Optimal Control
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Lecture - 28
Solution of Nonlinear Programming Problem using Interior Penalty Function Method (Contd.)

The last class we have discussed how to solve the nonlinear programming problem optimization problem using the interior point method. That means what are the algorithm states are involved here, first you take a point inside the feasible region, that is we call the interior point.

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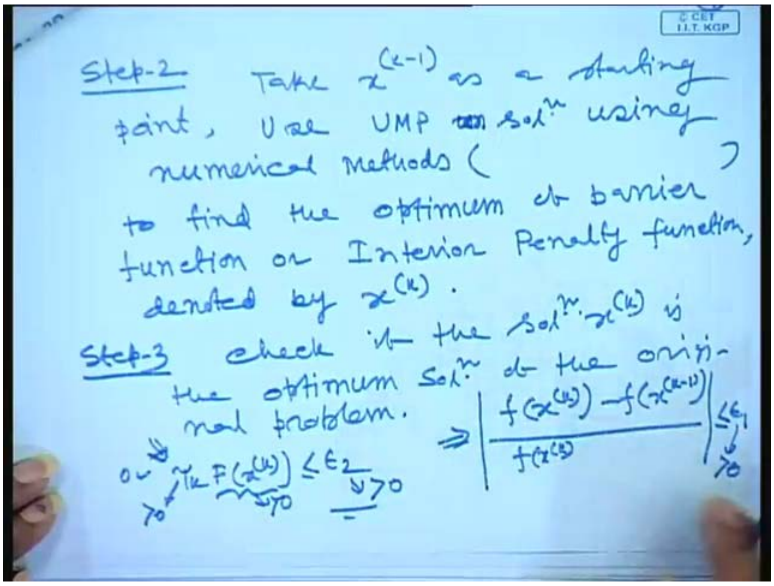

Then, you write what is called the necessary condition and this necessary condition whatever you will get it depends on the number on decision variable involves in the optimization problem. If the decision variables are n, then we have a n such type of equation necessary condition equation set of equation which may be in linear or nonlinear depending upon the objective function and the constraints. Next step is you have up to solve these set of this equation which is obtained from the necessary condition by either analytically or iterative method, or you have to solve by numerical methods that we have mentioned.

If you solve in analytical method, express variables, and if you want to express the variables x 1, x 2 in terms of the penalty coefficient it is very tax for higher decision variable and number of decision variables. So, after solving this one you check the what is called optimality condition, this is the optimality condition if the function value does not change. It is less than the epsilon, then you can stop or you check tau k f of k is less than equal to epsilon tau k is greater than 0 and f of x k is also greater than 0.

The way we have defined f k, once this is satisfied, you can stop that means we have reached to the optimal solution using the interior point method. That means it is approaching to the boundaries of the problems feasible region which is near to the boundary of these problems. Next, if you does not satisfy this condition, then go for next iteration by assigning the penalty coefficient tau k plus 1 is equal to c into t k and c is less than 1.

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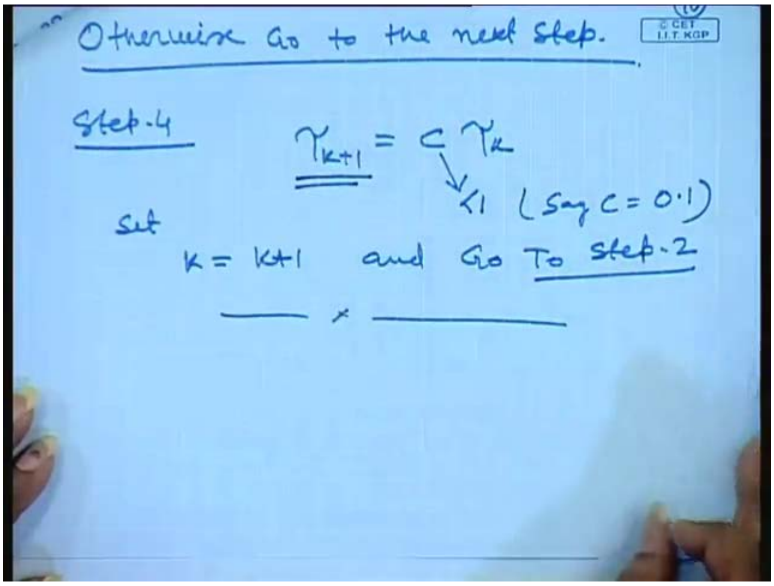

If the initial value of c is selected, then you multiply by for say 0.1, you reduce the penalty coefficient in turn you are approaching towards the boundary of the feasible region. Then you interment the sign k u k plus 1 and go to the step 2 and repeat the process to obtain the optimal solution of nonlinear programming problem using interior point method. Let us take one example and see how one can solve this using or adapting this procedure, so our problem is solved example.

