Electrical Engineering Optimal Control
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Lecture - 12
Post Optimality Analysis, Convex Function and Its Properties (Contd.)

So, last class we have seen that how to solve what is called optimization problem using the KKT condition, necessary and sufficient condition, and we also seen that there is a inequality constant is there and equality constants are there.

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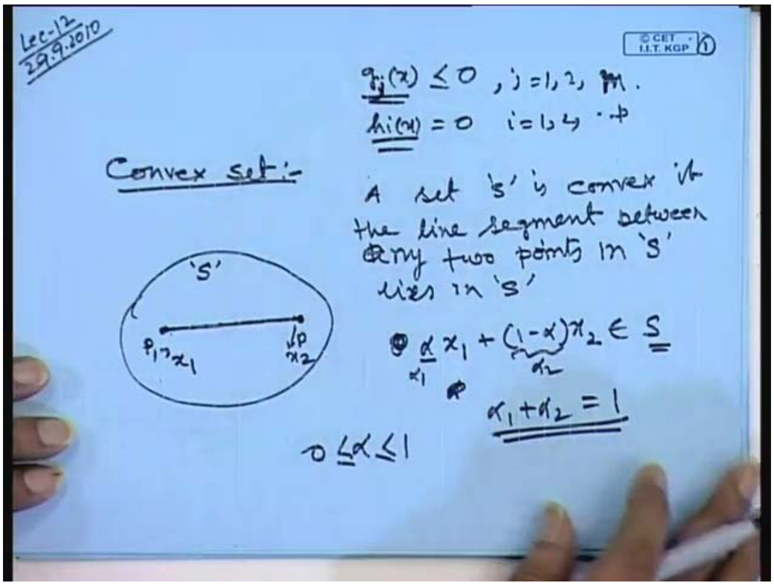

If in equality constants that x of k is less than 0 for j is equal 12 to n, sorry 1 to p m j, if this constant this 0 is part up with a small positive numbers. Then also the inequality constraints is this right hand side is part of with some positive number i is equal to 1, 2. There is such p equality constant, then what is the effect of objective function value of this, we have study it in other words. We have studied the sensitivity analysis after what is called postanalysis, we have studied it and also we have studied the effect of cost functions scaling on the Lagrangean multiplier effect of cost functions scaling on the Lagrangean multiplier.

Also, we have studied what is the effect of scaling on constraints that constant this are the constant if you what is the effect of scaling if multiplied by these p i and h hi is multiplied by m pi and g i g j of x in equality constants multiplied by mi. Then what is this effect on the Lagrangean multiplier that we have studied, so today will start what is called convex set and convex function. So, what is convex set, a set is a convex supposed this is a set this set will be convex if any point on this set.

This is a set S, the set is S is a convex set if we take any point on this setany point and form a line segmentthen this any point on the line segment belongs to that set, then will call this is the convex set. So, what we can write is a set S is convex if the line segments the line segment between two points, any two points lying in any two points in S in that setlies the line segment between any two points in S. Then we will call that set is a convex set, so let us call we have in this set is there, we form a line segment by joining the point p 1 and p 2. So, these two points that p 1 and p 2 you join any point on this line segment belongs to that is if they are that set is there in that set.

Then, will call the set is convex set, so in convex set we can always write it, let u call x is any point here this p is a point was that vector is x 1, this is a vector of x 2. These two points than linear combination of theta or you can say that alpha into x 11 minus alpha into x 2 belongs to that set. So, if you consider it is a single variable case for this x 1 x 2, there are two points are there you joint with this two pointsa than any points on this line. If it belongs to that set, which is what it is, call a set it can be in n dimensional variables x 1 x 2 to show only the condition is if you consider this is alpha 1 and this is alpha 2.

So, alpha 1 plus alpha 2 must be equal to 1, then when alpha 1 is let us call here when alpha is 0 when alpha is 0 is nothing but the x 2 point. This point when alpha is 1 is nothing but a x 1 point any point alpha valueany point between 0 to the alpha value is less than equal to greater than equal to 0. So, any point alpha between 0 to 1 indicates the corresponding point on the line if all the points on the line belongs to that set, then this will be called as a convex set, so we can see this one.

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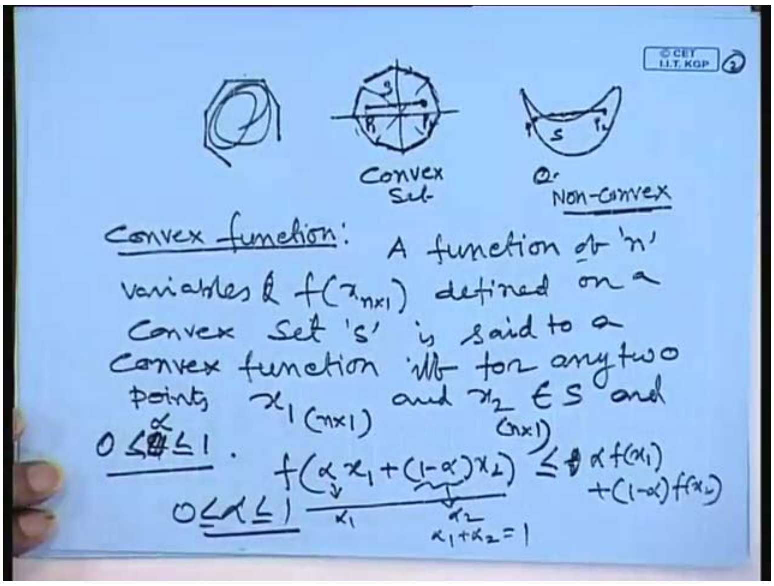

