Optimal Control

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Lecture - 16
Solution of Linear Programming Using Simplex Method – Algebraic Method

So, last class we have seen that if you have given a minimization problem, and is subject to linear equations. An objective function is linear and subject to the constraints the constant also linear, then it is called is a linear programming problem. And we have given a basic idea how to solve this type of problem by considering the two variables two class of variables. One is basic variables another is non-basic variables, then non-basic variables, which we will assigned as a non-basic variable values were assigned with 0 and in turn we get a basic variables values.

Then we shift one of the non-basic variable values will convert into a basic variables and one of the basic variables is converted into non basic variables, and which non basic variable which basic variable will be converted into a non-basic variable and non-basic variable that variables which one will convert it into a basic variable, that we have discussed yesterday. Let us see that what we can do with the what is call in algebraic approach matrix and vector form. So, numerical, numerical method of solving the linear programming is known as the simplex method.

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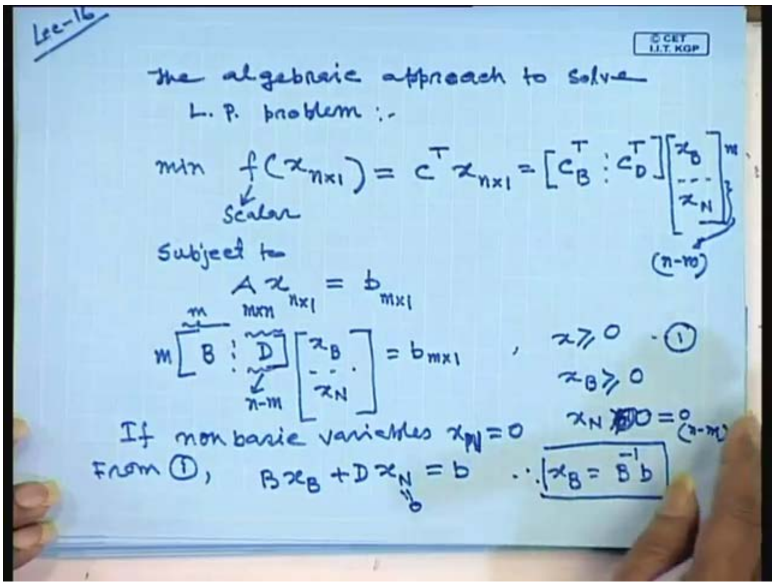

So, let us call the algebraic we have just discuss the algebraic approach to solve linear programming problem. This method is algebraic approach the algebraic approach or numerical method the way we solve the linear programming problem is known as the simplex method. So, let us call our minimation problem minimize f of x, x dimension is n cross 1 and decision variables are there and that we have written into a C transpose x n cross 1. Immediately you know what is the dimension of the f is a scalar function objective function is a scalar we have given.

So, this we partitioned into two parts this C transposed matrix we partitioned into two parts, C B transposed partitioned C D transpose that C B transpose will be associate with the basic variables, x B and x N non basic variables which is denoted by N. The dimension of this x B vector is m and this dimension from here to here this dimension is n minus m so and subject to what, this objective function is linear function and subject to our constant and that constant is a set of linear equation in terms of x variables. And x is a n components so we have A x which is a n components are there is equal to be and we have a m number of linear equations. So, this dimension automatically it will be m cross n.

So, this also we have partitioned accordingly, what is called basic variable associate the basic variable and non basic variable the matrix A is partitioned into two parts, B partitioned d then it is x B then x N is equal to b cross 1. And this dimension immediately you see this is m rows and this is b has m rows and this has a you can say these has a m minus m rows so this A is partitioned b into and d so and we are when x all components of x is equal or greater than 0.

In other words you can write these x is partitioned into two parts x B which is a basic variable that values are non-zero and x N which is a non basic variable this values are also greater than equal to this value is equal to 0 this is n is equal to 0 this. So, this we have partitioned that x values are greater than 0 then we have partitioned x B and x N. So if we consider that our non basic variable are x plus x minus n variables are non basic variables that values will be assigned to 0. So, this dimension is n minus m into 1, agree?

So from this let us call this is a set of equation which is a set of equation number of one. So, if we consider the non basic variables x N, x capital N is equal to 0 then from one we can write it from one, from one, we can write it see B into x B plus D into x N is equal to b, but this value is we have considered non basic variable which is equal to 0. So, we will get a basic solution and that basic solution must be a feasible solution, feasible solution means x is greater than equal to 0, and we have to all constant must be satisfied and this naturally it will be satisfying this one. So, our therefore, our x B is equal to B inverse into small b. So, this is our basic variables we have got it so our basic variable solution now so our basic variable solution are.