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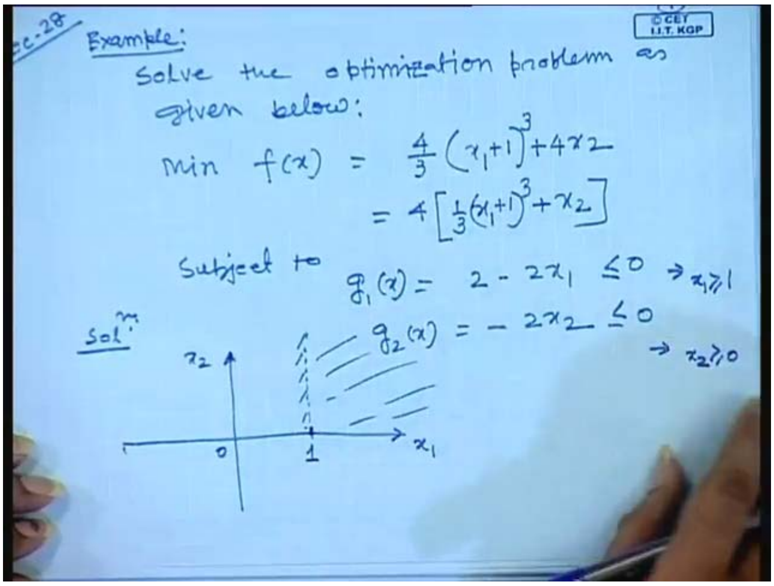

Solve the optimization problem as given below minimize f of x which is 4 by 3 x 1 plus 1 whole cube plus 4 x 2 which you can write it 4. Then you can write it this, what is called 4, if you take it common, then it is one-third x 1 plus 1 whole cube plus x 2 is equal to this. This is our optimum minimum value of this minimum, we have to optimize this one subject to g 1 of x, we have a 2 minus 2 x 1 which is less than or equal to 0 and g 2 of x is equal to minus 2 x 3 is less than or equal to 0. So, from this equation we can say the x 2, this indicates the feasibility x 2 is greater than equal to 0 and x 1 here from this equation you can say that x 1 is greater than equal to 1.

So, this is our that region feasible region of this one, if you plot it this one, the solution is solution you can say if you plot this one that our x 1 is in this direction and x 2. If you write it in these directions, then our feasible region is that part of this one, here is the feasible region of this thing. That means x 2 is greater than 0 the upper part and x 1 is greater than u 1 greater than equal to 1 because this is the feasible region.

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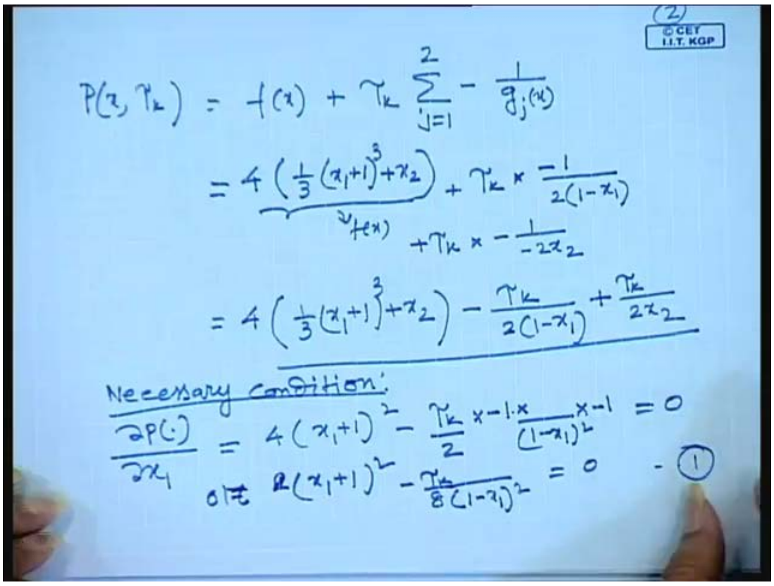

Now, see the values of this this, now we according to this you find out the solution is penalty function x tau of k, what is this the objective function plus tau k summation of j is equal to this. There are inequality conditions two inequality conditions are there at j of x this, so this we can write for our example this one it is a x 1 third x 1 plus 1 whole cube plus x 2. This is our f of x is nothing but our f of x plus tau k i is j is equal to 1 j is equal to 1 n is minus 1 and g j is equal to 1. This value is 2 minus 2 x k, so it will be a 2 common if you take minus 2 x 1 means x 1, this plus tau k into minus g 2 of x g 2 of x is equal to our g 2 of x is minus 2 x 1.

So, minus 2 x 2, this is g 2 x 2 2 x 2, so ultimately it will come like this way 4 that one third x 1 plus one whole cube plus x 2 bracket closed minus tau k 2 1 minus x 1, then minus plus this. So, this will be a plus agree plus then tau k divided by 2 x 2, so this thing is our penalty function for this one and tau k is the our penalty coefficients. So, our when mean inequality constants is there, it is obvious that if you have a equality constants there, we cannot directly apply this here like this one in this movement.

So, let us see that one how to solve that quantity now our necessary conditions, so del p with respect to dell x 1 if you differentiate this with a dell x 1, 3 x plus 1 plus 1 3. So, 3, 3 cancel this is 4 x 1 plus 1 whole square, then x 2 will not come into the picture, then x one is involved here that minus tau k then if you go up it will be a minus one into that tau k by 2 is constant a began minus 1. Then it will come 1 minus x 1 whole square than differentiation of that minus 1 minus x 1 is minus 1.

So, ultimately it is coming is 4 if you consider that this equal to 0, so I can write this one x 1 plus 1 whole square, I divided both sides by 4 minus tau k tau k by 8. This minus plus, so this into 1 minus x 1 whole square is equal to 0 or this one, so that is the equation we got. Let us call this equation is 1, then next equation because we have a 2 decision variables are there that next equation Del p of this necessary.