Let us call we have a simple octagon regular octagon 1, 2, 3, 4, 1, 2, 3, 4, 5, this is the drawing up to like this way, this is the circle. So, these this 3, 4, 5, 6, 7, 8, so these are the octagon, so if this octagon belongs to a set S let us call this belongs to set S any point on this line any point on this octagon. If you join together let us call p 1 and this p 2 this point is p 2 any point on this line two points x 1 and x 2 or p 1 point p 2. We join together and any on this line segment any point belongs to that set, then it is a convex set. So, the octagon of this method is a convexthis is a convex set convex set, so let us call this is a another setlike this way this this belongs to this belongs to this set S.

In this case, any point on the set does not belongs to that set you see any point if this is p 1, this is p 2 if you make a line segment like this wayand the any points on this segment this point. These points are not belongs to that set, so this is not a convex set this is not a convex set what will call non convex set. So, in this set, this is the point if you joined this two point thinking of just considering is the chord d you have joined it. So, any point on this line does not belong to that set, so this is not a convex set it is a non convex set. So, geometrically if you have a set, it just joined two pointsin that set and it will form a chord.

If all the points on the chordbelongs to that set that it is a convex set in geometrically you can say or between two points in joined a line a line segment if any points on the line segment belongs to that set that it is a convex set. Next is convex function and convex set, this condition must be satisfied any point x 1 and 2 and it is alpha 1, alpha 1 is 1 minus alpha the alpha 1 plus alpha 2 must be equal to 0 or equal to 1. Now, if you have this, you can write it now a convex function if function is said to be a function in general of ndimensional a function of n dimensional variablen variables.

So, let us call this function is f of x and cross 1 variable defined function is defined on a convex set.That function is defined on a convex set, the function is defined variable define on a convex set S is said to be convex function if and only if necessary. The sufficient condition for any two point, any x 1whose dimension you can say n cross 1 and x 2 any two points on the set and x and set is your S on the dimension of this is x 2 dimensional n cross 1. There is a scalar quantity theta is greater than equal to this or let us call since I have used alpha use alpha is greater than these.

Then, one can write it this one, f of alpha into x 1 plus 1 minus alpha into x 2 equal to less than equal less than equal to f alpha less alpha into a f of x 1 plus1 minus alpha f of x 2. If this condition is satisfied, then we will call the function is a convex function in the convex set this condition is satisfied one can easily see this one by a single variable case. That can be extended for a n variable function case, so what is the definition of convex function, a convex function in a function of n variable is set to be convex function if it in your set S if and only if this condition is satisfied.

If you consider the alpha is let us call alpha 1, this you consider considered is alpha 2 that alpha 1 plus alpha 2 is equal to 1. So, two points that now seen this one this to when alpha is 0, it indicates the x 2 points in n dimensional vector x 2 point when alpha is 1. It indicates the x 1 point in n dimensional vector alpha in between 0 to 1, any value that indicates that any value on this vector that x 1 minus x 2 within this.

With these two point that matter what will generate, it indicates any point on that vectors for any values of alpha from 0 to alpha from 0, 2 greater than 0 to 1. So, if this condition is satisfied, then we will call the function is a convex function in that convex set. So, let us see with an example of this one this that this condition is there, then the function will be said to be convex function.

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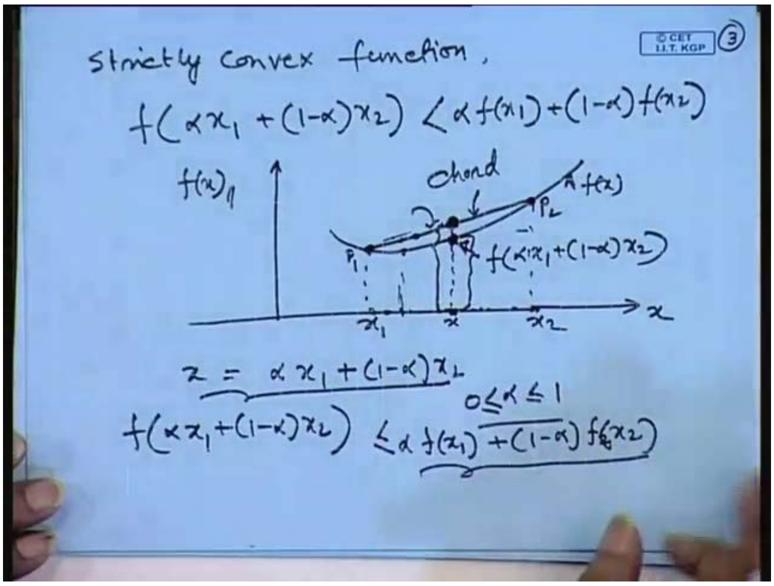

It is a strictly convex strictly convexfunction converts when this condition is satisfied alpha into x 11 minus alpha into x 2. This is less than equal to not equal to less than equal to previous it is less than convex from. Now, I am telling it is a strictly convex if this function value is less than equal to alpha into a f of x 11minus alpha into a f of x 2 values. So, what is this just safe for a single variable case what is the meaning of that one, so suppose this is the function of f of x, now you consider this the single variable case x is a 1 variable. So, single variable this is f of x this the function, so we take a two point here, let us call p is the point whose coordinates are let us called x1. This is p 2, there two points on the line on the chordfunction is x 2, so I told you that with these two points draw a straight line which is nothing but a chord.