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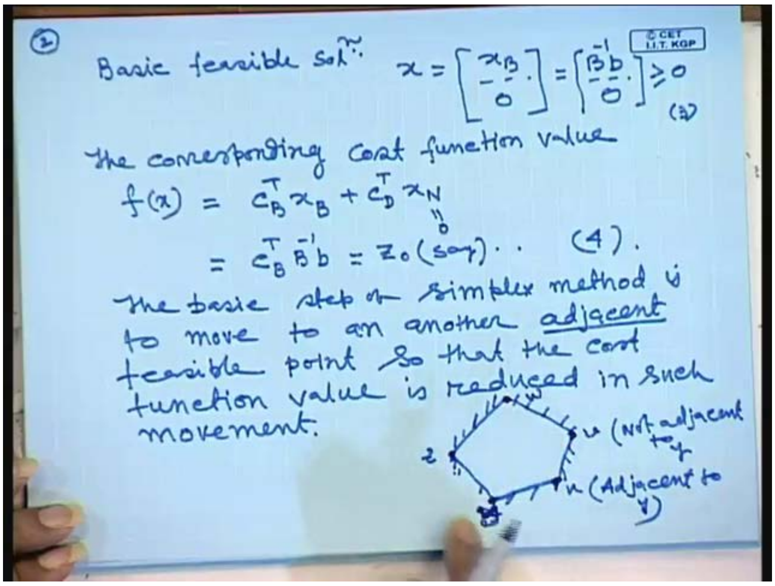

The basic variables basic feasible solution x is what x B and this value you know at this moment and this is x B is b transpose and b is inverse B and this x N which equal to 0 these non basic variables. So, this is these values are greater than equal to 0 this one. So, let us assign this equation number is what we got it this x B value this is two, and this is equal to or this equation will be two forget about this, this equation this equation number you will recall this equation once again, so and this equation is equation number three.

Now, we can find out that whether there is a next question is what is the corresponding cost function value f of x because we have assigned out of n variables. If you see we have assigned n minus 1 variables is 0 and the remaining m, we can get it by taking the inverse of B inverse and that is called basic variables also basic feasible solution that what we got it, basic principles reason is this one. Then corresponding function value you can find out, what is this function value?

If you can see the our objective function expression this is C B transpose into x B plus C D transposed into x N. So, x N value is 0 and this we got it is C B transpose and x B value you got it is B inverse small b say that value is 0 z 0 say and that is equation number 4. So, basic state of simplex method is there we are now having a feasible solution x from there that vertex, if you move to another point in a feasible region then we want to see whether that point will reduce the function value from the present, but x or not. If it is reduce than will accept it, agree?

So this is our basic idea was what is called the simplex method, what are the feasible solution is there now basic feasible solution will move to another adjacent feasible point, or feasible vertex and at the vertex, if the function value is less than the previous point previous adjacent point function value, then we will accept that point. So, let us see how logically will proceed that one, so this basic state of simplex method is to move to an another adjacent feasible point. So, that the function value, means cost function value is reduced, in such movement that is the basic idea of simplex method.

So, let us call we have a set of linear equations are there and this is a vertex of this one and these are the all feasible region is inside this polygon, and this vertex. So, let us call these vertex x u or you can write y this is y u b w z this is y. So, if you move the adjacent vertex I can move the adjacent vertex is here or at this point. I know the function value at this point y point, if the adjacent point if I move it really if the function value is less than the value the function at the point y, then we will accept that feasible solution of this on one of the feasible. But that feasible solution may not be what is called minimum. So, you have to look that anyone of these vertices will give you the minimum value of this function this functions, agree?

So, this is not the adjacent, not adjacent to y adjacent to why is u, u is adjacent to y this is adjacent to y z, but w is not adjacent to y. So, you will move immediate adjacent to our present feasible solution, whatever we got it. Let us say with this one that what happens next question will what happens, if the one of the non basic variable x N all non basic variable we have assigned 0, if you change one of the non basic variable to a basic variable means, non basic variable all variables is 0. If one of the non basic variable we make it non- zero at the than whatever the function value will increase.