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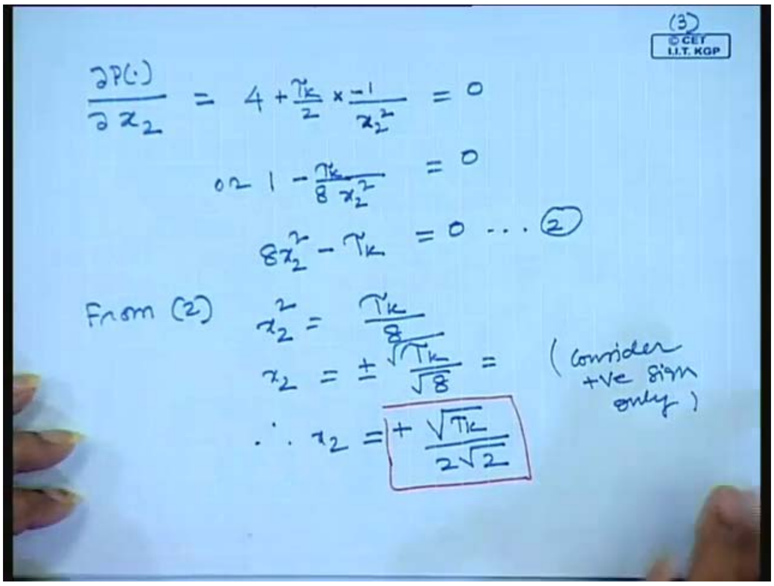

Next necessary condition is x 2 is equal to your are differentiating this with respect x 2, so there only one x 2 is here another x 2 is here, so this will be a 4 minus tau. Now, let us see this one, 4 then x 2 is here tau k by 2 tau k by 2 if you divide and then x 2 1 by x 2 differentiation plus tau k by 2 1 by x 2 is differentiation is minus 1 x 2 square x 2 square that equal to 0 or again divided by 4 both sides.

So, it will be a 1 that is minus tau k by 8 x 2 square is equal to 0, this is tau k, so if I multiply by both sides 8 x square, then it is x 2 square minus tau k is equal to 0. So, this is let us call equation number 2, so from 2, one can write it x 2 from 2, one can write it x 2 square is equal 2 tau k by 8. So, x 2 is equal to plus minus tau k y 8, so this is the solution, now question comes here which sign that means this is root 8 and root a.

So, this is which sign this x 2 is plus minus root over tau k by root 8, then which sign you will consider we have seen if you just see the problem of our original problem our x 2 is greater than 0. That is what we have seen x 2 greater than 0, so you want to make x 2 is greater than 0, 8 is a positive quantity tau k is an also positive quantity. So, we have to consider the positive sign of that one, so consider positive sign only and this is we come conclusion from the what is called feasible region is given x 2 is given x 2 is greater than 0 on this one. Therefore, considering the x 2 is equal to plus tau k square root of tau k divided by 2 root 2, this is our that one, now we can get it the x 1 value from equation 1. If you see this equation 1, now that one from equation 1 is this a 0, so x 1 plus 1 whole square is equal to I can write it this from equation 1.

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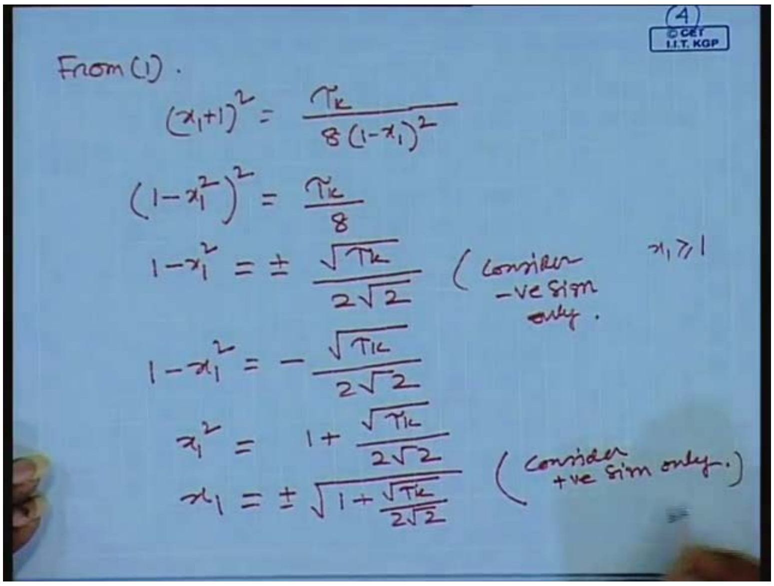

I can write it x 1 plus 1 whole square is equal to tau k 8, 1 minus x 1 whole square, now if you just do it 1 1 plus a plus b a minus b, so it is a coming this and this will come a x 1 square whole square is equal to tau k by 8. So, 1 minus x 1 square is equal to plus minus root tau k divided by 2 root 2, now look at this one which sign, I will consider now this 1 if x 1 is less than 1 if let us call x one value is less than 1. This quantity will be positive if x 1 is greater than 1, then this quantity this quantity will be negative, but our feasible region is x 1 if you see the our statement x 1 is greater than equal to 1.

So, this quantity will be a negative for that feasible region, so we will consider negative sign only, so our 1 minus x 1 square is equal to minus square root of tau k divided by 2 root 2. So, from there I can easily find out x 1 square is equal to 1 plus tau k divided by 2 root 2 and x 1 is equal to once again this quantity is tau k is positive.

