Optimal Control
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Lecture - 26
Solution of Nonlinear Programming Problem Using
Exterior Penalty Function Method (Contd.)

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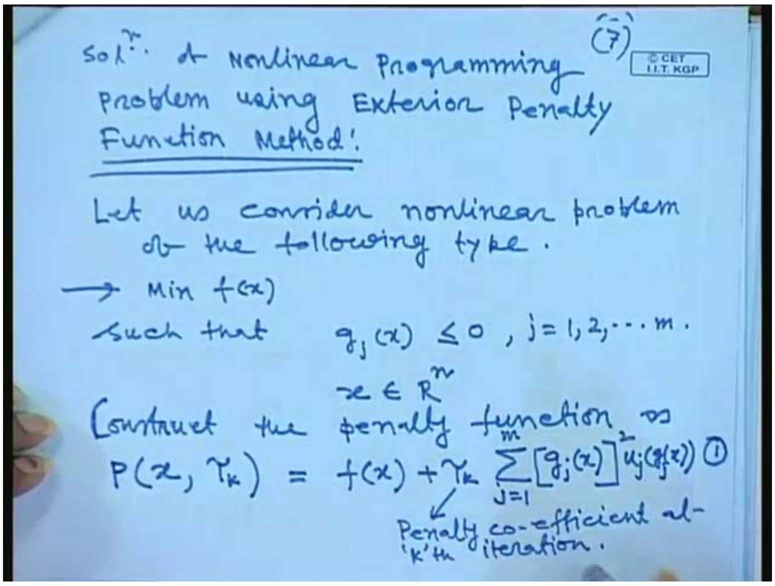

So, last class that how to solve the non-linear programming problem using exterior penalty method. So, our problem statement is like this way, minimize this function f of x, f of x is a non-linear function. And it can be a constant g of x will be also non-linear so our job is to solve this problem using the exterior penalty function method. So, the constant first the penalty function as like this way that means, whatever the constraints are there that constant function that square, you have to do the square and multiplied by a penalty coefficients this one then add with the objective function.

And this is the penalty function at the term associate this one is called the penalty term and this and this tau of k is called the penalty coefficients that this term mu j g j this value this values will be either 1 or 0. If the constraints are that g i of j if these constraints does this phi in the other sense that if the point is on the feasible region, then this value of mu j of this value will be 0. If the constant r outside the feasible region when a infeasible region the value of mu j g is 1, so that is what we have discussed last class this one.

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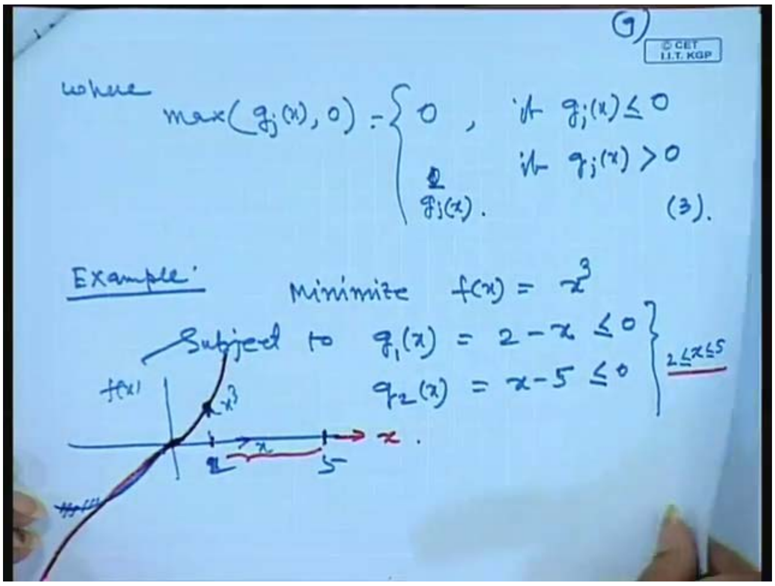

Let us take an example and explain how this problem can be solved by using the exterior penalty function methods. So, first see that what is called the penalty coefficients value, how to select the penalty coefficient values. So, let us call this is the non-linear equation, we have to minimize this one and subject to g 1 x is equal to 2 minus x less then or equal to 0 another is x minus 5 is less than or equal to 0. This 2 inequality constraint then we turn into this form that x from here you can see both side you multiplied by minus 1. That means minus 1 minus 2 plus x is greater than or equal to 0, so x is greater than 2 from here x is both side you multiply by x, you see x is less than 5 that is this so these 2 constraints equivalent to this.

If it is given one can write into 2 constraints x 2 minus x less than or equal to 0, take x in this side and x minus 5 is less than or equal to 0 so if you plug this one it is a cubic function like this way, that x is in increasing x is 0 constant value is 0 and x is increasing positive relation this function is strictly increasing of the power of x cube. Similarly, here also this function will be increasing like this way because x is equal to negative, the function value is negative is involved. You can say it is odd function of this one, so this is our in x in iteration our feasible region if you see lies between 2 to 5, this is our fusible region of that one. So, this that is what we discussed last class so we can equivalently.