The next question which non basic variable will increase out of n minus x variables which non basic variable will increase it. So, this is the question, so if you increase that if we just move that point naturally function value will change, increase because function value will increase that which variable will move it depending on this one function value may increase may decrease also. So, you have to take the judicial this decision that which variable of non basic variable will increase its value.

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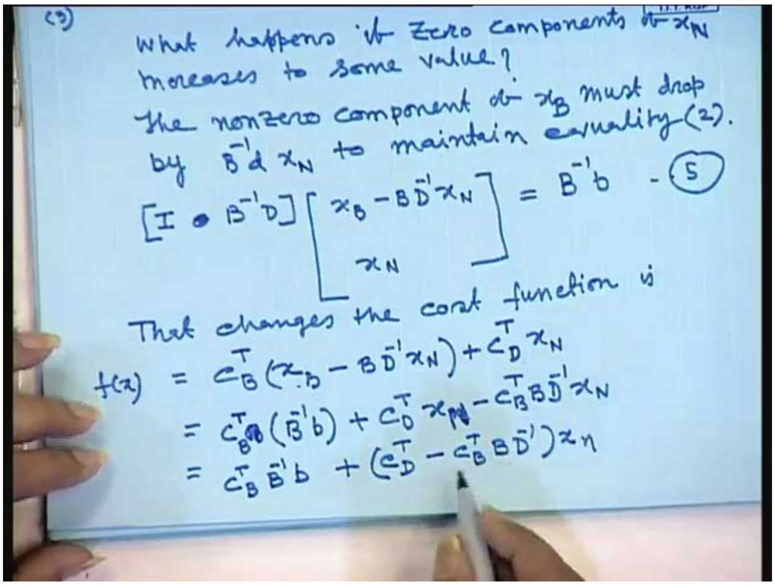

Now, question is here what happens if 0 component of x N, x capital N it has n minus one component all components are 0's increases to some value. So, what happens if 0 component of the increases to some value the non-zero component of x, you see the non-0 component of x B the basic variables, the basic variables values are non zero the non-zero component of B must drop by the B transpose, B inverse d x N to maintain equality, maintain equality constant equation two.

So, equation two you see this one so if because you see now this D is D is a matrix all elements of x is 0 out of all this the one element if I increasing, the increasing the corresponding column that corresponding column of D which multiplied by that element of the risk x N, agree? These corresponding if multiply by x 1 that value will not be equal to be until unless there is a drop in this components, agree? So we can write it now from this equation that is what we have written it, I minus B I not minus I B inverse x of B minus B D inverse x N and x N must be equal to the right hand side B inverse B see here what I did it both side I multiplied by B inverse. So, x B now it is I think B inverse D x N agree and right hand side is B inverse B.

Now, I am telling you out of n minus m element one of the element is non zero, so naturally the corresponding element multiplied by that corresponding column of that one this product, agree? This vectors will is added with this one previously it was 0 its contribution was 0, but now it is added with this one. So, in order to satisfy right hand side B there must be some component of this one it has to be dropped out. So, this is now coming to the picture, so this I can write it now if you see this one that x N into B if you multiply B into this that is B inverse way will be inverse D x N, it will b cancelled and ultimately it is able to b inverse of capital B inverse B.

So, this part is increasing and from the x B this part is drop out, so this is that condition is there that if you change one of the variables of x N, if you change it there must be a drop in from x B some component of x B must be drop out so this, so now total change or that you can say that changes, the cost function is see this, what is our C B transpose into x B, what is x B, x B now this whole one x B minus B D inverse x N plus C D transposed these our performing objective function C D transposed into x N this one.

So, if I one of the element of x N that out if one of the element if we change it there must be from x N some x, x B vector some has to be dropped out so that this equation is satisfied this is the thing. So, if you re adjust with this one I can write that C transpose B and x B you can write it this x B is nothing but a B inverse small b plus this term minus this into this C transpose x N minus C B transpose B in D inverse x N this is x N, this is x N. Now, this is, C suffix this B so this is C suffix B, b inverse B, this is is a our previous point objective function value, what are the previous function value if you change x N to some other value then this changes are coming C D transpose minus C B transpose B D inverse whole x N, agree?