You can say it is chord, so this indicates you see if it is the function is convex according to the definition that any point on this line between x 1 and x 2.Any point I can expresslike this way any point on this line, let us called x any point x1 to x 2, any point is equal to I can write alpha into x 1 plus 1 minus alpha into x 2or alpha values is less than to 1 or greater the equal to 0. So, any venue of alpha from within this range the x will lies in a where between the x 1x 2, similarly when alpha is 0 is nothing but a x 2 point when alpha is the extreme limit when alpha is one is nothing buta x 1 point.

So, alpha value in between this range it lies between x 1 and x 2 so that any value between x1x and x 1and x 2, the function value is that one and what is this value x any value in between x 1 and x 2 is that one, that expression. So, what is the function value alpha into x 1 plus 1 minus alpha into x 2, these will this coordinate, this is the coordinate of that one that x the frequency of the x. Then this is the coordinate of it, you can x and y direction and that that one that is a function of alpha into x 11 minus alpha into x 2 this value edit and this value always less than any point corresponding x.

That is the value what is value of the chord value of this equation value of the chord this, so this quantity always greater than this quantity definitely not how will findthis quantity that is simple geometrical configuration. One can find out that the equation of the straight line once you know the equation of the straight line that any point x what is the coordinate of this one. You can find out which in turn it will be at it can be expressed like this with this quantity is less than equal to you see a, when it will be equal to add the extreme points of the chord the function value and the chord value at the extreme point of p 1 and p 2, both are same that cycle to sign other than this point is two points these chord any point on the chord function value.

Function value that is always greater than the function value of f of x any point, here this function value whatever the function value is the record with at this point is greater than this one equation of this equation of this one. If you know because I know they equation out of how you will know the equation you know the ordinator of that one change in ordinator and changing x.

So, slope you can find once you know the slope and it is a function of x in to find out the wide this function equation. So, we can write with this one always alpha into x of 1 plus 1 minus alpha x of 2 that is f of x of 2 how to get this one this point value this point will validate. I told you find out the equation that what is the equation of the straight line, and then you can you find the equation of this straight line. Then you find out the chord value of x what is ordinator of this one so that you can find out and after signalling this one you will get this one. So, if this condition is satisfied, it indicates that function is convex function and belongs to the in convex set that is based. One can easily verify that one by the finding out the equation of the straightline or chord equation betweenpoint 1 and point 2. Find out the value of these function value of this chord value that this equation value at x equal to x 1 is nothing but that one after signalling. So, this is our convex what is the definition of concave function definition of concave.

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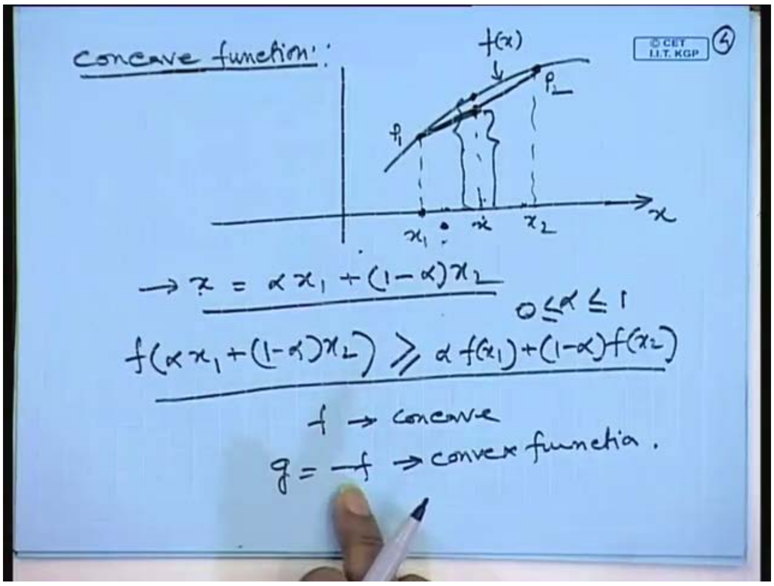

Concave function is similar way one can find out the similar way, one can find out that the two points are there; let us called this is the function f of x and is also single variable case we are explaining that we can extend for n dimensional case. So, this point is p 1 and this point is p 2, p 2 this p 1 draw a chord between these two points a, this is x 1 and this is x 2. Now, you can find out that linear combination of this two points and any points on this line between x 1 and x 2, any points this one I can express like this were x is equal to alpha into x 1 plus 1 minus alpha into x 2.