This is positive 1 plus tau k is also positive, so this will be a plus minus square root of 1 plus square root of tau k divided by 2 root k. This whole quantity is positive quantity because of tau k is positive quantity, now once again what value of plus will consider or minus will consider naturally we have considered plus because x 1 value is greater than or equal to 1. Since, this is positive if we consider negative that x 1 value is coming negative, so consider what is called positive sign only based on the our feasible region of this one, so our solution now x 1 is coming.

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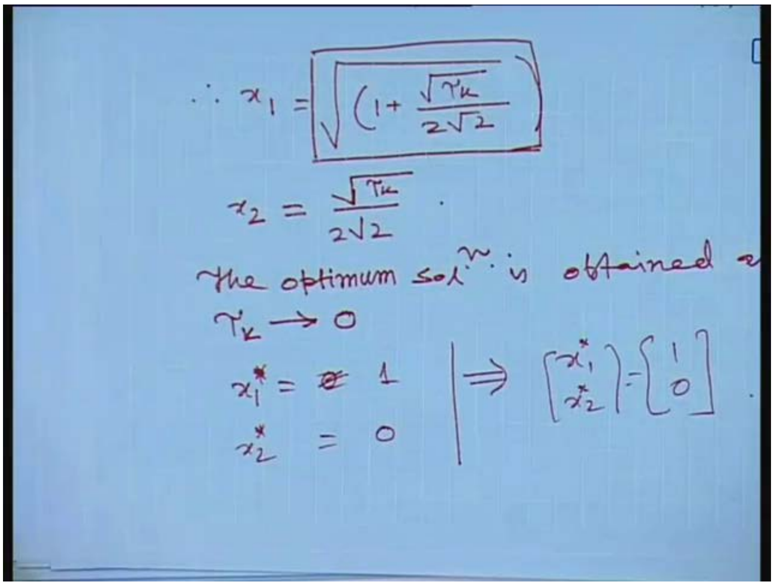

Now if you see therefore, our x 1 is square root of bracket 1 plus square root of tau k divided by 2 root 2. So, this is x 2 and we have already got x 1, we have already got what is called x 1 value is if you recollect this one, I t is 1 by 2 root 2 square root of tau k this is the thing. Now, we have seen there that that if you penalty coefficient if you start from the high value and then decreasing this value to this it is approaching to the boundary of our called optimization problem, we are approaching there.

Now, to get the optimal solution optimum to get the optimum solution is obtained by setting or by setting x tau k tends to 0. So, if tau k tends to 0, this is 0 of this 1 x 1 is 0 and this is 0 means x 2 is 1, sorry this is 0, 1 is there. So, this will be one, so this is a 0, so our optimum solution is x 1 x 2 star is 0 and if you want to find out and corresponding objective value function one can find this one. So, this is analytical approach we have solved, but I have told you once again if you number of what is called the variables are n, then you have a n number of what is called that necessary condition.

So, you have a n number of necessary conditions, but to express s x 1 x 2 x 3 in terms of that what is called the penalty coefficient it is almost what is called tough job. You cannot express it how we can solve this problem what we have set of equations you have got it from the necessary conditions. It can solve our standard numerical methods solving the set of equations optimization problems by numerical methods and that we have san seen you can solve it by new tons absent method that is a conjugate gradient method.

Then, you steepest method and then modified new tons absence method all these thing you can apply to solve a set of nonlinear equations to get the optimum value of the function. So, this is let us call once you got the expression with a iterated this is way than finally, we got it x tau k tends to infinity 0 this when exterior point we have start from the low value of the interior point.

That means, we are starting from a far way function it is approaching to the boundary of this one when tau k tends to infinity. But here is the interior point is different both the methods from the exterior point we take the point from the outside the feasible region, and it gives a sequence of solution and ultimately we are approaching the optimal solution of the problems. Whereas, an interior point method we take a point in the inside the feasible region and ultimately a sequence of iteration. It will give you what is called optimal solution of the non-linear application problem that means, we are approaching to the boundary of the feasibility chain this one.

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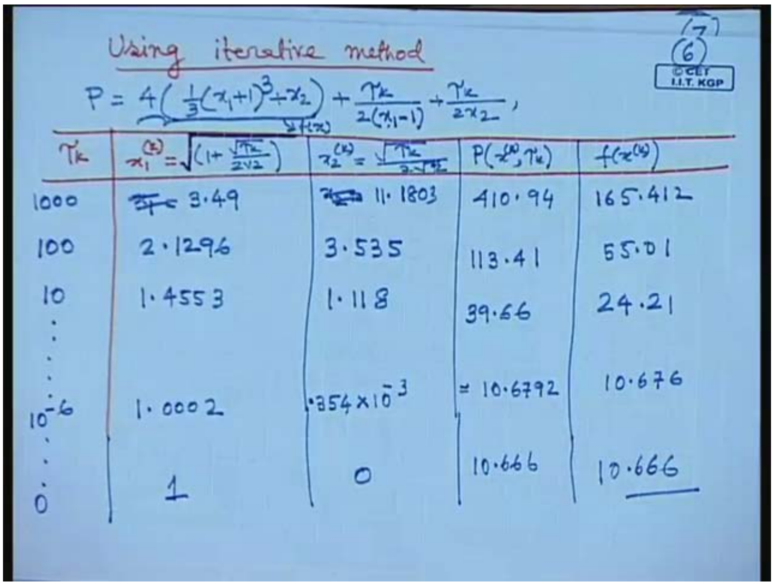

Let us solve this thing is in using iterating method similar as x is a point using iterative method you know the x one x two expression in terms of what is called the penalty coefficients. So, that table you form it like this way table tau k then x 1 superscript tau k is equal, now we have computed that expression one plus square root of tau k divided by 2 root 2 bracket. Then you find out x 2 of k that will be a tau k divided by 2 root 2 is x 2, then you find out the penalty function value at each iteration that one, then you find out the objective function value as given in the problem.