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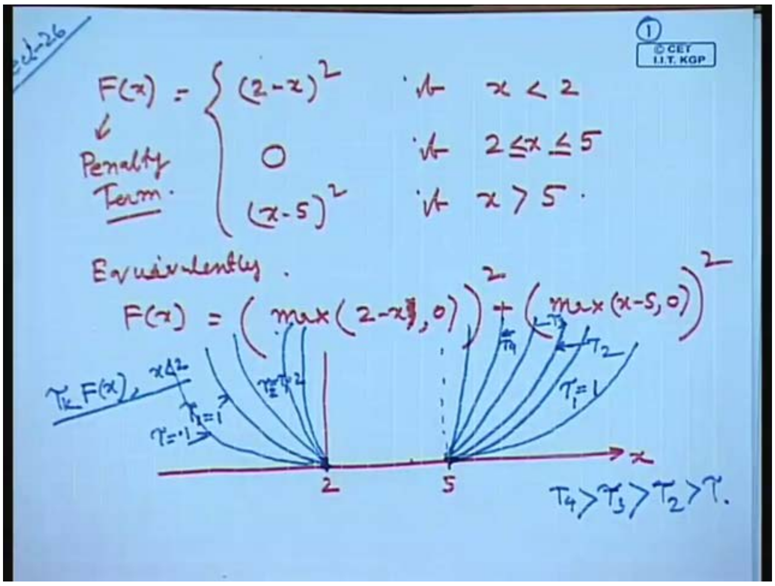

You can write it that the f of x is equal to if you see what we have defined f of x is the penalty term, this term is this term is the penalty term f of x. So, that one I am just writing it, now so f of x write it this equal to 2 minus x hole square if x is less than 2. That means, if x value is not in the feasible range than this belongs to this that one, then if is in the feasible region that f of value that means, when x is less than or equal to 2 and greater than or equal to 2 less than or equal to 5 that function value is 0, when it is because it is a exterior point penalty function method. If you see by the things.

So, when it is when this is greater than 0 means it is not in the feasible region that this m of x is g of x more specifically, if you see this expression that will be because when g of x is greater than 0, then maximum of this one maximum of this one means, which one that g of x because g of x is positive quantity now so, that we have writing. In other region we can write it x minus 5 hole square, if x is greater than 5. So, this f of x function which is called penalty term penalty term can be region in infeasible region is nothing but a function square g 1 this is another infeasible region this is g 2 square, but in feasible region f of x the term value is 0.

So, this mathematical this 2 expression equivalently f of x is equal to max 2 minus x sorry 2 minus x coma 0, this whole square plus max x minus 5, 0 hole square. Now, you see when it is in the region see here when x is less then this quantity positive maximum of this one you can take, but whereas, this one x less than 2, but this quantity is negative maximum of this one is 0 this is not coming into the picture, when x is less than this. similarly, when is x is greater than this, this will come into the picture when this part will not come into the picture because this quantity negative, maximum of this one will be 0, this only contribution penalty term condition is coming from this part.

So, if you plot this curve now you see let us see this our if you look this one our feasible region is 2 to 5 and this is our x, 2 to 5 is infeasible region. Then when it is x is less than 2 is in this region, when it is less than 2 in this region then this value is what? You see this value is this square, square term, so this value when x is equal to this value is what when x is less than 2 let us call 1.9 something that slowly, it will increase and it is squarely it will go like this way because it is x minus 2 2 minus x whole square like this way.

And let us call this function it is multiplied by tau, so if this tau value is 1, tau is one then this is the expression. Now, this tau value if I increase it then what will happen this is the same function increase means, actually I am plotting it now tau of k F of x of k in the region x less than 2, I am plotting this one so tau of k I am considering one. Now, tau of k i am increased by 2 let us call this function value will be same only it is multiplied by 2 point by point. So, it will be like this way this is tau 2 is equal to tau 1 is equal to 2.

Similarly, if you go on increasing it will be like this way and ultimately, when tau is infinity this will come to a picture at this point here about this will be a infinite values of this one. So, it will be a like this way now similarly, x is equal to when it is 5, x is equal to 5, this is 0 this is 0 that means this when x is greater than 5. That means when it is not in the feasible region similarly, it will go like this way, this is tau is equal to 1 and then you increase the value of tau, increase the value of tau, and this way.

And if you consider this is tau 1, this is tau 2, this is tau 3, this is tau 4 then you can say that tau 1, tau 4 is greater than tau 3 is greater than tau 2 and greater than tau 1, then this is the tau. That means, when tau is in infinity then it becomes a what is called just a this value very stiff value of function value will come this way. So, if you see this one when tau value is let me tau is less than one it will be like this way. Similarly, here tau is less than one it will be like this way, so when you if you start the tau value very small value an approaching to increase in the value and approaching to infinity, then we are approaching from infeasible region to feasible region.

