delta x f is equal to 0.

(Refer Slide Time: 35:05)



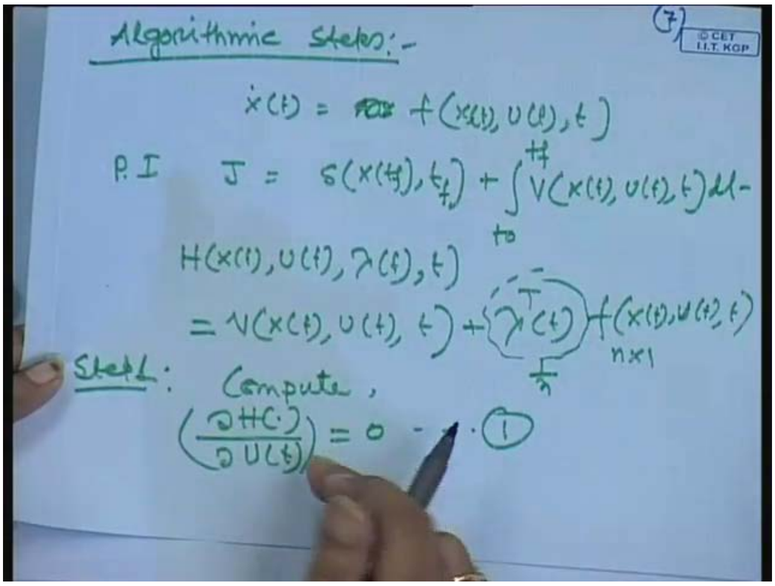
So, our like boundary equation is this, now see what we did it? This is the necessary condition when express in Lagrangian function. And when Lagrangian function is expressed with a what is called Hamiltonian function and some other terms, we replace it l by this and do simplification. We got that condition del h del lambda, del h del lambda is equal to x dot that is what we got it. Then this expression when you replace by Hamiltonian function l expressed then we got it del h del u is equal to 0.

Similarly, the boundary condition what we did it here, when we just replace that expression boundary condition with in terms of Hamiltonian we got it this function that is x dot. So, our simplified form not this expression that is just now we have calculated; this expression in terms of Hamiltonian. So, this is the important boundary condition in terms of Hamiltonian functions.

Now you will summarize this point, so if you see this one, this is the basic 3 equation you need to solve now. This is the equation 7, 8 del h del x equal to minus lambda dot t when the system dynamics is expressed in terms of what is called state space form. And then we can solve this one by using the Hamiltonian function where, the Lagrangian function is expressed in Hamiltonian function form. If you express in that form then del h del x star is equal to x dot what is called x dot lambda dot of t that one expression. Then del h del u is star is equal to 0 then another expression so this dimension is m cross 1, this dimension is n cross 1.

Then next equation what we got it here that, so this is another equation we got it, the del h del lambda is equal to x dot of t, so it is a state equation of this that one can get it from Lagrangian equation of this one, then this is the boundary condition. So, this is the equation number 9, this is the equation number 10, let us call. So, if you solve the equations 7 to 10 then you will get the trajectory of control input as well as which in turn, it will give you the state trajectory path in optimal path of this one, which will minimize the our objective function or performing index.

(Refer Slide Time: 37:55)

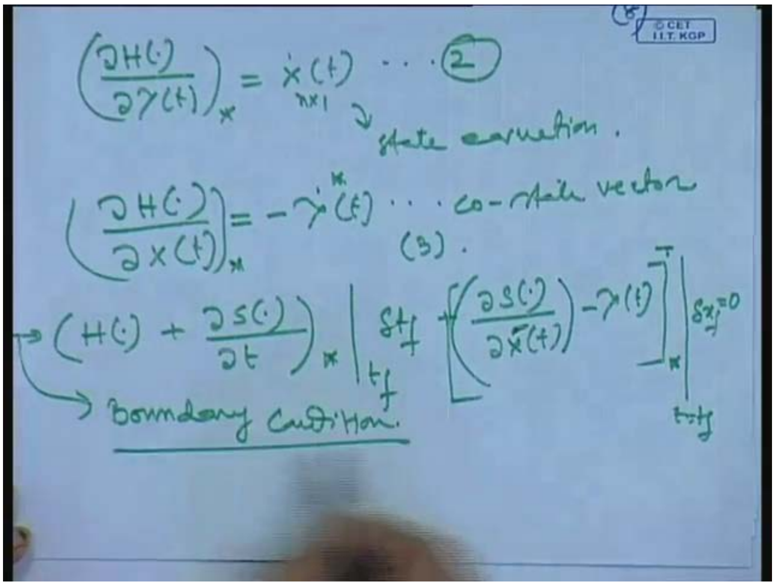


So, we will summarize the results like this way. So, if you see our problem algorithmic steps now, if you see our problem is like this way, given the function or our problem is x dot is equal to plant is given is given x of t u of t and t and performing index is j, the terminal cost is x of f t f t. You can write it x t comma t t f is equal to t is equal to t f this plus t 0 to t f v x of t u of t of t this d t. S, what you have form it you form first step is form the Hamiltonian matrix x of t, u of t, lambda of t and t.