I think here I have written C B so this is not a C B inverse D B inverse D so this is if you manipulate see B D inverse this agree B inverse D B D inverse not D B inverse just a minute. So, what we got that one, this is B inverse sorry here just make it correction because this is B inverse this is B inverse this is not inverse drop this is B inverse this is not the drop out agree when we have just x B value is this one then this is your B inverse this is not their, this is B inverse this is not there agree. This is I made a mistake here B inverse. So now you see this, this term if you see this term f of x is coming…

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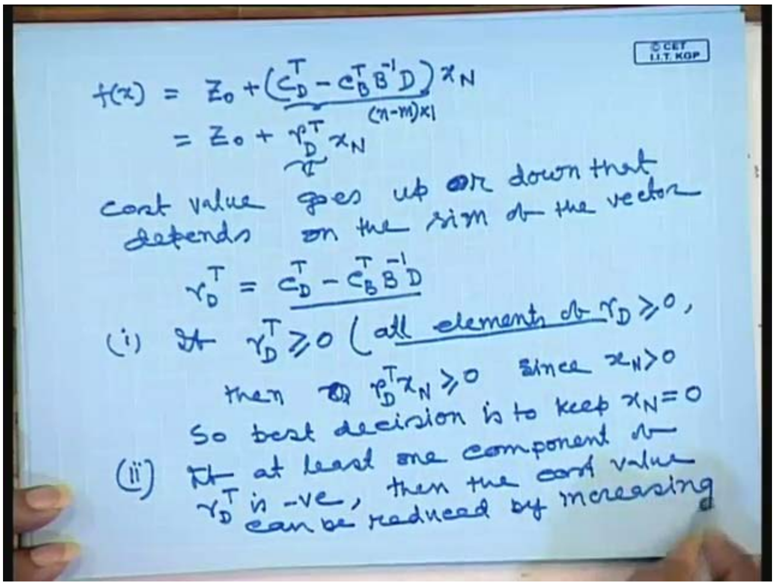

f of x coming z 0, the original point objective function value plus that your C D transpose minus C B transpose b inverse d into x N, and let us call I considered this is a r D this is the row vector, this is a row vector of dimension n minus m into 1, this because of this dimension is n minus 1, n minus m in n row this is column vector and this is a row vector it is a r D transpose into x N. Now, see this one this function value if, I change the non-basic variable value form 0 all values are 0 from there, if I change its value some of the element if I change its value then the total value of the this one is coming in the next point is z 0 plus r D transpose into x and this quantity is scalar.

And now it depends on whether the function value will decrease or increase it depends on the row vector elements of a row vector of r D transpose because one element, we have to increase because r x and x n non variable is greater than 0 we have to increase it. So, this function value will decrease when if corresponding element of x N corresponding element in r if it is negative because x N is x N that which element you have consider is positive, that corresponding element in r D transpose must be negative than there is a possibility of reducing the function value.

That means so we have at you this you can write it, the cost value you can write the cost value goes up and down, up or down that depends on the value of depends on the elements of or you can say the sign of the vector. That means, r D transpose is equal to C D transpose minus C B transpose B inverse D this one. Now, let us call if this row vector if you consider the all elements of this row vector, all elements of the row vector r D transpose all are positive agree.

And the x N how many elements of their if x N vector non visible there are n minus m elements were out of this one element let us call, I want to change its value from 0 to its value I want to increase agree, then this function product of this function will be a positive because all elements of element is positive than this product will be positive. So, objective function value is increasing, so we are now away from the our goal, our goal is minimizing the objective functions this one.

So, if you think in the other way if r D transpose is less than equal to 0 or the elements at least one elements of r D transpose is one is negative, and corresponding element of x N element of x N is positive then product of this one will be a negative one. So, there is a scope of finding the value of f x is finding the value of f x reduced agree again or to reduce the function of value will f x if we moved that point. So, you will think whether function value will increase of decrease that depends on the r D. So, in short now we can write it that if r D transpose is greater than equal to 0, that means all elements of r D is greater than equal to 0, then r D then r D transposed x N is greater than equal to 0.

Since, x N is greater than 0 you consider one at least one of the element is non zero so the best decision is to keep x N is equal to 0. So, x N is suppose r D all elements of this and if you all elements though then x want to increase it then this product will be greater than 0 that is function value is increasing, that it is greater whatever the point you are you are earlier at that point that point you keep it as an optimal point of this feasible solution, agree?