Similarly, in this case when this is the tau is let us call is 0.1 and tau is further increasing that means, we are from away from the feasible region. And if you increase the tau value you are approaching to the feasible value boundary of the feasible. So, in our iteration what will do tau value we start from a small value, which is far away from the boundary layer boundary region and the next iteration if we increase the tau value. That means penalty coefficient value increase it in approach to the infinity in other sense, it tells we are approaching from infeasible region to feasible region and we are getting and then we are getting the optimal solution of that one.

In our example if you see this example it is straight forward, what is the minimum value of the function in this region in this region if you see that one in this region, what is the minimum value of this function just here. So, if you see this one what is the minimum value of the function at this in this region at x is equal to 2, we will get minimum value of this function x is equal to 2 because of this region is 2 to 5, you increase the value x in this region function value is decreasing it is obvious from the figure.

Also you see you are approaching the values of tau from low value to increasing you are approaching to the optimum value of this functions and at what time we will get the optimum value of this function that we are getting. So, let us work out some problems, so before that I will just see because when you solve this problem by analytically, we have to first find out the necessary condition, what is the necessary condition for the non-linear problem and then you find out the solution of this necessary conditions. A set of equation you will get it and from there you have to solve the values of that decision variables.

So, from the figure it is clear that F of x is really differentiable using chain rule. Now, see this one this is the continuous function this one, so this function is differentiable so if you differentiate F dot of x this, then what is this function this I will do that differentiation to the x then what will do this one? Then twice this will come twice, then this differentiation need to do twice max, max 2 minus x, 0. And the differentiation of that minus x this is square differentiation minus x is minus 1.

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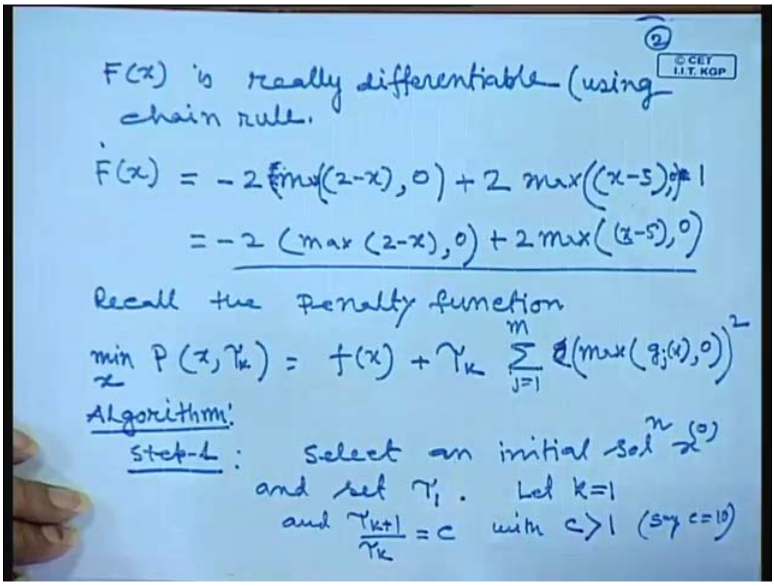

So, it is a minus differentiate minus 1 plus the second term differentiation x square 2 term will becoming 2 then differentiation of x minus 5. That means one that will come max of x minus 5 into 1. So, our values is now is like this way minus 2 max of 2 minus x that is m minus bracket is z here max, so this is not there. So, the max f this is these 0 here max of x minus 5 comma 0 bracket closed. So, these is plus twice max bracket x minus 5 of 0 this one. So, this is the differentiation of that one the that is the important relation, when you are going to find out the non-linear problem solution or linear problem solution we find the necessary conditions first.

What is the necessary function? The gradient of that function, when you find out the gradient of this function this expression will come in the max expression of F of x. That means, what is called penalty terms this is the penalty terms, so this so our if you equal our problem equal the penalty function, what is p x what is this one p x of k is equal to f of x tau k penalty coefficient submission of how many inequality condition is there, j is equal to 1 to m inquilines coefficient of that max, max g j x comma 0, this then this max whole square.

So, this is our the penalty functions and this is the penalty term and this is the penalty coefficients, so max of g out of this which one big also our initial guess is the infeasible point, so this value will be greater than 0 and feasible region means x j of x is less than or equal to 0. So, this and our problem is minimize this penalty function for x find out the value of x for which this is minimize, but tau k will supply the value of tau k from various norm value agree. So, this is the our problem, so what is our algorithm states to solve such type of problems that non-linear algebraic problems with constraints using exterior penalty function method, then what are the steps we have to follow.

And the steps are simple because first you have to take a one point, which is the outside the feasible region in the infeasible region that point and then you find out the necessary conditions for this one second time, necessary conditions solve it in terms of lambda sorry tau k. So, our first step first step is select an initial solution x superscript 0, which is outside the feasible region, which is outside and set tau one tau 1 value is set very low value. Let iteration starts with k is equal to 1 and next iteration k plus 1 by tau k this value is c with c greater than one say c value is 10.