So, what is p, I am writing p is if you refer to our penalty earlier pages, then you will see four that is one-third x 1 plus 1 whole cube plus x 2 that is our p that is f of x. Then tau k divided by that x 2, see this one p what we have written p that p is written here. You see this minus that one that is coming minus tau k 1 minus x k, so it is that you can write it tau k x 1 minus this is our problem is this one these things.

So, that will come tau k that means x 1 2 x 1 minus 1 minus plus tau k 2 x 2, see this one what is this it is actually minus, I just change this order x 1 minus x 2 minus plus, I will change the order x 1 minus 1 this and f of x value you know already this is our f of x. This is our f of x this value, so let us call we started with a high value of tau k 1 1 and immediately I can find out the value of this because tau k value is 1,000. This is known, so x 1 value is coming that 1,000 means it is a 10 root 5 all these, so this value will come 3.49 values and this value square root of 1,000, 2 by 3.

That will be become x 2 values will come 11.1803 we will just omit x 2 here because x 2 under this column only. So, x 1 value is this one and x 2 value is this one, now immediately we can find out the f x this is our f x. So, this f x values is coming 165.412 put the value of x 1 2 in this in this expression that is our f x if you see that is our f x and you will get this only once you get this one f c you add these two terms with the f x tau k. You know x 1 you know these three points 4, 9 x 2, you know, so you can find out and that you will give you the 410.94. So, next what will go this is our and you see if you see our value of f of x is that one.

Now, you reduce the value of penalty coefficient by let us call one-tenth, so it will be a 100, put the value of 100 is this divided by 2 root 2 plus 1 square root of that one. That will come 2.1296, similarly x 2 put the value of lambda tau k 100, then you get this value is 3.535. Once you know the x 1 x 2, the f x value is that 1 this value will come we will get it 55.101 and similarly p you can calculate 1,100 and 13.41.

In general I told you if you are number of variables are n you have a n sets of what is called necessary condition and that x 1 x 2 all decision variables. You cannot express in terms of our penalty coefficient and this method you cannot apply either iterative method or that analytical method this one. What we have to do once, you get it the set of equation necessary conditions of set of equation on the necessary condition that you solve numerically and get the results for the optimal value of the function.

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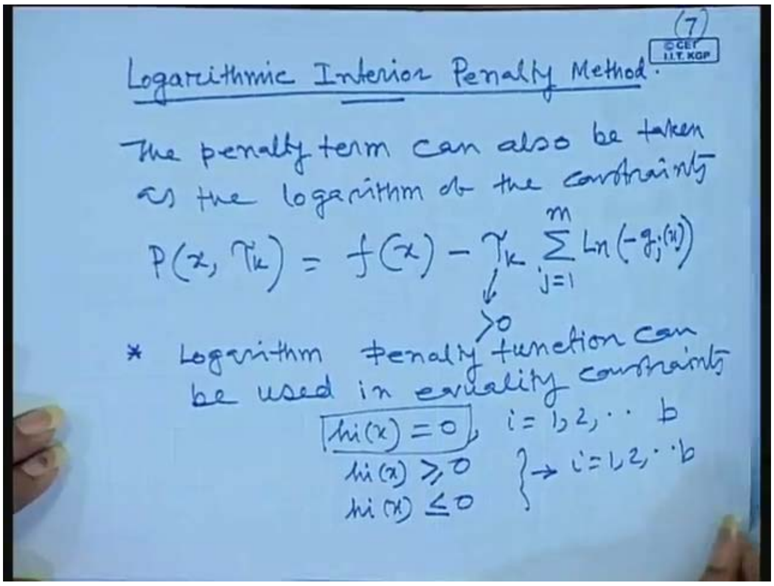

So, this is about that iteration and there is another method is there what is called record is what is call logarithmic interior penalty. So, if you have a now in more general this logarithm penalty function structure is like this one the penalty term can also be taken is same as algorithm method 1 1 by sum mission of residual of one by g i minus. Only we are expressing in terms of logarithm that as the logarithm of the constraints, so our penalty function is like x and then tau k and then f of x minus tau k. Then you summation of j is equal to 1 to m we have a m such in equality conditions minus of g j of x this.

So, this inequality constraints I can convert into a 2 inequality constraints of x another is this and both are I is equal to 1, 2, 3. So, inequality constraints you convert into this and then this multiplied by minus 1 both sides it will greater than or equal, so one you write it in place of that one you write it in terms of h i is greater than or equal to 0 minus h i of x is a again greater than or equal to 0.

So, replace equal to sign by this then apply logarithm penalty function method this. So, let us see this one directly, you cannot used the h of this here one just like a exterior point method you cannot use directly this one. So, let us see what the term associate to you what the meaning of that term penalty term of that one you just investigate this term.