So, another choice is together all the elements of this if at least one element, one component of r D transposed is negative at least one out of how many elements of there n minus one elements of r D out of this at least one element is negative. Then corresponding x N element which is multiplied by x element because when you change one element that element if you change from 0 to non basic variable to basic variable to 0 to increase this value, the function value will decrease that is why, if at least one component of r D is negative. Then the cost function cost value can be reduced by increasing by increasing the corresponding component of x N.

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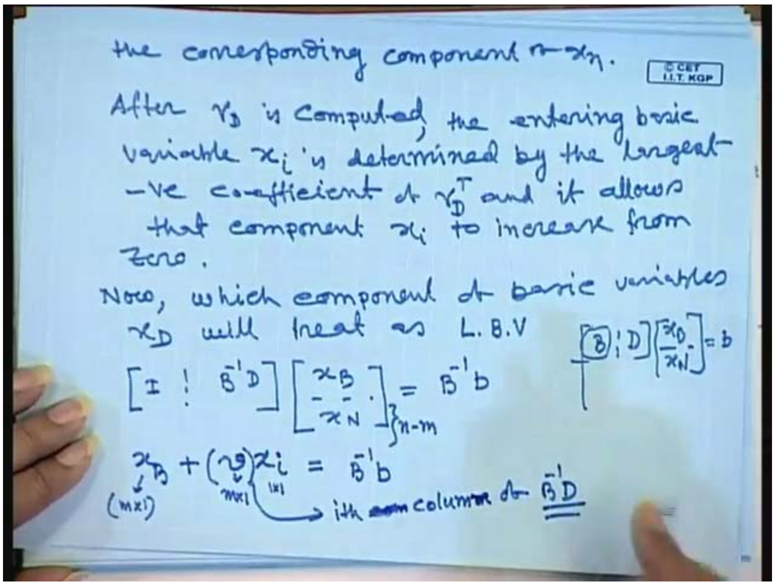

So, now if you find that there is a the r D that row vector r D transpose row vector if there is more than one component, more than one component having a negative sign of that r D transpose that where row vector element more than one element is a having a negative sign. Then which corresponding I mean variable of that r D you considered as a basic variable the most negative, a most negative sign of that numerical value will consider is that basic variable because that will give you the reduction of the function value, much than the other negative value of in r D x in r D transpose.

So, after r D is competed sorry is competed after r D is competed the entering point entering basic point basic variable. Let us call x i is determined by the largest of most largest negative coefficient of r D transpose and it allows that component allows that component x i to increase from 0. Next is we have to increase this one, so just now you may mentioned it here if you recollect. So, one of the basic one of the non basic variable is now chosen as a basic variable, now one of the basic variable now we have two select as a non basic variable out of an basic variable which basic variable will consider as a non basic variable that we have to select it.

Next is how now which component of basic variables x D will treat as living basic variable that means that basic variable is leaving and entering as a non basic variable which basic variable consider that is the next question, entering is that this which basic variable living and entering as a non basic variable. Now, recall the our basic variable that is I be trans B inverse D x B x N is equal to B inverse b that as our basic that is our what is called equality constants at the we have written if recollect B this one we have written B partition D, the than we have a basic variable is exhibiting then we have a non-basic variable x N is equal to b.

So, both side inverse this is a square matrix this is a square matrix both side will be the inverse of it, so it will come to you see a I have this if you consider the ith component of x N is non zero remaining element is how many components and there here x N, n minus m out of this ith component of x N is non zero other is 0. So, if you multiplied by B inverse D with x N agree if you multiplied by this B x, so first component of this one x N will multiplied by first component of B inverse D plus second component of x N will multiplied by second component of the B inverse D and so on.

So, I have consider the ith component of what is called x N will be multiplied by ith component of B inverse D that will be ith component multiplied that will be non zero. Other remaining element will be column will be 0, so I can write it now let us call ith component of ith component of B inverse D is equal to this small b into I am considering ith component of this one multiplied by x i, agree? This than what is this component will near here x B plus is equal to B inverse b. So, this dimension if you see it is a m cross 1 and this dimension is your m cross 1, this is one, one cross one but ith column 1.

So, this one so this is you can write it this agree is what please is the ith component ith column ith column of B inverse D. So, this is equation number we have come up to a equation number five, six let us call up to five, six this equation number six. Now, we can write it see this one when this component will be 0, when you see this ith component of ith column of B inverse D, I multiplied by xi because xi is non zero corresponding column of B inverse is multiplied and this is B inverse capital B inverse b when this component this component that is it has a m components there, out of m components out of m components and components you see which components because this x B components will be 0 when?