We have seen in this example if you see in this example, when tau value is in is in this one this function value that is what is this function value that infeasible region, that penalty function is approaching to the optimal value function that we have seen it. Now, you consider c is equal 10 this one so first iteration tau k is equal to 1 tau 1 I have selected some small value then next iteration tau k plus 1, you select ten times of tau one that is one that is first L P is like this one.

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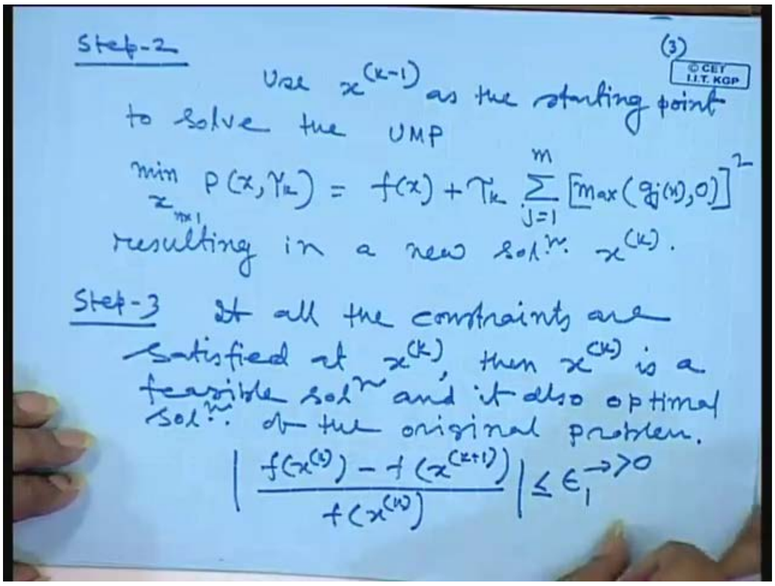

Next is step 2 use x superscript k minus 1 that first iteration, what was the initial guess we have starting point we have considering using x value use x k as the starting point to solve the unconstraint minimization problem U M P unconstraint minimization problem. That means, minimize x p x tau of k is f of x that objective function plus tau k you the penalty coefficient and summission of our inequality constants and that is written as max of g j, g j of x among this whole square, this is the objective function minimize this one. Solve this minimization problems that optimization problems resulting if you solve this one resulting in a new solution. Let us call x superscript k you got it.

Now, how we will solve this one there are different ways already we have discussed or earlier lecture that if non-linear problem is there with constraints one can convert into a on constant optimization take optimization problem, then you can solve this problem on constant optimization problem by using Steffen Desen method Newton's method modified Newton's method and what is called the conjugate gradient method, agree? That different techniques you can solve this problem, so after solving this problem step three, that how we will solve this one you know this function then you find out the necessary condition. Find out there gradient of this function because we have a n decision variables x 1 x 2 dot dot x n this x is a dimension is n cross 1.

So, we will get a n such type of equation n equation which is associate to with max term agree, each equation having a m max term because m max square term is that each equation. So, step three if all the constraints are satisfied at x k that indicates then x k is a feasible solution and it also gives an it also and also, the optimal solution of the original problem. So, that means if all the constraints are satisfied all the constraints means g 1 of g j of x less than or equal to 0 because if it is not constraint is satisfied it indicates that we are now still are in infeasible region. Then we have to do our iteration process we have to keep on doing, until or unless all the constraints are satisfied.

In other words until and unless we have reached from infeasible region to a feasible region, then that gives you the optimal solution of the problem. So, the whether we have reached or not one can check it that one function value f of x that function value minus that is the stopping criteria f of this f of x this mod this should be less than or equal to epsilon. An epsilon is the various small positive quantity and another stopping criteria gives there you find out the penalty function.

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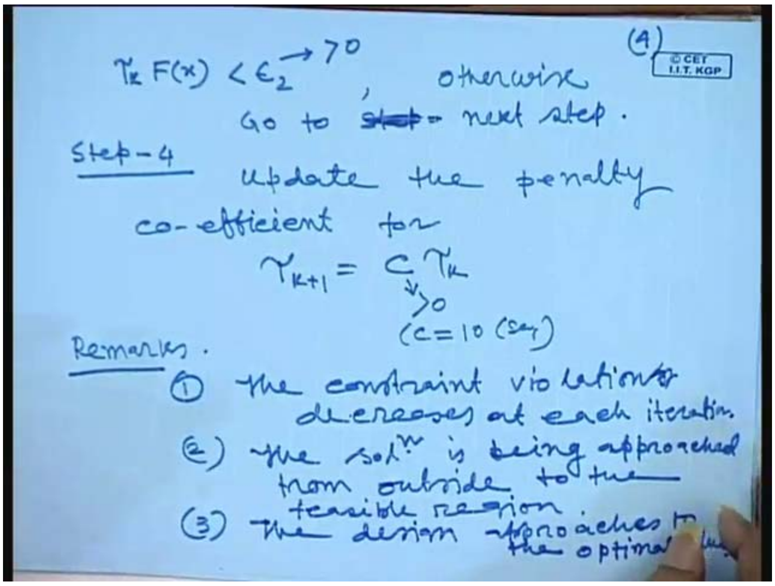