What is the Hamiltonian matrix? This integrant plus lambda transpose of t into f of x of t u of t and t and this dimension n cross 1 and the whole dimension is 1 cross n. So, this is the scalar quantity so this is free from x dot, this Hamiltonian function. Once you know the Hamiltonian function then what is called step is there, step one compute del h del u is assigned to 0, let us call this is equation number. Once I know the Hamiltonian function from the performing index given and the constraint, this is the constraint.

Our problem is finally we have to find out u of u t or u star of t such that this performing index is minimized subject to the equality constraint and that equality constraint may be linear may be non-linear dynamic equations. So, this is the first equation, second equation the state equation that we have written it del h dot del lambda of t. This if you like you can put it because we have to solve it and what is the solution we get that is, the optimal star means optimal solution you get or you can omit this star, it does not matter because you have to solve this equation.

(Refer Slide Time: 40:55)



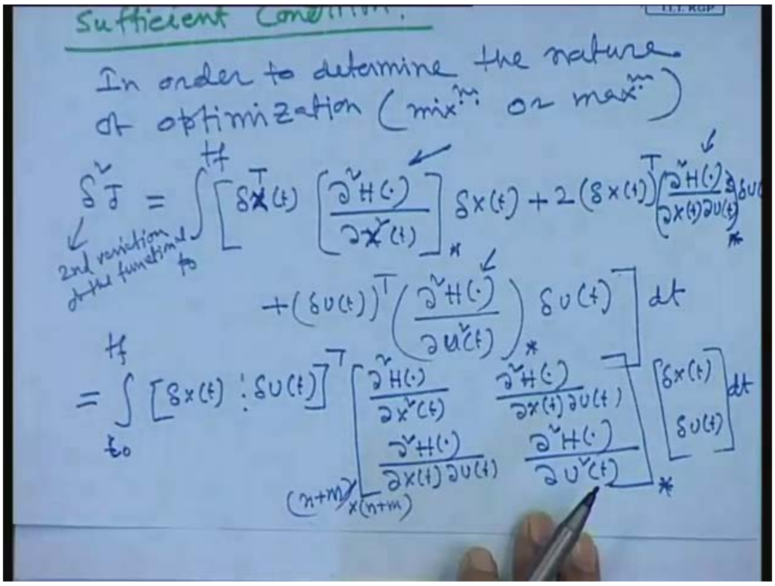
So this equal to x dot of t, so this is also n cross 1 let us call this is equation number 2. And this is the call state equation then third is lambda del h del x partial differentiation with respect to what is called Hamiltonian function? You differentiate with respect to x, which is equal to minus lambda dot star and this is your co state vector. See this one what we got it from the Lagrangian equation vector that expression is the third, so this is the equation number 3.

And then we have to solve this 3 equation by using what is called boundary condition. And our boundary condition is h Hamiltonian function del s, which is a function of x t and del t whole star t is equal to t f delta t f plus del s dot del x, x of t this minus lambda of t whole star t is equal to t f into delta x f is equal to 0. So, this is a vector, so transpose is there that so this transpose you do not forget to give transpose that one, so this is the boundary condition in terms of Hamiltonian function. So, if we just recall once again that given the plant dynamics this x dot is equal to f of x or performing index is that way, this is the terminal cost this is the integrant part of what is called cost function.

Then our first step is find out the Hamiltonian function, that Hamiltonian function is nothing but the integrant part of this integral, this one plus the Lagrangian multiply into f of x that is, we have seen it how we have converted a constant optimization problem into unconstant optimization problem using, what is called Lagrange multiplier technique. Once you find out the like Hamiltonian function then differentiate that partial differentiation of h with respect to u because h is the function of x t u t and lambda t and t. So, next is once you find out then differentiate that h with respect to lambda t that will be a x dot of t that we have derived this one.

Next is differentiating this with respect to x t partial differentiation of h with respect to x t that will become lambda dot of t. So, this will be giving you what is called boundary condition of this one and this is the 4 equation you have to solve simultaneously. So we take a simple example before that question is after solving this one, what is the guarantee that the objective function or what is called your that; performing index will give you the minimum value of the functional or maximum value of the functional.