When this equal to this when this equal to this, and how many components of this you be you have a m components are there. So, it will be at this B 1 x i B inverse of b if this is equal same x B you will 0. Now, out of all thing which element of x B, the and the basic variable will force it to make it 0 agree that is our next question.

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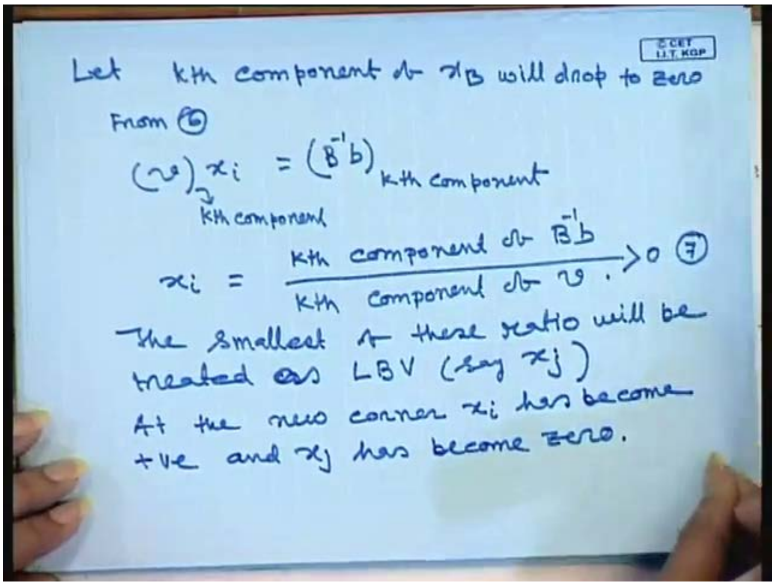

So, let get component kth component of x B, so how many components of their here m com component out of this kth component of x B will drop to will drop to 0, agree? Then than from equations six from six, what you can write it the k th component of this is assigned 0. So, will write it v bracket x i B inverse of b again I told you here I just mentioned it kth component of this these as a m variables are there, out of m I am just considering component of x B is forcing to the 0.

So, if it is forcing to 0 than this is equal to this and that is also of the consider kth component of the B and here also we have to consider kth component of B inverse small b. So, that is why I am writing here the kth component and this is also kth component that this vector is you having a m cross 1 column vector of m elements out of this that the mth component I make it the x B m th component m forces to the 0. That means in other words that basic variable is entering as a non basic variable that is kth component x B I am assigning it to 0.

So, this equation in this equation must satisfy, so than I can write it xi is equal to kth component of B inverse of b divided by kth component of v and this must be greater than equal to 0, that is let us call equation number we have given six that is seven. This is why it is greater than equal to 0 because denominator power cannot be negative, why it is greater than 0, when you will compete the basic variables expression from here you see when you compete the basic variable expression here this, that i th component of the this one I can write this kth component of this 1 minus, if you take that side minus kth component of the kth component of v into x i.

Since, x i value is greater than 0 this is the minus v i and x i this will be 0, this will be 0 provided the that v this coefficient kth component of the this one is positive if kth component of this one is positive, if you take that side is scalar quantity kth component of b is scalar quantity if it is positive, but if you take it the other side it will be negative agree. And the k xi is your positive so this positive quantity minus whatever that this quantity minus of this quantity, it has a chance of making this is 0. And if you and if it is a negative this one agree, if it is a negative of if the component is negative and it goes to the right hand side is a positive multiplied by positive quantity because x i is now is the basic variables. So, this added with the positive quantity agree.

So, this value will increase will not be able to force it to 0 that one, so that is why this is greater than 0 this we have to consider. The smallest so we have a such type of things there may be possibility, we have a such type of x i choices m choices there out of this the smallest of this ratio will be treated as will be treated as LBV living basic variable say x j. So, x j say x j, so our new corner point or new vertex will be what at the new vertex new feasible solution you can say, our new vertex our new corner point xi has become has become positive and as j has become 0, agree, with this as so keeping all this in mind which will see how to solve this problems, how to solve that our linear programming problem using the simplex algebraic approach, simplex method using algebraic approach.