You find out the penalty function F of x, so long it is in the feasible region infeasible region that F of x value is positive, agree? So, this value if it is less than some epsilon, epsilon is greater than 0. That means if you see the our objective function or penalty function if this comes if this comes is very small quantity agree, if this term is very small quantity then you can stop that iteration stop that one. That means, this quantity F of x into tau sorry if f of x into tau is small quantity then you can stop the iteration. That means we have reached the optimal solution that means whole thing is very is small it is adding with the f(x) you see it is adding with the f(x) penalty function, this is small whole quantity and then this is f(x) plus very small quantity. So, we have reached to the optimal value of the functions.

So, this and last step suppose is this to criteria is not satisfied or all the constraints are not satisfied not feasible region or points are not in the feasible region, then what to do that means we are not list the optimal point. And otherwise that means, if the constraints are not satisfied at these are not satisfied, then you do or this conditions are not satisfied. Otherwise you do you go to step two, what go to next step.

What is the next step is we have to do because we have already got the values of into value of x 1 and x two that from the point the further stay away from the feasible region that is now approaching to the feasible boundary of the feasible region. So, what you will give it now update, the penalty coefficient for tau k plus 1 is equal to c tau k, tau k is assign already at you multiply by c whose values is greater than 0, we have a sign c values is a say 10, ten times you multiply it and start the process once again. So, regarding remarks that iteration that constant that the constant values is this first remarks is the constraints violation, constraint violation decreases at each iteration. This first for me each iteration constraint violation slowly it is decreasing means, it is approaching to the feasible region.

Second is in other words what we are saying is the solution is approaching from outside to the feasible region, the solution is being approached from outside mainly infeasibility region to the feasible region. That is two things and ultimately that design approaches, the design approaches the optimal values. Finally, design approaches, the design approaches to the optimal value, so this is the things first iteration if you go on increasing the iteration, that it indicates that constant values decreases.

Next we can observe we are approaching from infeasible region towards the feasible region from infeasible region to towards the feasible region, and ultimately we will approach to the optimum value of the function or optimal or at what point the optimal optimum point will occur that we will achieve. So, next is let us call the solve this problems, example of problem.

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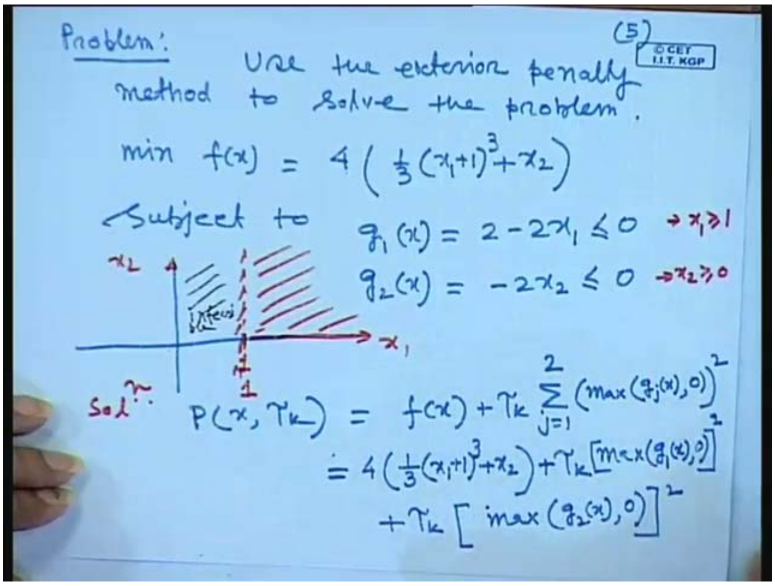

Use the exterior penalty method to solve the problem non-linear linear problem. So, all the non-linear problem, what is the non-linear problem? Minimize f of x is equal to 4 one-third x 1 plus 1 whole cube plus x 2 this, subject to g 1 of x is equal to 2 minus 2 x 1 is less than or equal to 0. Another constraints are there is minus 2 x 2 is less than or equal to 0, our function f of x is a non-linear function and the 2 constraints are linear inequality constraints. If you see carefully what this two equation indicates this one, our region is if you see carefully our region is here that, this we can write it that minus 2 plus 2 x 1 is greater than or equal to 0.

So, x 1 is greater than or equal to 0 this implies this one is nothing but a greater than equal to 1 that is one that x 2 is greater than or equal to 0 this equivalently we can say this is this one is that this equality. So, if you see our region is x 1 value if this is x 1 in this direction and this is x 2 then x 1 is greater than or equal to 1, this whole region is feasible region and x 2 is greater than x 2 is greater than 0. That means the upper portion of this one is 0, that means our region feasible region is our that whole portion, this whole portion x cannot be x 2 cannot be negative.