(Refer Slide Time: 45:42)



So, that is called sufficient condition, so the sufficient condition if you see this one, in order to determine in order to determine the nature of optimization either functional value is minimum or maximum, this is the nature. We must confirm this one, that can be done by using sufficient condition, we have already derived what is the sufficient condition is there in terms of Lagrangian function.

And if you replace that l in terms of h and some other terms, we have shown it if you replace l by Hamiltonian function plus some other terms then you will get it the second variation of the functional this equal to t 0 to t f delta x of t transpose then delta square h of this delta x square capital x. Let us call capital x because we have started with capital x vector and the star. And delta x of t plus twice delta x of t whole transpose then delta square h delta x of t then delta u of t multiplied by delta, that whole thing compute at along the optimal trajectory del u of t then plus del u of t, whole transpose del square l dot del u, u square of t whole this del u of t that whole bracket and that you differentiate with respect to d t.

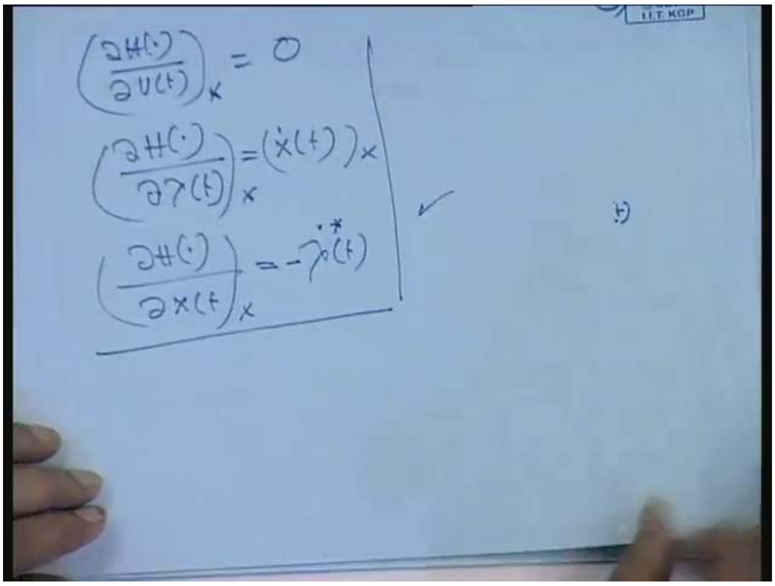
So, we know this one the delta square of this must be positive, if the functional value is minimum, if this value will be negative then the integration value is negative, if the integration value is negative then functional value is maximum. So, this you can write it one can write this thing into a matrix and vector form t 0 t f del x t. And this is what this matrices this is the Hessian matrix, this is symmetric matrix we will get it, so that we can write it now del u of t transpose, this whole thing is a scalar quantity that we can write it in quadratic form.

So, you are writing del h square dot del x square of t del second derivative of h with respect to del x del u, the order can be changed del u del x this does not matter, the results are same. So, del x del u of t then del square h del u square of t that whole thing at along the optimal trajectory u t x star multiplied by delta x t and del u of t this and then differentiating with respect to this.

So, the second variation of functional, if it is a negative that second variation is a scalar quantity, if it is a negative indicates that this integrant part that matrix, whose dimension is this matrix dimension is n plus m into m plus n matrix dimension and that is the symmetric matrix and that matrix must be a. If it is greater than zero that matrix must be a greater than zero means positive definite, if it is a positive definite then we will call that our what is called functional value is minimum.

And at what point along the trajectory u star and x star and that u star solution and x star, this is got once you find out u star x star also. You can find out and that is the optimal trajectory that you obtain by solving equation. Just now we have mentioned it by solving equation that algorithmic steps, if you see that by solving the equation that means a just our equation is del h del u. That means the 3 equation you have to solve it del h, del u is equal to 0 that means if we write del h, del u is equal to 0 then del h, del lambda is equal to x dot of t that you have to solve it.