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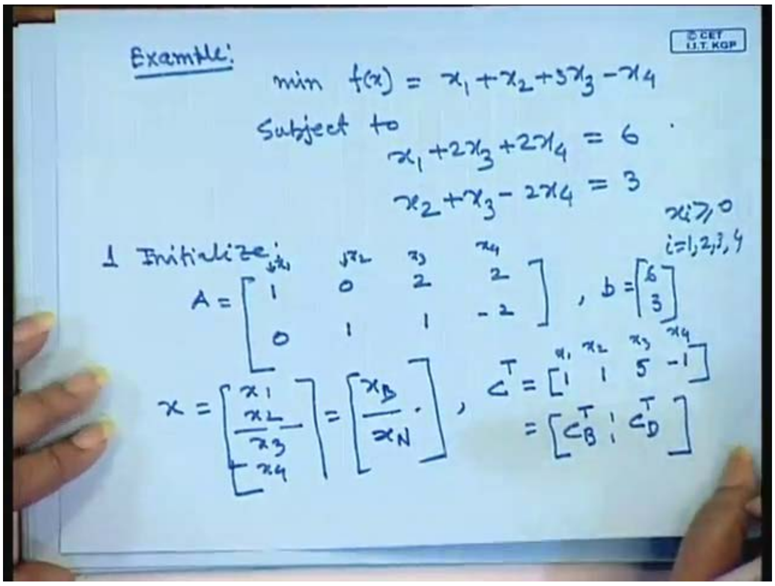

So, we will take our earlier problems what we have discussed that minimize f of x is equal to x 1 x 2 plus 5 x 3 minus 5 x 4 and subject to x 1 plus 2 x 3 plus 2 x 4 is equal to 6 and x 2 plus x 3 minus 2 x 4 is equal to 3, so this is and your xi is greater than equal to 0 i is 1, 2, 3, 4. So, if you recollect that last class we have just considered that problems this problem where xi x 2 x 3 x 4 is greater than equal to 0, that is as we have considered last class was that of the one in again and we have seen that which component, we have to consider in non basic variable and basic variable to reduce the function value from previous point to new point, the function value must reduce this one.

Now, will say that this algebraic approach how can be applied to solve these problems, so first you identify the matrixes A matrixes and our other matrices. So, first is initialize, so this is our with our A matrix is that one quickly can write it 1 0 2 2 that is 0 0 1 1 minus 2 and you can think of it this is x 1 coefficient x 2 x 3 and this column when you think of it x 1 into 1 x 2 into x 3 into 2 x 4 into 2 is equal to b, b matrix is your b vector is 6 3.

Similarly, you can think x 1 into 0 x 2 into 1 x 3 into 1 this one plus x 4 twice is equal to 3. So, this one is our is our first impression, which one is our basic variables basic variable we will consider these one, which are in the canonical form this our canonical form x 1 is involved equation one x 2 is involved only in equation two. So, these two we can think of as a basic variable a non basic variable is our x 3 and x 4 so you are now partitioned x 3 our x 1 x 2 our basic variables x 3 x 4 are our non basic variable. So, this is written as x B so x N, agree?

And what is our c, C transpose matrix you see one, one this so our C matrix is if you see here 1 1 5 minus 1 and this corresponding to x 1 and this corresponding to x 2 and this corresponds x 3 x 4, this we can think of it x 1 into 1 plus x 2 into 1 x 3 into 5 this one x 4 into minus 1 is equal to our objective function or cost function. So, if you see the C B this matrix is corresponding to the basic variables and it is x 1 and x 2 are the basic variables. And these corresponding to our non basic variable x 3 and so these partition and you know our C is what and v is what I can just partition now.

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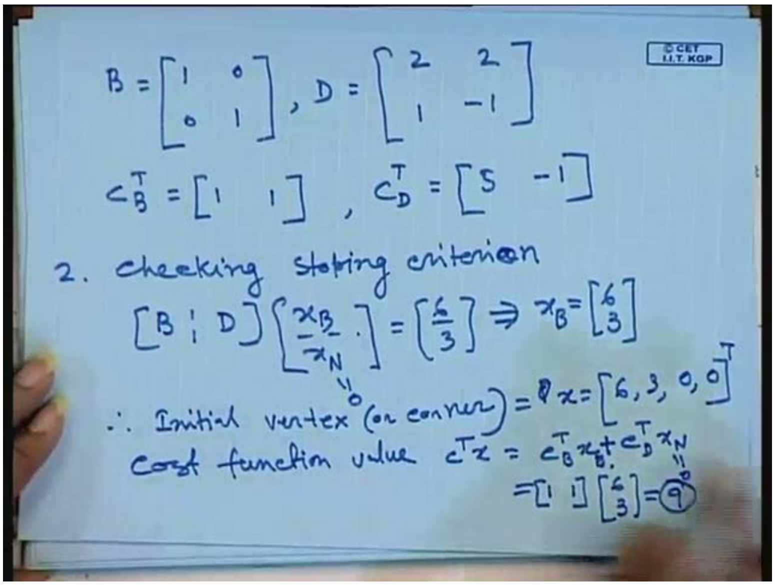