And x 1 where is x 1, x 1 is greater than or equal to 0 the whole region is this is the our feasible region, but our problem we have solve the problem, but using what is called penalty function methods exterior penalty functions method then what we have to do this we have to convert into a what is called the standard penalty function form. So, our solution what is the standard so our penalty function is like this way x minus tau k is equal to objective function f of x, there are 2 inequality constraints are there tau k summation of j is equal to 1 to 2 inequalities are there. And this is penalty terms that take in the account max g j x 0 this whole thing square.

When it is in the feasible region, this portion g j is negative quantity and maximum of this is 0. So, this term coming to picture when this g of j is greater than 0 g j is greater is 0. In other words it is in the infeasible region, that means when it is in this region. This is our infeasible region, when it is in this feasible this term will come into the picture. If you write more other clearly this one what is the f of x this is the our f of x the 4 one-third x 1 plus 1 whole cube plus x 2, I have written the plus tau k what is the first term max i j is equal to 1 x i write it max g j g 1, x coma 0 whole square.

Therefore, first I have written plus tau k max g 2 of x coma 0 this one whole square, this is a two term I have written like this way. Now, next you see this one because it is in the bigger our exterior point means, infeasible point we take and in this situation g value exterior point g 1 of x value is positive, g 2 value of this is positive. So, maximum so this and this at g 1 x to 0 so g we will write it and that square. So, this I can write it next is like this way.

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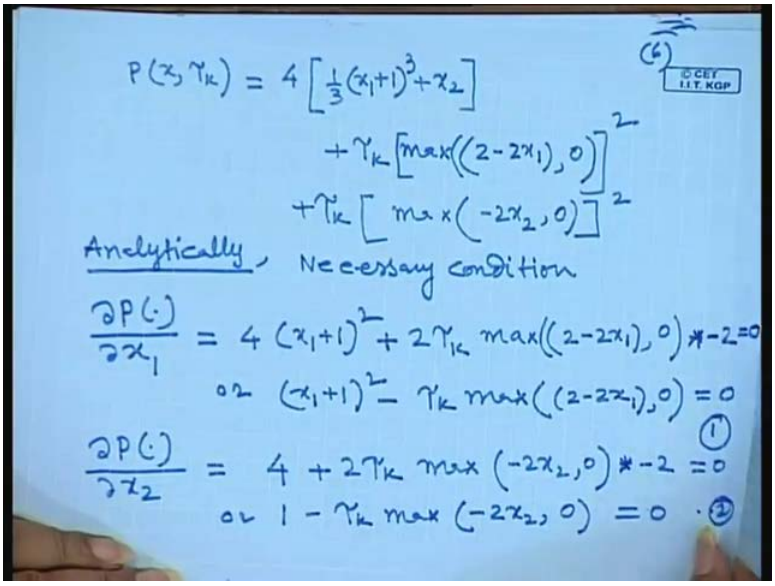

So, now you can write it this that P x tau of k 4, then one-third x 1 plus x cube plus x 2 this is plus tau k tau k then if it is x 1. I am just writing the same thing, where the tau k this minus 2 2 x 1 that is you say I am writing the value of g 1 k because this is know in feasible region, in feasible region this value is greater than 0. So, maximum of this one will come g 1 value and what is g 1 value that max of this coma 0 out of this which one is maximum, then that square plus tau k max then it is minus 2 x 2 0 then that square.

So, that is our penalty function, we have to minimize that function with respect to x because tau is we have assign some value very small value of tau k that why very large value of tau k will assign initially, because we are starting from the far away from the feasible region that what tau k is a penalty coefficient will start initially very large value. So, now you see this one, analytically sere that one tau value we have considered very low slowly we are increasing an approaching to the what is called feasible region.

So, analytically the necessary condition, so we have converted the non-linear optimization problem into a non constraint, constraint non-linear problem is a converted into a non constraint optimization problem by improving the what is called the adopting the exterior point penalty function method. That means we will take the install guess the outside the feasible region an exterior point and then slowly approach to the feasible region. So, our necessary condition is what dell P dot dell x 1 because lambda one is known to us this what is this values if you differentiate this with respect to x, then what will this coming that this that means 4 it will come x 1 plus 1 whole square first term because it differentiate it is x 2 is will not be there.

The second term will come twice tau k is constant then max 2 minus 2 x 1 coma 0 bracket this one, I am differentiating with respect to x into that it is minus 2 if you differentiate this minus 2 that equal to 0 that is this term there is no x. So, ultimately it is coming 4 this 2, 2, 4 cancelled. So, I will write it x one plus one whole square minus this is minus, minus tau k max this twice minus 2 x 1 twice minus 2 x 1 coma 0 this so this is equal to 0. So, let us call this equation is equation number one.