This is the definition of our convex set, if you recollect whatwe have discussed the convex set is belongs to convex is belongs to these our convex set of this one with this expression and this one alpha varies from 1 to 0. Once again, when alpha is 0, this is x 2 when alpha is 1, this is x 1 to any point on this lineany point on this between the two points. I can take the alpha values in this range greater than 0 less than 1, so let us call what is the function value you see in this case function value at any point between x 1 and x 2.

Let us call x 2 is always greater than what is the value of these on the chord what arethe value of these equation of this straight line what is the y value at this point. It is always less than the function value are f of x function value, so one can write it for convex function in a convex set can write it alpha into x 1 same expression 1 minus alpha into x

2. This is the function value at this point it is greater than equal to the greater to this point function value this function value and this quantity is that one in this quantity.

I am writing alpha into f of x 1 plus 1 minus alpha into f of x 2, this one, so convex function if this condition is satisfied, then I will call the function is concave function in the convex set this one. This is you see graphically also this nature of the chord is concave chord provided this condition is satisfied. This means the x belongs to any convex set and a and if this condition is satisfied that function is a convex function and how one can do it in the similar manner.

I can do suppose f is a function, I will define if f is a function which is a let us call concave function I multiplied by let us call g is equal to minus f. So, this function g function of the convex function if it is a convex multiplied by f minus is a convex function. I know how to if I can check the g is a convex function in the set S, in other words I can say f is a concave function in the set just multiplying f by S.

So, definition then convex function is that we can write with the definition for concave function definition of concave function to this. These are the chord, in the beginning also I told also the chord joining these two points, so definition of convex function a function f of n variables defined.

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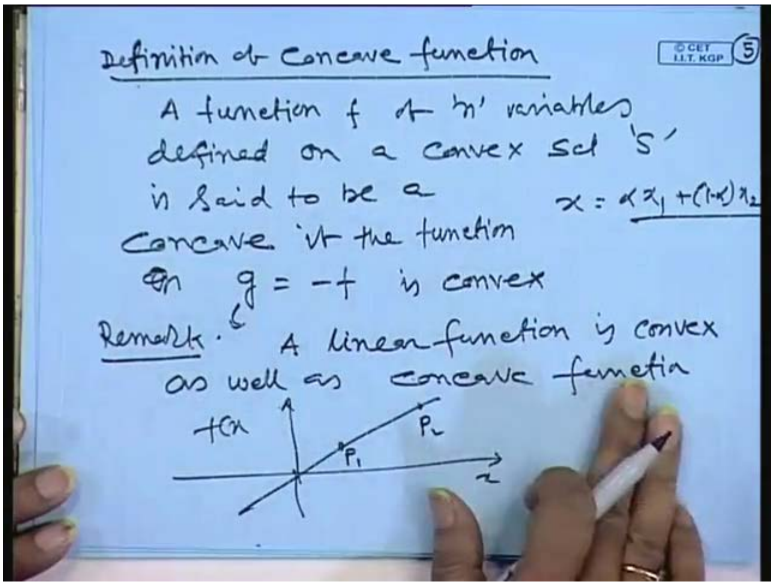

On a convex set S, if it is a divine on a convex set x than any point on the set, I can express any point on the on this at x. I can express alpha into x 1 2 points if you know 1 minus alpha into x 1 x 2 and that x 2 belongs to that set when it is a convex set in the set S is said to be concave if the function that our f is multiplied by f minus 1. That g is convex if minus f is a convex, then I can f is our concave function on that is in belongs to that set, convex set again. So, this next is remark was remarks a linear function is convex as well as concave functions just say a linear function is convex as well as concave function, and a linear function.

I can treat as a convex or a concave function that is called this is again it is a single variable case of you see or this is our linear equation that belongs to in a convex set. Now, whether it is a convex function or not or concave function and not if you take two points on this set that is p 1 and p 2 joined to this. It is just the same line as the original straight line, so naturally it may beat convex as well as concave function with equality sign is now it is belongs, it is just valid equality sign both cases. So, you want just remember a linear function it can be a convex function or concave function. I can say both this one next is your the properties of that what is called convex functions some properties will just discuss before we discuss the optimization problem using the what is called convex optimization problems.

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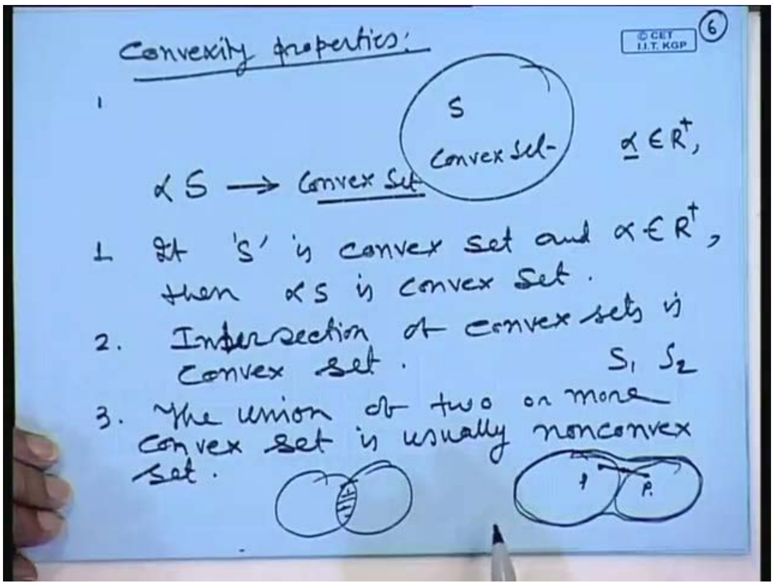