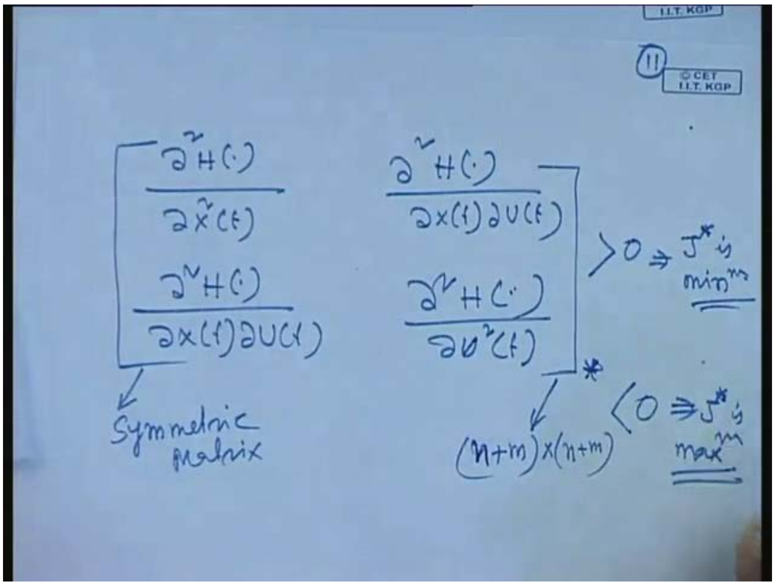
(Refer Slide Time: 52:20)



And del h del x is equal to this one is equal to lambda minus lambda dot star. This is your lambda dot star dot star; you can write this three equation along with that boundary condition. Just now you have written the boundary condition and this boundary condition agree 4 equation number that means equation number 1 del h del u, equation number 2 del h del lambda x dot. And equation number 3 del h del x is equal to minus lambda x dot and equation number 4 is necessary to solve the a set of non-linear equation that boundary condition is required. So, that way we can solve it this one. Once you solve the optimal trajectory in order to check, whether the functional is a optimum means minimum or maximum.

To show this one nature of the optimization that this matrix the Hessian matrix del h del dot del x square t del h dot del x of t, del u of t del square h dot del x of t, del u of t, del square h del x del u square of t. So, this matrix if it is greater than zero means positive definite matrix agrees and this matrix is a symmetry matrix. If it is greater than zero it implies that functional value is minimum, the j star is minimum, functional value is minimum. If it is less than equal to 0, the negative definite this matrix is negative definite along the optimal trajectory, what we got by solving?

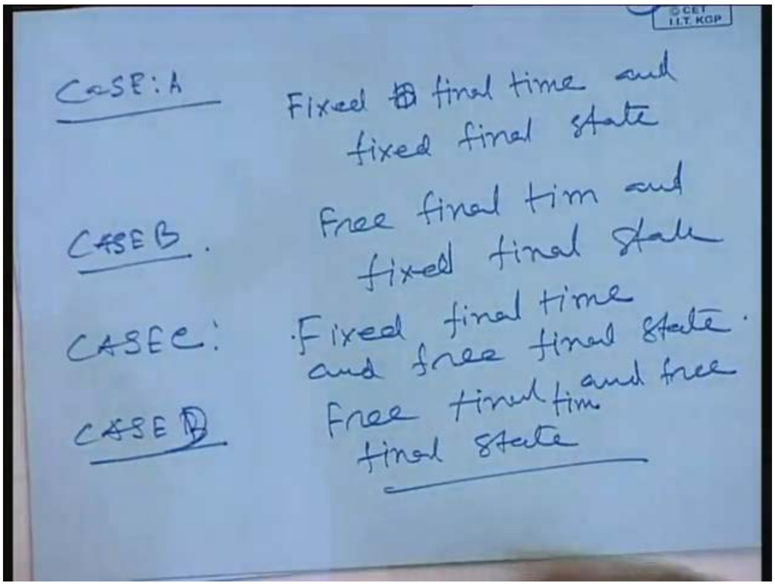
(Refer Slide Time: 54:01)



Just now mentioned equation number 1 2 3 and boundary condition that one and put it here in this matrix. If you get a negative definite this implies that j star functional value is maximum. So, this is the necessary and sufficient condition you have to require to find the trajectory of what is called trajectory of control u of t and x of t.

Once you get it optimal trajectory of u star of t control u star of t and x star of t optimal trajectory of this state then next question is to know the nature of this optimality, whether the objective function or functional value is minimum or maximum. And you have to test with this matrix and that matrix dimension you see n is the number of states m is the number of inputs. So, it is n plus m into n plus m that matrix you have to check it if it is a positive definite j z star is minimum, if it is a negative definite j star is maximum. So, this is the basic theorem of this one.

(Refer Slide Time: 56:32)



So, we will take an example and see that how to solve a practical problem of this one. You can see this one we can just consider the 3 cases, case A as we discussed earlier fixed final time and fixed final state and then case 2 you can see the fixed final time. And you see the case B free final time and fixed final state. Similarly, case c we can say the fixed final time and free final state and last one case D both are free, but free final time final time and free final state.

So, whatever we have considered the boundary conditions all these thing; that is one of this cases will be a special case of that general boundary condition. So, we will stop it here next class we will just take an example and show how to solve this problem by using the Hamiltonian function method to get the optimal value of the functional, to solve the control problems. So I will stop it here.