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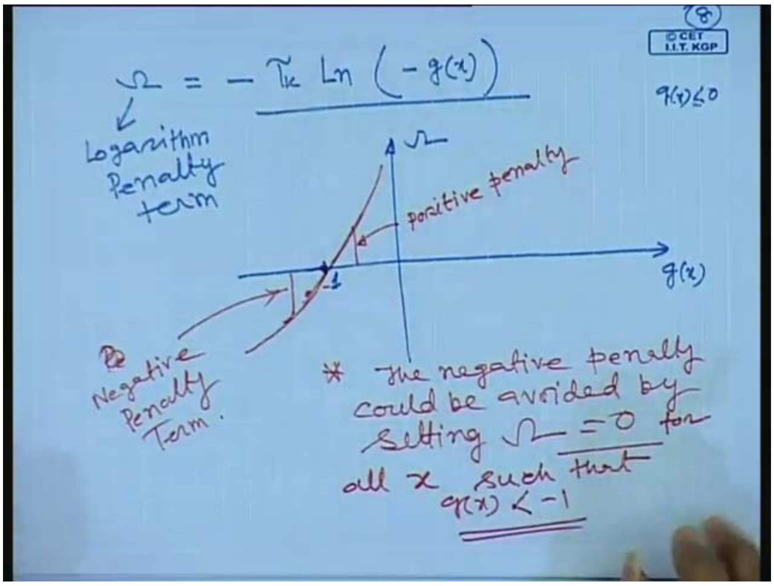

Let us see this whole term we will investigate so that whole term, let us denote is this is this term is a logarithm penalty function term that is equal to minus tau k l n minus g of x k this this is that part. Only I have denoted by capital omega this part, let us investigate this part, so I am plotting this one in this direction g k and in this direction is omega e plotting. So, this function if you see when g k value is when g k g g of x value is constraint is g of x is less than equal to 0.

So, let us call g of x value is minus 1 minus 1 minus plus then this value is 0. So, at one this function value is here, then further if it is less than 1 less than minus 1 minus 2, so this will be what minus 2. Now, let us see then how to solve the problem by using the logarithm penalty function methods using the interior point interior logarithmic interior point method interior point penalty function method.

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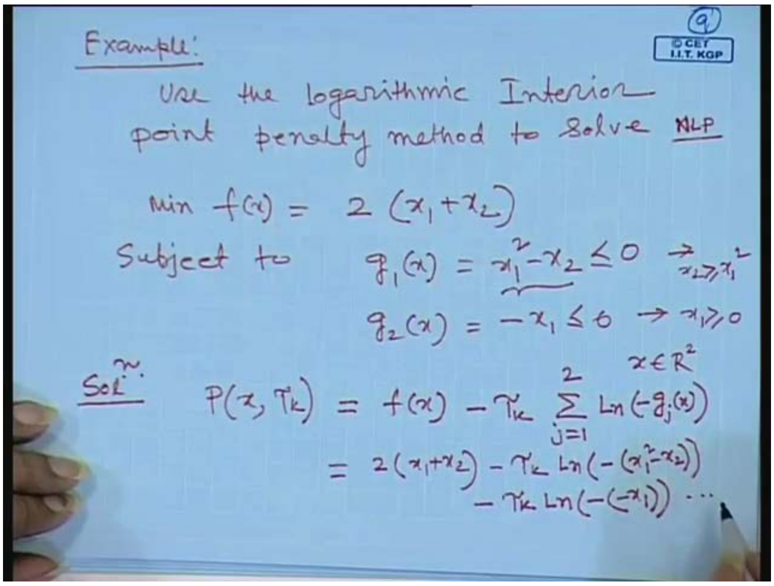

So, example use the logarithmic interior point interior point penalty function method to solve to solve non-linear optimization problems to solve non-linear and l p problem NLP problem. So, this example is we have taken minimize f of x is equal to twice x 1 plus x 2 this objective function is linear whereas the constraints subject to the constraints g 1 of x is equal to x 1 square minus x 2 is less than or equal to 0. Our g 2 of x is equal to minus x 1 is less than or equal to 0, so this shows first this shows that x 1 is greater than 0 feasible region x 1 is greater than 0 and once x 1 is greater is greater than 0. This shows that x 2 is both side you multiply it by minus 1 minus x 2 plus x 1 is greater than or equal to 0.

So, minus you take it, so x 2 is greater than x 1 square since x 1 is greater than 0, similarly x 2 also will be implies greater than 0. So, that is our and x as the dimension real and which dimension is 2 cross 1 two decision variables are there. Then if you want to solve by what is called logarithm method interior point method, then first what we have do exactly in same manner you generate or from a penalty function for the given problem which you are going to solve by interior point method. That means you are considering a feasible point inside the region and that feasible point is by interior point and starting from this one point. You are now solving the set of equation which is form which are form from the necessary conditions of the problem.

So, this penalty function I am writing is f of x plus here is minus tau k summation, now j is equal is 1 to 2 l n minus j g j of x whole this 1. So, f x you know this 1 2 is x 1 plus x 2 minus tau k l n l n then minus sign is this 1 g k g 1 of x. this 1 u 1 is what x 1 minus x 2 bracket closed. Then minus j is equal to 2 l n minus is minus x j is equal to 2 is minus x 1 bracket closed this is our penalty function.