So, the properties convexity properties convexity properties, so one thing if S is a convex setif S is a convex set S is that our convex setand alpha belongs to our real positive quantity alpha is any positive real quantity than S is our set which is a convex set. This is a convex set this S is a convex set, and then if you multiplied by this convex set by alpha this also a convex set what I am telling if you S is a convex set.

If you multiplied by that set either you enlarge or you just too admitted this by alpha may be greater than 1 less than 1, but positive quantity. So, this one then alpha into S is also convex set that means if alpha is greater than 1, and then set is enlarged, if alpha is less than this set is contracted.

So, this one property what is this properties if S is a convex set convex set and alpha belongs to that r plus 1, then alpha into S is convex set. This is another with property is there if there are two convex set is there any intersection of two convex set is also a convex set. We have two sets of the S 1 and S 2 are the two convex set a intersection of these two convex set is also a convex set, but union of two convex set is not a convex set it is a non convex set. So, we will write it this intersection of intersection of convex sets is convex set the two sets of their let us call S 1 and S 2.

The two sets intersects which one is the intersects of this one, set is there that belongs to that common that convex set that the union of two or more convex set. The union of two or more convex set is usually non convex set that means S 1 is a convex set S 2 Conway union of this one. Let us call this is a convex set and this is a convex set something than you see this is the union of this one is not necessary a convex set because you take a 1 point p 1 is here p 2 is here p 3 is there.

We joined this two points any points on the line of this one must lie in the convex set, but it is these points are not belongs to the convex set S. So, this is not a union of two or more convex set is not usually not a convex set whereas, the intersection of two convex set the intersection of two convex with this one belongs to it that our convex set to this third point is this one. Now, if the if f is a convex functionf is a convex if you if you multiplied by f by a scalar quantity which is greater than positive quantity which is positive quantity than result is a convex function in that convex set.

So, another property is that if you have it two convex functions are there two convex function f 1 and f 2 convex function, they are some of these two convex functions is also is a convex function. So, last two properties like this when if f 1 is a convex function f 1 multiplied by some real positive number this f 1 the resultant is a convex function i n that convex set.

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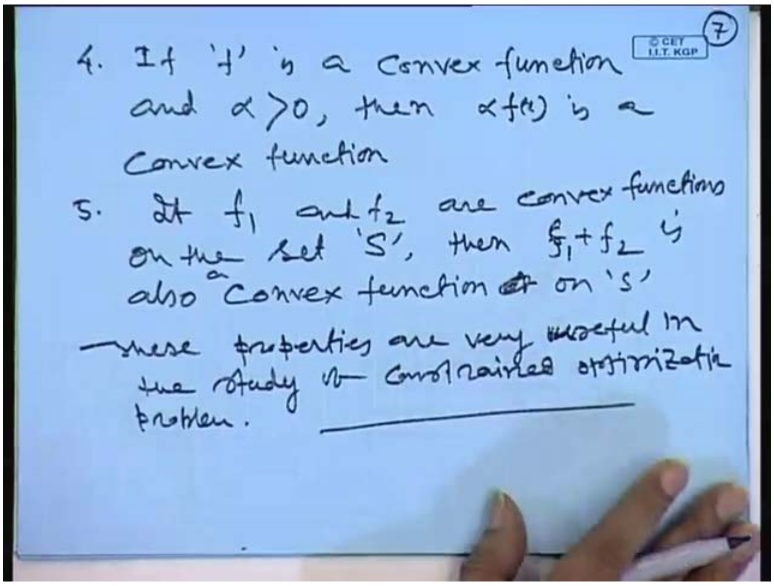

So, our fourth point is if f is a convex function and alpha is a real positive number then alpha into f of x is a convex function in that set. Then fifth properties are if just now you mention if f 1 and f 2 are convex function convex functions on the set S. So, then S 1 f 1 plus f 2 is also it also a convex function on S this promo you can do simple by the definition of our convex functions that in the definition of convex function from that one the 4 and 5 you can easily verified this one.

So, f is a function of convex functionand multiplied by positive quantity, so that function also belongs to that convex set. Now, you write it because f 1 of f 1 and f 2 are two convex function on the set, so you take a any two points on that set what the condition for f 1 to be convex set toward the condition f to be convex set that add to the these two things. This result in you sees whether you will be able to express in this form if you cannot express this form than it is a convex function of that one. That can be easily proved, so these properties are useful in the study what is called constraint problems concentrating optimisation problems.

So, these properties are very useful properties are very useful in the study of constant optimization problems mentioned problem. So, let us say some of the facts of convex optimization of convex functions a, so let us call f is a function that belongs to c 2. This indicates a c 2 means it is continuously twice differentiable function is continuously 2 means twice differentiable.