So, you see here you see here when it is infeasible region this quantity, this quantity is positive in feasible region means, when x value is less than 1, when it value is less than one this value will be positive quantity. I will take this quantity only so let us now there next is there are 2 decision variables are dell x 2 is equal to if you differentiate this with respect to x, there will be a 4 x first term there is no x 2 term x 2 is here then it is twice tau k max minus 2 x 2 0 there into minus 2 is equal to 0. So, again if you 4 you cancel it 1 minus tau k max minus 2 x 2, 0 is equal to 0 let us call this is equation number two. So, this is the necessary condition is the this you got it, then one can write it from equation two and one and two. So, let us consider we are in the as we started with a infer feasible region means exterior point.

Assume 2 minis 2 x 1 is greater than 0, this implies x 1 is less than 1, so what is you can say this indicates this is our g 1 x is greater than 0. That means, we are taking a point which is outside the feasible region that is that whole thing. So, this implies that first constraints is not met is not met and this leads leading to max 2 minus 2 x 1 0 is equal to 2 minus 2 x 1, agree? So, this because this is a positive quantity in the infeasible region so max of this one is this.

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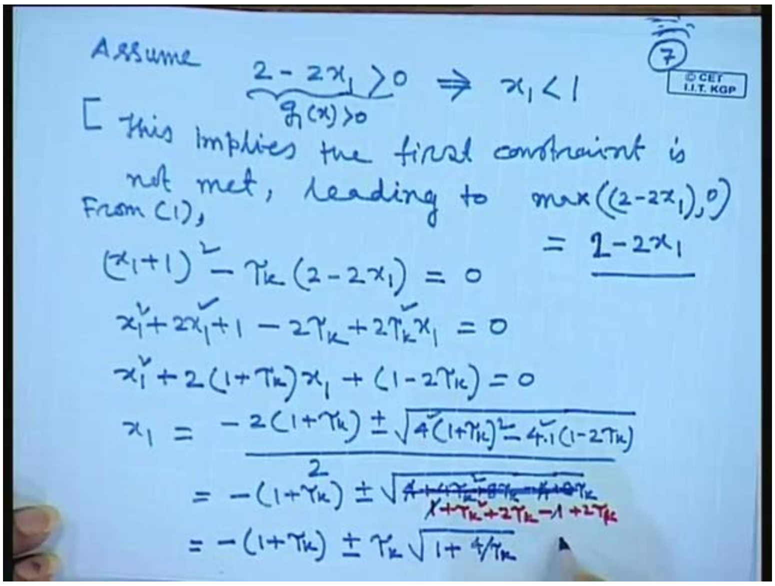

Now, from one from this equation from this equation what you can write I can write the value of this is nothing but 2 minus 2 x 1 from one I can write it x 1 plus 1 whole square from equation one from one agree minus see this one tau minus tau k into that one 2 minus 2 x 1 minus tau k 2 into 2 x 1, this equal to 0. So, this is this is known in the sense do iteration method otherwise this is unknown I can easily find out the value of x 1 in terms of tau, agree? And that tau slowly if you increase the value of tau k each iteration and when tau k is very large, we will approach to the what is called the optimal solution. That is what we have shown it here, if you see this one when tau is increasing we are approaching the what is called the optimal value of this function.

Now, see this expression how x one can be retained in term of x 1 can be retained in terms of tau k. So, this is x 1 square plus twice x 1 plus 1 minus twice tau k plus 2 tau k x one visible region, agree? Now, you see x 1 square plus x 1 square plus 2 so if you take the 2 common, 2 common if you take then one it will become 1 plus from this one and this one, 1 plus tau k x 1 then what is the remaining part is left 1 minus 2 tau k is equal to 0.

So, I can express x 1 in terms of tau k so what is this 1 minus this 1 minus b 1 plus tau k plus minus this square 4 1 plus tau k whole square minus 4 this co-efficient is 1 and this is one minus 2, this if you take it common 1 minus 2 tau k divided by twice so 4 you take it out. So, it ultimately come 1 plus tau k plus minus then what will be there 4 this 4 cancel if x 1 this one if x 1 this one 4 and this 4 will e cancel from here you will get 8 from here 8 here is plus 8 agree so in total you will get it, if you consider log x 1 this one plus 4 plus 4 tau k square plus 8 tau k minus 4 plus 8 tau k, agree?

So, this is cancelled and this 4 tau square 4 tau square and this is the 8, what is this? 2 8and this is 4 into 2 8 plus 16. Now, we are taken this out, 4 we have taken out agree so it is left to it that one already, one plus tau square plus 2 tau k minus 1 plus 2 tau k, agree? So, ultimately I will get it this one is 1 minus 1 plus tau k plus minus if the tau square this, this cancelled, so tau square is there tau k and you will get it root over 1 plus 4 2 to 4 tau k. Now, out of this plus minus which one will consider.