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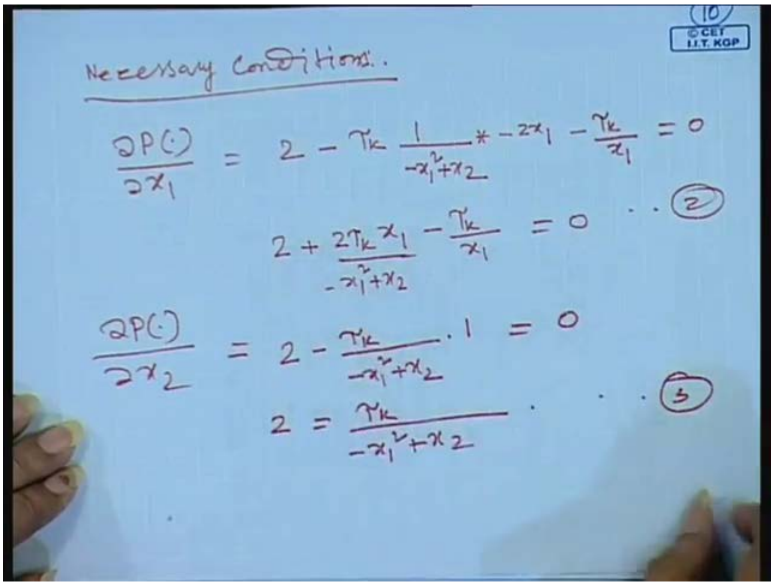

So, let us call this is equation number one or you can say that one if you can rewrite that that one if you rewrite this one or say instead of rewriting this one write equation one then our necessary condition for this one necessary condition. So, del p dell x 1, now see this one it expect to this x 1, so there is two is equal to 2 and x 1 is involved here x 1 is involved here and x 1 is involved here there is minus plus this. So, if you differentiate with respect to this one tau k divided by 1 by x k minus s in, so third second what you will write it see this one second term minus tau k as it is minus tau k.

Then, this one is reciprocal of that one reciprocal of that one differentiation of this one means minus x 1 square plus x 2 and that differentiate the differentiation of that one is equal to into minus twice x 1. So, this minus plus and there is another term is here if the difference is with respect to this minus plus this 1. So, divided by this one, so it is minus x 1 plus x 2 into differentiation of x 2 is 1, so this equal to 0, now you have 2 is equal to 2 tau k divided by minus x 1 square plus x 2, let us call this as equation number 3. So, from 2 one can solve the equation for x 1 because see from here, so you see tau k minus x 1 square plus x 2 tau k divided by x 1 square plus x 2, this question I can replace by 2 this equal to I can replace by 2.

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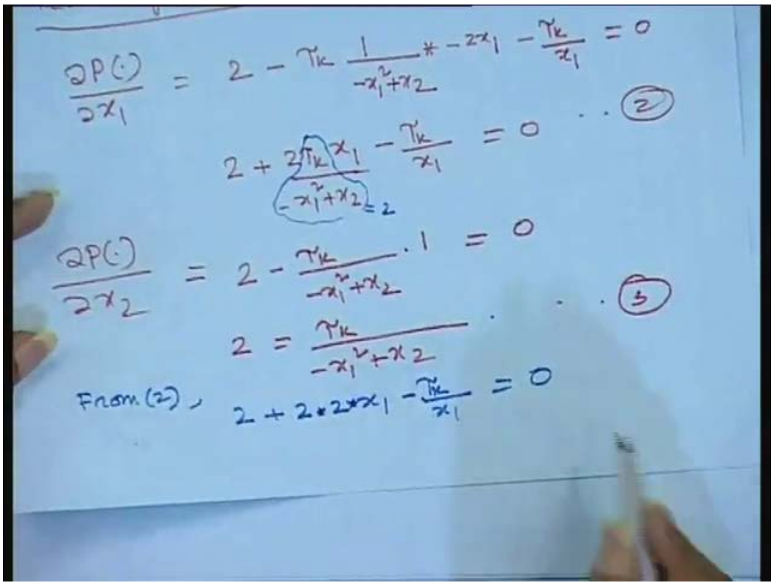

So, if you replace from two now from 2, so 2 plus 2 into 2 into 2 into x 1 into 1 minus tau k divided by x 1 is equal to 0. So, if you solve this one this equation because this is now a value x 1 you can express in terms of tau k. So, now our solution of x k, I am skipping this one you will get a quadratic equation like this.

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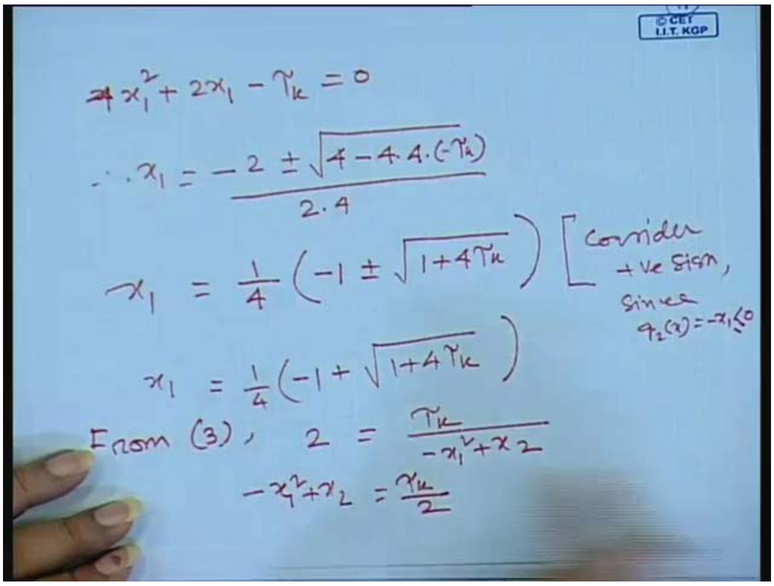