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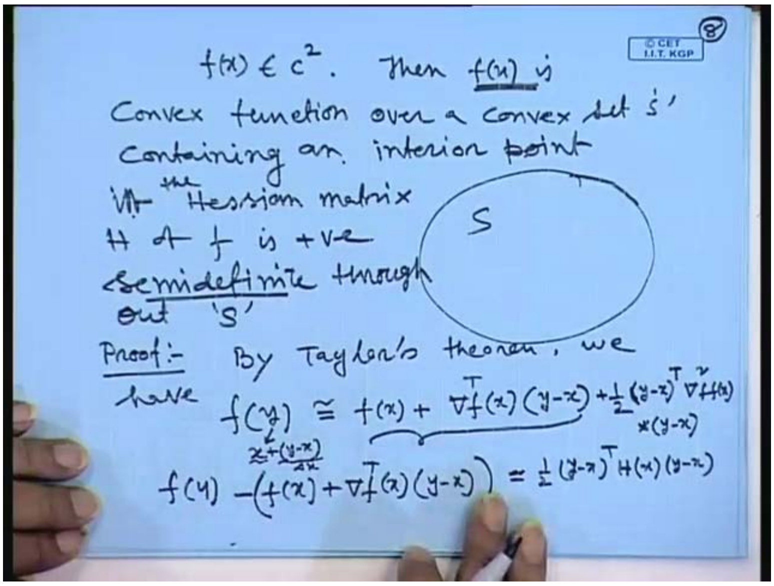

If I write this is a 3 times continuously differentiable this function, so this is a the function f of x is continuously twice differentiable see continuously power to means twice differentiable. If this f of x is convex function over a convex set S of the f of S is convex set over a convex function S containing an interior point. So, we have a f of function is there belongs to that set and there is a interior point one point in the in the set in there point different if only if this function is a convex set. If a normally over the set S containing a interior point one interior point if and only if the Hessian matrix the Hessian matrix h of f is a function.

We know where earlier lecture how to find out the Hessian matrix that means second partial differentiation of f with respect to x the Hessian matrix if and only if the Hessian matrix is positive semi definite. It is positive semi definite throughout S set many convex set in the convex end any point on the convex at any point on the convex set. If the Hessian matrix of f Hessian metrics of f means second partial derivative of the f this is f in Hessian metrics over the points of the set if it is a positive semi definite metrics or positive definite metrics.

Then, I will call the function is a convex function, so let us see this proof of that one that is once again I repeat that one. Suppose, if you have a function f of x and which is belongs to c 2 means continuously twice differentiable. Then function is called convex function about the convex set S containing that interior point if and only if the function if you do the second derivative that twice. If you differentiate this one, the function with effect to S switches the Hessian matrix and this Hessian matrix is positive semi definite over the convex set.

Then, we will call the function is a convex functions or the Hessian metrics is positive definite metrics also it will be the convex function proof. So, let us call we have since we have saying that there is a interior point is there in the set of this one let us consider by Taylor's expansion now by Taylor's theorem we have supposed we have a f of y f of y. I am writing y is there is a interior point around this interior point there is a y x is in turn around this one is y. So, I am writing y, I am writing as if it is x this is delta x, now this equal to by Taylor's expansion. I can write nearly equal to f of x plus gradient of this function transpose of f of x at this one again multiplied by incremental change y of x delta x plus the second terms in the Taylor.

Third terms in the Taylor of second order comes of this one is half that y minus delta this is a delta x transpose into that Hessian metrics that f of x multiplied by y minus x whether x is the vector scalable basis to this is nearly equal to this 1. So, f of y if you bring this part in that site left inside f of y minus this part is equal to this quantity and this quantity will be positive, right hand side will be positive or negative depending upon the value of the Hessian matrix over the that our set S. If we can say that Hessian metrics value over the set S is a positive semi definite metrics it indicates this quantity is a positive semi definite. This quantity means that that this quantity is positive semi definite that means this value of this one either it will be 0 or it could be greater than 0.

This one, now this I can write it, now if just see this one f of y minus f of x plus gradient of this f of x into y minus x. This quantity is equal to nearly equal to I can write it half y minus x transpose and I am writing the gradient of that one again and gradient of this one let us call x that, that is not to that is Hessian matrix of this on h into y of x. Now, this side value right hand side value will be positive or 0 when this metrics value over the set S if it is that positive or 0 when it will be this Hessian metrics should be positive semi definite.

It mean this indicates that this positive means this indicate f of y is greater than this quantity. So, let us say what is this meaning of this one a f of y is greater than this quantity what is this means, so I can write it is clearly, I can write from the equation that or this x for any value of x in this because I told you this is our x and this is our y any point.