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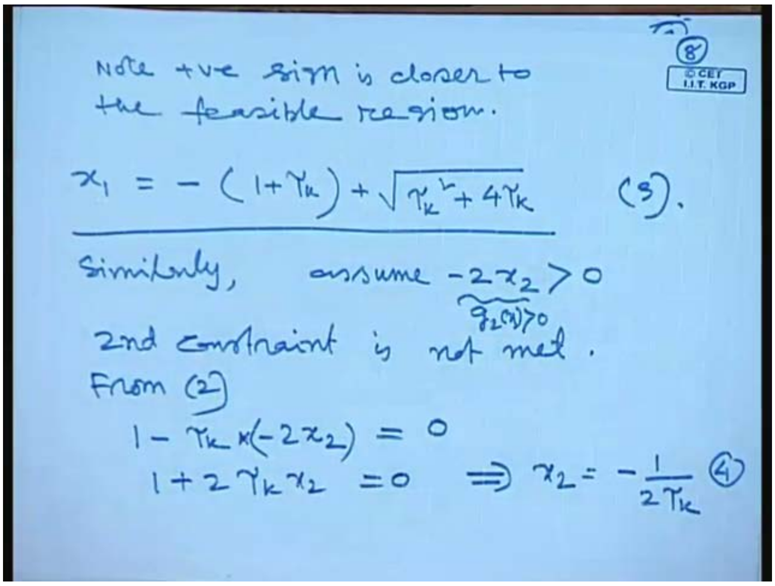

Note the positive sign is closer to the feasible region rather than negative sign. So, we will consider the positive sign in this case, so our x is equal to x 1 is equal to 1 minus 1 plus tau k plus that you can write it tau k square plus 4 tau k. Suppose, if you go take it inside this it is like this way so we have taken tau k in that sense. Means, the plus sign is you consider then then x 1 is closer to the our what is called the feasible region x 1 is closer to the feasible region. Other than you consider minus and it is far away from the feasible region so similarly, we can do similarly, assume 2 x 2 is greater than 0, the second that is what g 2 x is greater than 0 means, it is in feasible region so if g 2 is greater than 0, the second constraint is not met.

So, from equation two that one it just see from this equation two, the necessary condition for equation two, we can write it 1 minus tau k because this quantity is positive. So, I will write this as minus 2 k so into minus 2 x 2 is equal to 0, agree? So this so that will be 1 plus 2 tau k x 2 is equal to 0, so this equal to x 2 equal to minus 1 divided by tau k. So, let us call this equation number four and this equation number two. Now, you see this one this and this when tau k approaching infinity very large values to be 1 so from two or from three equation.

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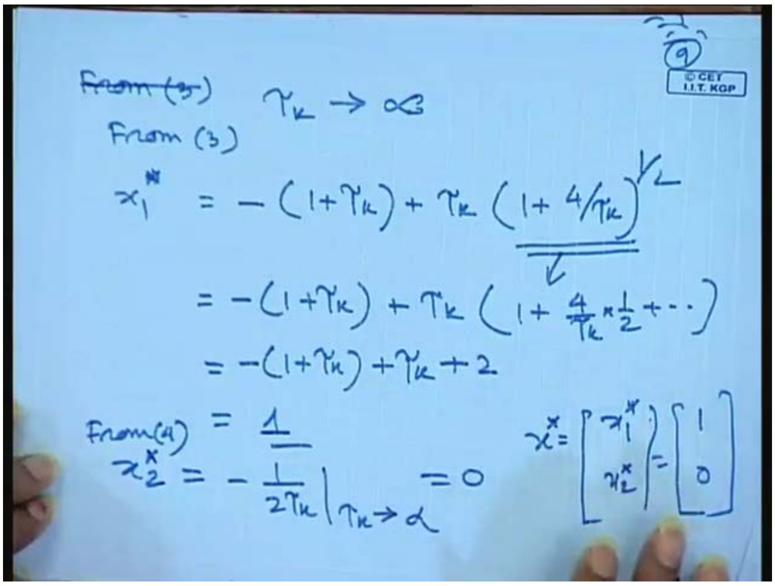

From three tau k when tau k is tensile infinity from three x 1 plus x 1 optimum value of this one is minus 1 plus tau k, that is I am writing from three is tau k 1 plus 4 tau k half if x 1 this thing. So, if you expand this thing binomial expression because it cannot put tau 0 to infinity because one is negative sorry infinity, one by infinity is equal to 0 that tau k is your infinity. So, you cannot do it do this what is called binomial expansion, agree. So, if you do this one it will be 1 plus tau k plus tau k 1 plus 4 tau k into half plus higher term.

Now, you push it out a so 1 plus tau k plus tau k then it is a 2 this is cancelled 2 this and other terms when tau k tends to infinity this will be 0's and other terms will be infinity 0's. So, this will alternately it will be a one. And similarly, from 4 from x 2 star is equal to if you see 1 minus 2 tau k is tau k tends to infinity this is 0, so our optimal value of this one x star is equal to x 1 e star x 2 star that value is equal to 1 0. So, this is the optimal solution by using the analytical expression using the necessary condition, this one and express x 1 x 2 in terms of penalty coefficient, agree? This is the solution, this problem one can solve it by iterative method also that we will discuss in next class.