Now, I can just partition like this way our B is your 1 0 1 0 0 1 B D is equal 2 1 2 minus 1, agree? This is our D similarly, we can say that our C B, C B transpose is equal to 1 1 and C D transpose is equal to 5 minus 1, so this we are using so now second state is checking kept checking the stopping criteria, checking the stopping criteria, agree? What is the stopping criteria how to check it, that you can find out the your r D transpose the row vector is a stopping criteria.

So, let us see B D multiplied by x B and x N this value is non basic variable will be considered to 0 is equal to 6 3 this one. So, our basic variables solution is x B is equal to 6 3, so if you get any component of x B is negative this indicates that solution is not feasible because our problem was x is greater than equal to 0, this one. So, now let us see that our initial, so you can write it now initial vertex or corner or initial feasible solution we got it vertex or corner is what x 1 is we got it that is x we got is 6 x 2 3 x 3 0 this got it.

Now, what is a cost function value questions cross function C transpose into x which is equal to C B transposed x C D x B C D transposed plus this is plus C D transpose x N agree. So, these values is 0 and these value C B transpose this 6 1 into 6 plus C B value is C B transpose is 1 1 and this is this C B is 6 3. So, if you do this one it is 9, so our objective function value at this point is 9. Now, we have to see whether we can if you move one of the non basic variable to basic variable and basic variable to one of the basic variable to non basic variable whether the function value can be reduced or not.

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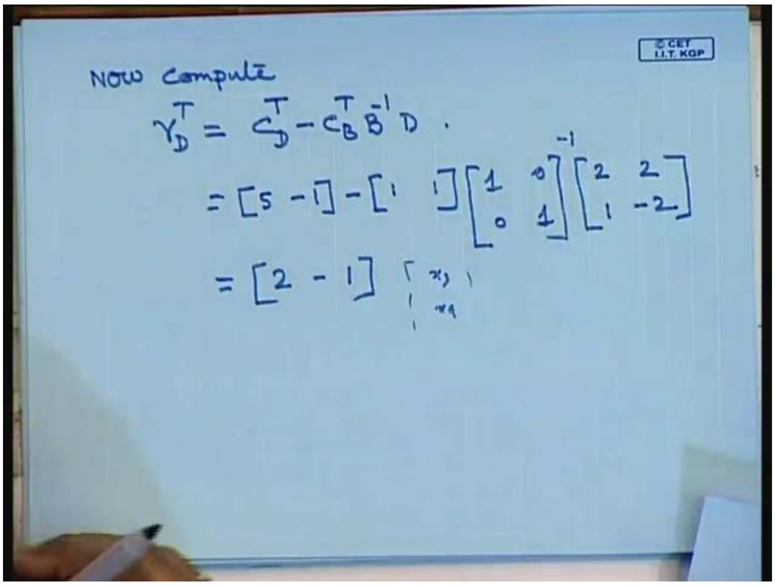

Now, compete to know this one or to show this one compete r D transpose if you recollect this on we have derived r D to C turns C D transpose minus C B transpose B inverse D. So, this equal to C D you know nothing but a 5 minus 1 minus C B you know it C B is 1 1 B transpose b, I know is identity matrix agree so D is your 2 1 2 minus 2, see this, this is our D. So, if you compete that one ultimately you will get it here 2 minus 1. Now, you see r D that this, this coefficient because r D multiplied by what is it x N if you think of it is ultimate basic x N means, x 3 x 4. So, it is multiply x 3 x 4 r D so there is minus sign coefficient reduction coefficient.

So, if you change x 4 value from 0 to some quality value, there is a possibility of reduction in cost function value our x 3 is 0, I have not changed it. So, there is a possibility if you do this x 4 value you change it to non zero values. That means some increasing value, so x 4 will go act as a non basic variable and x 3 will remain as a basic variable. So, I will stop it here today next class I will continue the remaining part of this problem.