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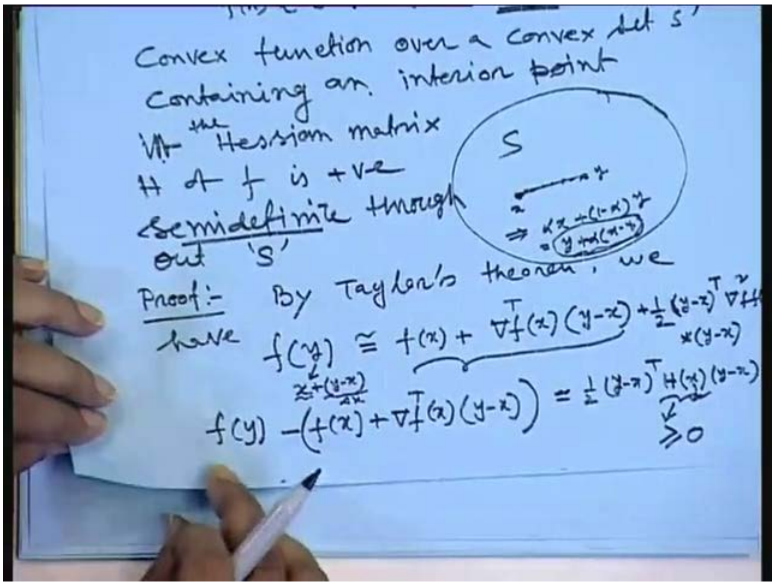

In this, I can write it alpha into x plus 1 minus alpha into y any point of this one any point in two points in this. So, I can write it this alpha into this if you can write itthis the y plus while that utility in this one. I can write out file into x minus y just see this one in this x is this point y is this point I multiplied by this one some polygon alpha means since this two points belongs to that set S any point joining this to point this chord any point on this line belongs to that set.

This means it implies that one, so y is that one than alpha into x alpha into x minus alpha into y minus alpha into y that one. So, I can write in place of anything here point of this one x will be now replaced by that quantity you can replace by the quantityany point on this chord on this set. I can replace, so this positive indicates this one positive indicates that this Hessian matrix this metrics should be either positive or 0 when the Hessian metrics will be positive semi definite metrics.

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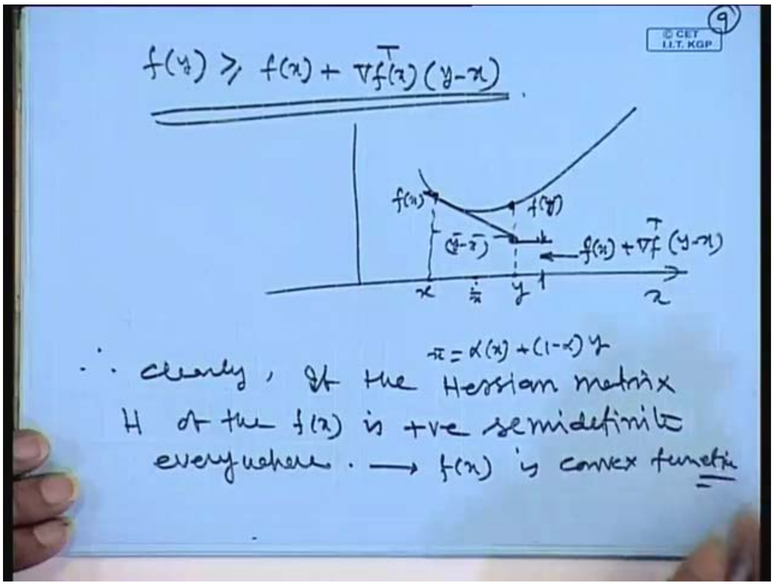

So, this indicates that f of y f of y this indicates that f ofy clearly this indicates that f and y will be greater than equal to f of x minus plus, sorry plus gradient of gradient transpose of f of x into y minus x. This indicates geometrically one can see because this is the condition if you see I am writing f of x is twice differentiable controller. The function is continuous the function is convex function over the set containing a interior point x if and only Hessian metrics of x is positive in every metrics. So, what is this one if you see this one for a scalar variable case x and we have at this type of functions is there so this is our f ofx this is f of x and y is in some point is here y.

This is f y this is y and any point in this and we can express by a convex set definition is alpha into x plus one minus alpha to y is equal to any point on this write it let us call any point on this one is ever x bar. I can write x bar this one, now see this one what is this meaning of that one f of y function value is that quantity is always greater. Then what is the function value is here plus this quantity and what is this this is the gradient at this point for scalar case is nothing but a slope at this point multiplied by changing this one. That means this, but this slope multiplied by this what you will get it that quantity this is that you y minus x again.

So, the y minus x and this is the slope of that one slope means this this quantity divided by this quantity the slope is given multiplied by this multiplied by this if you know the slope multiplied by this. You will get it this quantity to f x minus of this quantity means this quantity indicates this is f of x plus delta f of transpose y minus x is always greaterless than equal to thisis always less than equal to that one.

Therefore, if clearly we can say clearly if the Hessian matrix h of the function of the function f of x is positive semi definite everywhere positive semi definite everywhere than function than f of x is a convex function everywhere. Then f of x is a convex function, so this is that there is a important theorem is they did this one that one because this is a convex optimization problem this theorem is most important one that can theorem.