Therefore, x one is equal to minus b minus plus minus root to the power b square minus four a c c is minus tau k divided by twice a. So, ultimately we will get this one after simplification we will get it this 1 minus 1 plus minus root to the power 1 plus 4 tau k. Now, tell me this is the x 1 what value of x 1, I will select that if you recollect that our original problem we have considered here that this. So, x 1 is greater than 0, so if you consider minus sign this quantity is greater than 1, so it will be a negative. So, 1 is to consider this minus 1, but this is greater than 1, so you have to consider positive sign, so that our x square plus x 2, so it is you will get it minus x 1 square plus x 2 is equal to tau k by 2, now I can if you take it x 1 square that side.

It will be x 2 is equal to x 2 is equal to x 1 square plus tau k by 2 x 1 square, now you got it the x 1 x 1 is this one the square of that one. So, it will be a 16, 1 by 6 is equal to minus 1 plus root to the power 1 plus 4 tau k whole square plus tau k by 2. So, again now you put it tau k tends to 0 you have to get the optimal solution of this problem, so tau k tends to 0, we get x 1 star is equal to x 1.

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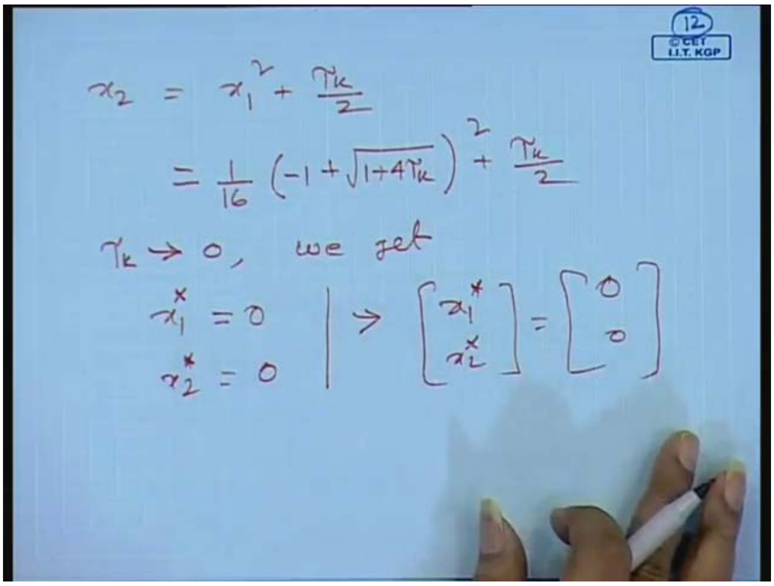

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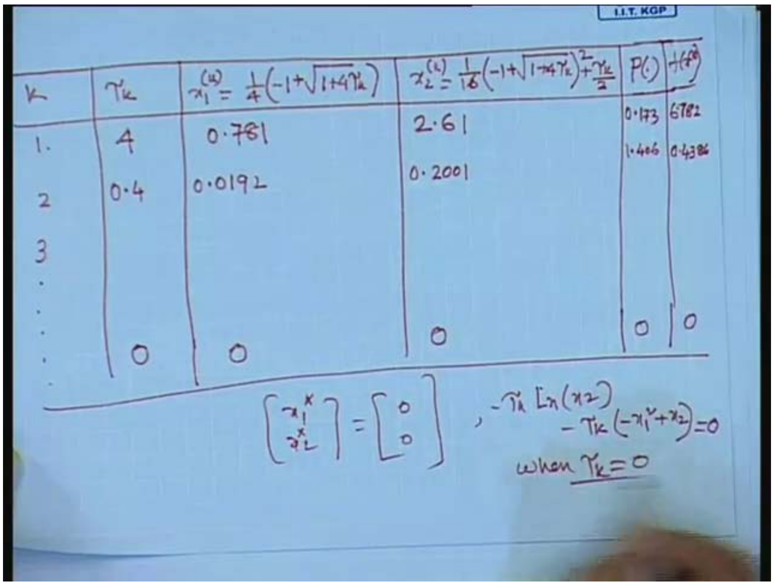

You see here from expression this x 1 tau k is 0 that means it will be 1 minus 1 plus 1 0, so that will be a 0 and x 2 star will be see x 2 star 0 this is 0. This is 0 and this is 0, so it will be 1 1 minus 1 0, so this is a 0, so our optimum solution for this one x 1 star x 2 star the optimum value of this one is equal to 0 0 and the corresponding things you can find out the value of and same thing. You can solve it by using the iterative method the same problem you can solve by using iterative method.

When this is problem when tau k u assign to 0 that, so we will stop it here, so next class we will just consider the following. So far, we have discussed the single variable optimizations problems, but in real fact is an application you will see the number of what is called objective functions may be more than 1, 2 or more than 2. So, in that situation how to obtain the optimal solution of a multi objective optimization problems, how to solve the for a nonlinear and linear optimizations problems, so that we will discuss next class.