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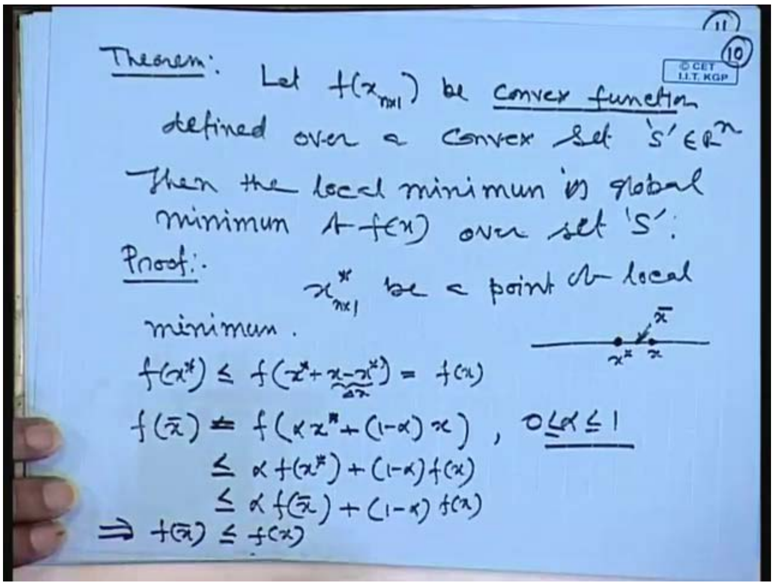

Let f of x whose dimension is n cross 1 big convex function defined this function is divine over a convex set this function convex function defined over a convex set S belongs to get in n dimensional case. Then the local minimum is the global minimum than the local minimum provided the function is convex function mind it if the function is convex function of the convex set S. Then local minimum of this function will give you the global minimum of this function global minimum of the function local minimum is global minimum of f x over the set over the set S. So, what is a proof is very simple if you see let us call x star x star n cross 1 be a local be a point of local minimum is the local minimum point can minimum.

That means we have this at this point is local minimum we got x star, so we can write it now f of x star is equal to less than equal to if you take any point around this x. So, effervescent since it is a local minimum point f of x is less than equal to x star plus x minus x star this is a. You can say delta x, this that the delta x you can say, so is less than this is equal to our f of x, this.

So, you can write it there is a let us call there is an n, now this two points at their and these two pints belongs to a convex set any point on this line I can write it to this one f of x bar this by definition of this one. I can write it the equal to what is what the let us call this point is x bar this point is x bar, so I can write it this x bar is equal to that x alpha into x star alpha into x star plus 1 minus alpha into x is x bar where alpha greater than 1 less than equal to 0. You can write it now this you can writeit by definition of convex function because is a convex function that definition we can write it that alpha into f star of x plus 1 minus alpha into that what is called f of x.

We can write that that one since f x is a convex function that one, so this we can write it since x bar if this is the x star is a optimal point minimum point x bar other than this. So, I can if I write it in place of x star, if I write it x bar this quantity x bar is greater than x star function value because x star is a optimal point plus 1 minus alpha f of x. So, if you take it this is that site this quantity that site this implies after taking this site this implied f of x bar is less than equal to f of x.

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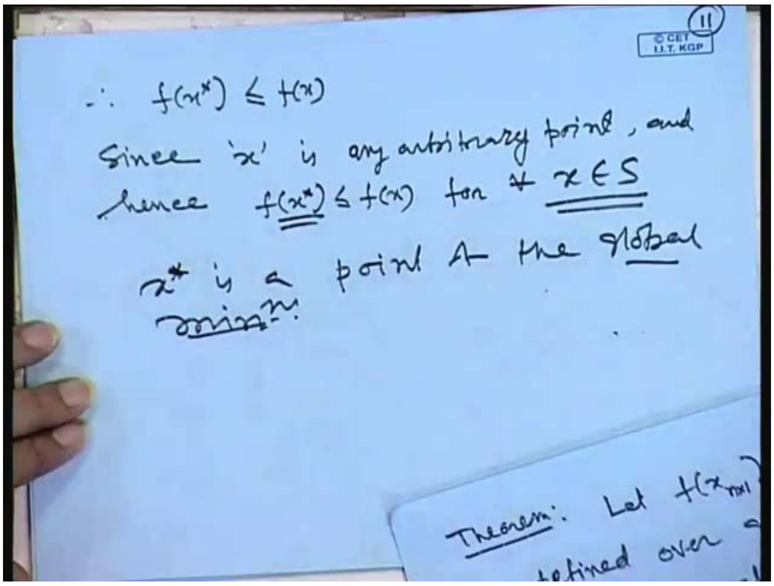

This we have shown to this implies and we know f of x star is also less than equal to f of x. So, since if you see since f of x since x is a any arbitrary point x is any arbitrary point hence and hence f star of x is less than of f x see this one x x that is what reacts is any arbitrary point this may be here. Also, this site because this belongs to that site, now this we can write it for all x belongs to that set. So, our conclusion is that x staris a point of the global minimum because this two points I have taken it this on this two points belongs to that is convex set and x maybe any added point.

Since, f x value is less than f x l a and we have shown you there f star is less than this one it indicates the f start of x is less than any point on the global means in the set itself or else this f start is less than that 1. So, it is a global minimum, so our conclusion is if f x is a convex function of what is set S and that minimum point that means meme of local minimum is nothing but a global minimum of the function if f of x is a convex function. Then only over the convex it set, so next class will just discuss other points will stop it here now.